DIFFERENTIAL FORMS ON LOOP SPACES AND THE CYCLIC BAR COMPLEX

Ezra Getzler, John D. S. Jones and Scott Petrack

Dedicated to the memory of Frank Adams

Contents:

Introduction

In this article, we present a model for the differential graded algebra (dga) of differential forms on the free loop space LX of a smooth manifold X and show how to construct certain important differential forms in terms of this model. Our motivation is an observation of Witten (described by Atiyah in [2]) that the index theorem for the Dirac operator can be thought of as an application of the localization (or fixed point) theorem in $\mathbb{T}$-equivariant homology, suitably generalised to the infinite dimensional case of the free loop space; here, $\mathbb{T}$ is the circle group. We can summarise our main results as follows. We are really concerned with equivariant differential forms and equivariant currents and we show how to reformulate these geometric objects as cyclic chains and cochains over the differential graded algebra $\Omega(X)$ of differential forms on X. In fact, the cyclic chain complex of $\Omega(X)$, if it is normalized correctly, is a sub-complex of the complex of equivariant differential forms on LX, and is a good enough approximation that it allows us to compute the ordinary and equivariant cohomology of LX, and to write down explicitly certain important differential forms and currents.

We begin by explaining the motivation more carefully. Let S be a Clifford module on X with Dirac operator D. Witten observed that it should be possible to associate to D an equivariantly closed (inhomogeneous) current μ_{D} on the free loop space of X. The basic property of this current is that the index of D is given by pairing μ_{D} with the differential form $1 \in \Omega^0(LX)$:

$$\langle \mu_{\mathsf{D}}, 1 \rangle = \operatorname{ind}(\mathsf{D}).$$

The source of this current is the formalism of path-integrals in supersymmetric quantum mechanics.

In [3], Bismut showed how to generalize this formula by associating to a vector bundle E over X, equipped with a connection ∇, an equivariantly closed (inhomogenous) differential form on LX.

This is the **equivariant Chern character** $\mathrm{Ch}(E,\nabla)$. The first basic property of $\mathrm{Ch}(E,\nabla)$ is that the index of the twisted Dirac operator D_E on $S\otimes E$ is given by the formula

$$\langle \mu_{\mathsf{D}}, \mathrm{Ch}(E,\nabla)\rangle = \mathrm{ind}(\mathsf{D}_E).$$

The second property is that if $i : X \longrightarrow LX$ is the inclusion of the point loops then

$$i^* \mathrm{Ch}(E,\nabla) = \mathrm{Tr}\exp(\nabla^2)$$

where ∇^2 is the curvature of ∇. In this paper, we give a formula for this equivariant Chern character from the point of view of our model for equivariant differential forms.

The formalism of this paper leads to an attractive formula for the Witten current μ_{D}, from which it is easy to prove the above index formulas (see [8]). These constructions make as explicit as possible the algebraic nature of the path integral for the Dirac operator. We hope that there exists an analogous algebraic version for the supersymmetric nonlinear-sigma model in $1+1$ dimensions; although this theory has not yet been shown to exist rigourously, Witten has demonstrated suggestive relations between it and elliptic cohomology.

Our model for equivariant differential forms on free loop space is based on Chen's theory of iterated integrals, in a version due to Jones [12]. However, in this paper, we give explicit formulas where [12] made use of the method of acyclic models, and we introduce several new constructions for iterated integrals on free loop spaces. For example, we give a new chain level formula for the product structure of iterated integrals, which we show to be an A_∞-structure in the sense of Stasheff [17]. We also show how to construct iterated integrals that are not invariant under the action of the diffeomorphism group $\mathrm{Diff}[0,1]$ on LX, by replacing X by $X\times\mathbb{T}$. These extensions prove to be vital to make the connection between this model for differential forms and its applications to path integrals and index theory.

Let us describe in a little more detail what we prove. Let $\mathsf{C}(\Omega(X))$ be the dga

$$\mathsf{C}(\Omega(X)) = \sum_{k=0}^{\infty} \Omega(X)^{\otimes k+1},$$

graded by

$$\deg \omega_0\otimes\ldots\otimes\omega_k = \sum_{i=0}^{k}\deg\omega_i - k,$$

with differential b as defined in Section 2, and with product the shuffle product. We will refer to this dga as the **cyclic bar complex** of $\Omega(X)$. Likewise, let $\Omega(LX)$ be the dga of differential forms on LX, with differential d and product the exterior product. In Section 2, we define the iterated integral map $\sigma : \mathsf{C}(\Omega(X)) \longrightarrow \Omega(LX)$, and show that it is a homomorphism of dgas. The kernel of this map is denoted by $\mathsf{D}(\Omega(X))$ and has been described explicitly by Chen [6]. Chen also showed that this kernel is acyclic if X is connected; we recall these results in Section 2. We define the **Chen normalized** chain complex

$$\mathsf{N}(\Omega(X)) = \mathsf{C}(\Omega(X))/\,\mathsf{D}(\Omega(X))$$

and we can summarize Chen's results as follows.

