A Study of the Haar Measure Problem

SURG | Natural Sciences and Engineering (NSE) | *Tags: Theory*

This cover page is meant to focus your reading of the sample proposal, summarizing important aspects of proposal writing that the author did well or could have improved. ***Review the following sections before reading the sample****. The proposal is also annotated throughout to highlight key elements of the proposal's structure and content.*

Proposal Strengths	Areas for Improvement
The proposal defines and explains key terms and concepts in simple terms, using real-world examples and analogies to help a reader visualize.	At the end of the methodology, we suggest including "metrics of success" that demonstrate how you will know when you've answered your research question. You can create an "If, Then" statement whereby you work a reader through a possible result and how you would interpret it.
The researcher explicitly identifies gaps in knowledge and makes claims for why it is important to fill these gaps using evidence from past research to support their assertions.	We do not recommend using footnotes even if it is common in your field due to space restrictions.
The methodology includes a timeline. Also, the methods are justified in terms of how they help to answer the research question/address the gap in knowledge.	While an aim/objective statement is present, rephrasing to create or including an explicit research question could help add clarity and comprehension for non-expert audiences.

Other Key Features to Take Note Of

With work that engages heavily with theory, it is often the case that part of your justification relates to the underlying logic/assumptions with which you are starting. It is critical to give the reader a sense of where those established definitions/assumptions/logic came from, citing your sources wherever possible.

The idea of integration underlies many important applications today, such as finding the center of mass of an object (integrating over its mass distribution) or computing the expected value of a random variable (integrating over the distribution of its values). In the 19th century, Riemann proposed arguably the first successful rigorous treatment, and probably the most well-known version of integration – Riemann integral. However, it was not until Lebesgue rephrased Reimann's procedure in the language of measures that integration obtained its full form today. This project will focus on a particular class of measures, called the Haar measures, and try to identify what are referred to as the non-measurable sets associated with these measures. Overall, this is known as the Haar Measure Problem, which asks the exact question in a rather abstract setting. This project will approach the general problem through identification of non-measurable sets in several specific examples, in the hope of gaining insights to the original problem in the most abstract setting.

The focus of the project is clear within the first paragraph

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In an attempt to generalize the idea of addition to continuous settings (for instance adding up the area beneath a curve), Riemann integral essentially breaks down the question onto finer and finer intervals and approximate the final result by adding up easily computed sub-results on the intervals. However, when applied to more intricate situations where the subject of integration is more fractural (for example involving a lot of discontinuities), Riemann's definition is prone to contradiction and confusions (Rudin 76-78). To address such deficiencies, French Mathematician Lebesgue proposed the following: there can be a more general notion of length applied to more general sets – not just the intervals but sets that can in some sense be approximated by intervals; then we break down the question further into these new sets. These new sets – an analogy to the original basic intervals – are what people refer to as measurable sets, and the notion of length defined on these "new intervals" their measures.

Lebesgue's construction turned out particularly successful. Not only did his theory fix many problems with Riemann's procedure, but it was also generalizable to very broad settings: given any set, people can define the intervals of that set and what the length of those intervals are, as long as certain technical conditions are satisfied. Take the natural numbers as an example, we can define any subset to be an "interval" (i.e. a measurable set formally), and its "length" (i.e. the measure) simply the number of elements it contains. It turns out in this case the Lebesgue integral is equivalent the usual notion of addition of natural numbers! Therefore expectedly, numerous different measures have been developed since the introduction of Lebesgue theory – for one, the entire modern probability theory is based on the notion of probability measures – and ongoing studies focusing on the properties of these new measures fundamentally drive the development of the relevant fields.

Identifies gap in knowledge

Justifies why gap should be filled

This project focuses on the Haar measures: measures that preserve the length of a measurable set when it is shifted leftwards – a property referred to as left-invariant. This may seem apparent for a traditional interval: shifting the interval [0,1] leftward to [-1,0] surely doesn't change its length. Nevertheless, when applied to more complicated constructions, this property may not hold in general. For example, given a random variable, the probability measure of [0,1] (i.e. the probability of the random variable lying in [0,1]) may indeed not be the same as the probability measure of [-1,0]. Therefore, the study of left-invariant measures is of interest in its own right, and in fact also serves to lay the foundations for other crucial theorems such as the change of variable formula often cited in calculus.

