EXERCISES, SEMICLASSICAL ANALYSIS AT MAQD 2024

References:

- Primary: M. Zworski, *Semiclassical analysis.* Graduate Studies in Math, 138. American Mathematical Society, Rhode Island, 2012.
- L. Hörmander, *The analysis of linear partial differential operators. I.* Distribution theory and Fourier analysis. Second edition Springer Study Ed. Springer-Verlag, Berlin, 1990.
- L. Hörmander, *The analysis of linear partial differential operators. III* Grundlehren Math. Wiss., 274. Springer-Verlag, Berlin, 1985
- A. Martinez, *An introduction to semiclassical and microlocal analysis.* Springer-Verlag, New York, 2002

Notational reminders at the end of the file.

Warning/hint: A good number of these problems have their solutions in Zworski's book.

Some problems will be done in the problem session.

(1) Recall $\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \sup_x \langle x\rangle^N |D_x^\alpha \phi(x)| < C_{N,\alpha}\ \forall\ N \in \mathbb{N},\ \text{multi-indices } \alpha\}$. Show that the semiclassical Fourier transform maps $\mathcal{S}$ to $\mathcal{S}$.

(2) For a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ and a function $f \in C^\infty(\mathbb{R}^n)$ with $|f(x)| \leq C(1+|x|)^N$ for some C, N, we define fu via $(fu)(\phi) = u(f\phi)$ for all $\phi \in \mathcal{S}$. Show, using this definition and the definition of the derivative and Fourier transform of an element of $\mathcal{S}'$, that $\mathcal{F}_h(x_j u) = ih\partial_{\xi_j}\mathcal{F}_h(u)$ and $\mathcal{F}_h(h\partial_{\xi_j} u) = i\xi_j \mathcal{F}_h u$.

(3) Let $\{u_j\} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence. If, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{j\to\infty} u_j(\phi) = u(\phi) \in \mathbb{C}$, then $u \in \mathcal{S}'$ and we say $u_j \to u \in \mathcal{S}'$.
Do each of the following distributions have a limit in $\mathcal{S}'(\mathbb{R})$ as $h \to 0^+$? Identify the limit, if it exists. (As in lecture, we identify a (sufficiently nice) function u with a distribution via $u(\phi) = \int \phi u$.)
   - $u_h = h^{-k} e^{ix/h}$, $k \in \mathbb{N}$
   - $u_h = h^{-1/2} e^{-ix^2/2h}$

(4) Let $m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Then $H_h^m(U) = \{f \in L^2(U) : \int_U |(hD_x)^\alpha f|^2 < \infty \text{ for all multi-indices } \alpha,\ |\alpha| \leq m\}$, with
$$\|f\|_{H_h^m(U)} = \sum_{|\alpha|\leq m} \|(hD_x)^\alpha f\|^2_{L^2(U)}.$$
Show that if $U = \mathbb{R}^n$, this is equivalent to the norm defined using the semiclassical Fourier transform, by
$$\|f\|^2_{\tilde{H}_h^m} = \int_{\mathbb{R}^n} (1+|\xi|^2)^m |\mathcal{F}_h f(\xi)|^2 d\xi.$$
Note: This second shows how to define h-Sobolev spaces for m not a non-negative integer. Moreover $H_h^{-m}(\mathbb{R}^n)$ can be equivalently be defined as the dual of $H_h^m(\mathbb{R}^n)$.

(5) Let $a \in C^\infty(\mathbb{R}^{2n})$ satisfy $|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$ for some m and all multi-indices α, β. Let $\phi,\ \psi \in C_c^\infty(\mathbb{R}^n)$ have $\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset$. Show that the operator $\phi A\psi$ defined by
$$(\phi A\psi u)(x) = (2\pi h)^{-n} \iint e^{i\langle x-y,\xi\rangle/h}\phi(x)a(x,\xi)\psi(y)u(y)dyd\xi$$

satisfies

$$\|\phi A\psi\|_{H_h^{-N}\to H_h^N} = O(h^\infty)$$

for any $N \in \mathbb{N}$. (Note that $A = a(x, hD)$, but do this directly.)

(6) Suppose $a \in \mathcal{S}(\mathbb{R}^{2n})$ (or, more generally, that $a \in S(m)$). Show that

$$a(x,\xi) = e^{-i\langle x,\xi\rangle/h} a(x,hD) e^{i\langle\cdot,\xi\rangle/h}.$$

(7) (a) Show that if $a \in C_c^\infty(\mathbb{R}^{2n})$, there is a $b \in \mathcal{S}(\mathbb{R}^{2n})$ (which in general will depend on h as well) so that for all $u \in \mathcal{S}$,

$$b(x,hD)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle/h} a(y,\xi)u(y)dy\, d\xi.$$

Moreover,

$$b(x,\xi) \sim \sum_{j=0}^\infty h^j \frac{(-i)^j}{j!} \langle \partial x, \partial \xi\rangle^j a(x,\xi) = \sum_{j=0}^\infty h^j (-i)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha a(x,\xi).$$