Identifies gap in knowledge

Justifies why gap should be filled

One final step towards the full formulation of the Haar Measure Problem is to note that: given a set, with a well-defined collection of measurable subsets and a measure, it is not intrinsically clear what the non-measurable subsets precisely are. Such subsets are those for which there is no well-defined notion of length, which could happen if a set is "spreading out" too randomly and densely. In the natural number example given above, all subsets are measurable. However, consider for instance the traditional notion of length of an interval. On the one hand, it turns out impossible to generalize this notion of length to all subsets of the real line (Solovay); on the other, it turns out also very hard, though accomplished, to find a non-measurable subset

Identifies gap in knowledge

despite people's awareness of their existence (Franks).[1] Hence the Haar Measure Problem is motivated: "Does every infinite compact group have a non-measurable subgroup?", where an infinite compact group is a general setting in which left-invariant measures make sense.

> Justifies why gap should be filled

A paper in 2008 successfully addressed the Haar Measure problem by proving the existence of the desired non-measurable subsets (Przezdziecki et al). However, the proof explicitly resorted to what is called the Continuum Hypothesis, which roughly assumes that any set containing more elements than the natural numbers, must have at least as many elements as the real numbers. Accepting or rejecting the hypotheses, though, would necessarily place one into two incompatible logical frameworks, and it is hence usually of interest to see that a useful theorem remains true in either case. **This project thus aims for the construction of non-measurable subsets for particular instances of infinite compact groups, such as spaces of orthogonal matrices, without explicit usage of the continuum hypothesis.** Specific constructions may provide a better idea on exactly at which point the continuum hypothesis generally kicks in, and potential guidance to circumvent the hypothesis in the construction of non-measurable subsets in more general settings.

> Background section narrows from broad topic to specific aims of project period

> While an aim statement is present, rephasing to an explicit research question would add clarity

Indeed, for certain measures constructions of non-measurable subsets without resort to the Continuum Hypothesis are already in place. The Vitali sets noted above is one such example. Other examples include the Lebesgue measure defined on the unit circle, where moving an interval leftward is interestingly accomplished by multiplication with e^{it} (Watson and Wayman).[2] Therefore, the first two weeks of the research cycle would partly be spent on analyzing past sample constructions such as the two mentioned here, which shall provide guidance for the intended constructions of this project on more complicated and abstract instances. The rest of the first two weeks would be used to get familiarized with certain more advanced topics in group theory, which shall promote a clearer picture on the Haar Measure problem overall. Then, equipped with enough techniques and guidance to attempt for new original constructions, the latter 6 weeks of research will focus on the main goal of the project: to construct non-Haar-measurable subsets without resorting to the Continuum hypothesis on specific examples, but those more complicated, general (than for instance traditional Lebesgue measure) and previously untried, including in particular groups of n*n orthogonal matrices where the left-shift operation is given by matrix multiplication. Ultimately, if successful, this project shall provide several additions to sample constructions of non-Haar-measurable subsets, favorably on more abstract examples. In the meantime, the specific constructions may unveil certain insights on the nature of these non-measurable subsets. For instance, comparing the construction outlined in the 2008 paper (which uses the Continuum Hypothesis) applied to matrix groups with a self-derived construction without using the Continuum Hypothesis may promote understandings on how the subsets interact with the hypothesis and even intuition on how to avoid using the Hypothesis overall.

> Timeline is useful in the methods

> Should include metrics of success, meaning, how you will know when you have answered your research question

Personally, I completed Linear Algebra during my freshman year, which provides sufficient background in working with spaces of matrices intended as potential research subjects for the project. Since then, I finished the entire MENU sequence in analysis which culminated in a careful introduction to measure theory and Lebesgue integration. Moreover, I will also complete the entire sequence in topology and MENU probability theory before summer, both helpful for the understanding of general collections of sets and measures other than the traditional Lebesgue or Borel measures. Currently, I'm looking forward to master's programs in math, such as part III studies typically offered at British universities, and I hope that a research opportunity over the summer may prepare me better for more advanced studies in mathematics and help me gain exposure to materials more on the frontier.

[1] A classic example is the Vitali sets, discussed in detailed in Introduction to Lebesgue Integration by John Franks, Appendix C.
[2] Another established example on Torus can be found in the paper by Saleem Watson and Arthur Wayman.

> Footnotes are not necessary

Works Cited

Franks, John. *A (Terse) Introduction to Lebesgue Integration*. McGraw-Hill, Inc.,1976.

Przezdziecki, Adam J, et al. “The Haar Measure Problem.” 11 May. 2018, https://arxiv.org/pdf/1709.02312.pdf. Accessed 13 March 2019.

Watson, Saleem and Arthur, Wayman. “Visualizing a non-measurable set.” April, 2003. https://www.maa.org/sites/default/files/Visualizing_a_Nonmeasurable18723.pdf. Accessed 12 March 2019.

Solovay, Robert M. “A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable.” July. 1970, https://www.jstor.org/stable/1970696. Accessed 14 March 2019.

Rudin, Walter. *Principle of Mathematical Analysis*. Providence: American Mathematical Society, 2009