(b) Show that if $a \in C_c^\infty(\mathbb{R}^{2n})$, there is a $b \in \mathcal{S}(\mathbb{R}^{2n})$ so that for all $\phi,\ \psi \in \mathcal{S}(\mathbb{R}^n)$, $\langle a(x,hD)\phi, \psi\rangle_{L^2(\mathbb{R}^n)} = \langle \phi, b(x,hD)\psi\rangle_{L^2(\mathbb{R}^n)}$. (Here $\langle\phi,\psi\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \phi(x)\overline{\psi}(x)dx$ is the L^2 inner product.) Moreover,

$$b(x,\xi) \sim \sum_{j=0}^\infty h^j \frac{(-i)^j}{j!} \langle \partial x, \partial \xi\rangle^j \overline{a}(x,\xi) = \sum_{j=0}^\infty h^j (-i)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a}(x,\xi).$$

(8) Show that if $a \in S(\langle\xi\rangle^k)$, then $a(x,hD) : H_h^k(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with norm bounded independent of $h \in (0,h_0]$, and if $a \in S(1)$, then $a : H_h^k(\mathbb{R}^n) \to H_h^k(\mathbb{R}^n)$.

(9) Let $m(x,\xi)$ be an order function that satisfies $m(x,\xi) \to 0$ as $|(x,\xi)| \to \infty$, and let $a \in S(m)$.

- Show that for any $b \in C_c^\infty(\mathbb{R}^{2n})$, $b(x,hD) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact. (Hint: You can show $b(x,hD) : L^2(\mathbb{R}^n) \to H_h^1(\mathbb{R}^n)$, and that the functions in the image of $b(x,hD)$ have support in a fixed compact set.)
- Show that $a(x,hD) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.

(10) Here are two ways to define the semiclassical wavefront set for a tempered family $u = \{u_h\}_{0<h\le h_0}$. ($u = \{u_h\}_{0<h\le h_0} \subset \mathcal{S}'$ is a tempered family of distributions if there are $k,\ l,\ N \in \mathbb{N}_0$ so that $\|\langle x\rangle^{-k}u\|_{H_h^{-l}(\mathbb{R}^n)} = O(h^{-N})$.) :

(1) $(x_0,\xi_0) \in \mathbb{R}^{2n}$ is *not* in $\mathrm{WF}_h(u)$ if and only if there are $\phi,\ \psi \in C_c^\infty(\mathbb{R}^n)$ (independent of h) with $\psi(\xi_0)\phi(x_0) \neq 0$ and $\psi(\xi)\mathcal{F}_h(\phi u)(\xi) = O(h^\infty)$.

(2) $(x_0,\xi_0) \in \mathbb{R}^{2n}$ is *not* in $\mathrm{WF}_h(u)$ if and only if there is an $a \in S(1)$ with $|a(x_0,\xi_0)| > \gamma > 0$ so that $\|\langle x\rangle^{-k}a(x,hD)u\|_{H_h^{-l}(\mathbb{R}^n)} = O(h^N)$ for all N. (Here γ is independent of h.)

For these exercises, assume u is tempered.

(a) Suppose $u \subset L^2$. (So k, l in definition above are both 0.) Show, under the second definition, that if $(x_0,\xi_0) \notin \mathrm{WF}_h(u)$, then for all $b \in S(1)$ with support sufficiently close to (x_0,ξ_0), $\|b(x,hD)u\|_L^2 = O(h^\infty)$.

(b) Show that the two alternate definitions given above are in fact equivalent. To make things a bit easier, take u tempered in L^2.

(c) Show that if $a \in C_c^\infty(\mathbb{R}^{2n})$ then $\mathrm{WF}_h(a(x,hD)u) \subset \mathrm{WF}_h(u)$. (Or do the same, with $a \in S(\langle x\rangle^k\langle\xi\rangle^l)$.)

(d) Fix $\omega \in \mathbb{R}^n \setminus \{0\}$, and $u(x) = u(x;h) = \exp(i\langle x,\omega\rangle/h^\alpha)$.
Check that

$$\mathrm{WF}_h(u) = \begin{cases} \mathbb{R}^n \times \{0\} & \alpha < 1 \\ \mathbb{R}^n \times \{\omega\}, & \alpha = 1 \\ \emptyset & \alpha > 1. \end{cases}$$

(e) Show by example that $\mathrm{WF}_h(u)$ can be empty even if u is not smooth for any (fixed) value of h in $(0,1)$.

Reminders of notation

- $h > 0$; idea is for h small
- semiclassical Fourier transform: $\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle/h} u(x)dx$
- inverse: $\mathcal{F}_h^{-1}(\phi)(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle/h}\phi(\xi)d\xi$
- multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$
- $D_x^\alpha = (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial x_n)^{\alpha_n}$, $(hD_x)^\alpha = h^{|\alpha|}(D_x^\alpha)$.
- $\mathcal{F}_h((hD)^\alpha u)(\xi) = \xi^\alpha \mathcal{F}_h u(\xi)$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$
- $\langle x\rangle = (1 + |x|^2)^{1/2}$
- Schwartz functions: $\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \langle x\rangle^N |D_x^\alpha \phi(x)| < \infty \ \forall\ N \in \mathbb{N},\ \forall \text{ multi-index } \alpha\}$
- tempered distributions: $\mathcal{S}'$ is the dual of $\mathcal{S}$
- $\mathcal{F}_h : \mathcal{S} \to \mathcal{S}$; $\mathcal{F}_h : \mathcal{S}' \to \mathcal{S}'$