Theorem (Chen). *The iterated integral map defines homomorphisms of* dga*s*

$$(\mathsf{C}(\Omega(X)),b) \longrightarrow (\mathsf{N}(\Omega(X)),b) \xrightarrow{\sigma} (\Omega(LX),d).$$

These induce isomorphisms on cohomology if X is simply-connected.

We are more interested in the version of this theorem which provides a model for equivariant differential forms on LX. Given a $\mathbb{T}$-manifold Y, this is the space $\Omega(Y)[u]$ of differential forms on Y, with an indeterminate u of degree 2 adjoined and differential defined as follows. Let φ_t be the circle action on Y, generated by the vector field T and let ι be interior product with T. The differential is the operator $d + u\tilde{P}$, where $\tilde{P} : \Omega^\bullet(Y) \to \Omega^{\bullet-1}(Y)$ is defined by

$$\tilde{P}\omega = \int_{\mathbb{T}} \varphi_t^* \iota\omega \, dt.$$

The operators d and $\tilde{P}$ satisfy the identities $\tilde{P}^2 = d\tilde{P} + \tilde{P}d = 0$, so that $(d + u\tilde{P})^2 = 0$.

The usual complex of equivariant differential forms is $\Omega_{\mathbb{T}}(Y) = (\Omega(Y)^{\mathbb{T}} \otimes \mathbb{R}[u], d + u\iota)$ where $\Omega(Y)^{\mathbb{T}}$ is the space of invariant forms on Y; see [4]. We will use the larger, but equivalent, complex $(\Omega(Y)[u], d + u\tilde{P})$, since the iterated integral map σ does not map into the space of invariant differential forms on LX.

It is precisely the behaviour of the iterated integral map with respect to products which is one of our main concerns in this paper. We must face the difficulty that the operator $\tilde{P}$ is not a derivation with respect to the exterior product on $\Omega(Y)[u]$, so that $d + u\tilde{P}$ is not a derivation on $\Omega(Y)[u]$ thought of as an algebra with this product. However, there is a simple way to perturb the exterior product so that $d + u\tilde{P}$ becomes a derivation with respect to the new product; this perturbation is similar to the perturbation $u\tilde{P}$ that we have added to d. We now face the difficulty that the new product is not associative.

There is a simple way to deal with this kind of situation; we use the theory of A_∞-algebras introduced by Stasheff [17]. In fact there is a natural A_∞-structure on the space of equivariant differential forms; we will give the precise formulas in Section 1. Our formulas for the product structure on iterated integrals lead to explicit formulas which strengthen the results of Hood and Jones [11] at the same time as simplifying their proofs; this is explained in detail in [9]. One of the reasons that this A_∞-structure on $\Omega(X)[u]$ may prove useful is that it enables us to make sense of the cyclic bar complex of $\mathsf{C}(\Omega(X))[u]$. We hope that this iterated cyclic bar complex will be a useful model for the complex of equivariant differential forms on the iterated loop space $L(LX)$ with the action of the torus group.

It is shown in Section 2 that Connes's operator B on $\mathsf{C}(\Omega(X))$ corresponds under the iterated integral map to the operator $\tilde{P}$ on $\Omega(LX)$. In Section 4, we generalize this by giving the formulas for the A_∞-structure on $\mathsf{N}(\Omega(X))[u]$ which corresponds to the A_∞-structure on $\Omega(LX)[u]$ of Section 1 under the iterated integral map. This leads to the following extension of Chen's theorem, whose proof occupies Sections 2–4.

Theorem A. *The iterated integral map defines a strict homomorphism of A_∞-algebras*

$$\sigma : \mathsf{N}(\Omega(X))[u] \to \Omega(LX)[u],$$

which induces an isomorphism on cohomology if X is simply-connected.

Here the term strict means that σ commutes with all the extra structure; this is much stronger than the natural notion of a homomorphism of A_∞-algebras. In particular, the theorem shows that

$$\sigma : (\mathsf{N}(\Omega(X))[u], b + uB) \to (\Omega(LX)[u], d + u\tilde{P}),$$

induces an isomorphism in cohomology if X is simply-connected, and that this isomorphism preserves products; but it gives much more precise chain level information than the corresponding statements at the level of cohomology.

In Section 5, we introduce a further generalization of the iterated integral construction, in which the manifold X itself carries an action of the connected compact group G. It is natural to replace the dga $\Omega(X)$ with differential d in the iterated integral construction by the dga $\Omega_G(X) = (\Omega(G) \otimes S(\mathbf{g}^*))^G$ with Cartan's equivariant differential d_G. In fact, in the case of the circle, we would prefer to use the A_∞-algebra $\Omega(X)[u]$ introduced in Section 1 instead of $\Omega_{\mathbb{T}}(LX)$. Its cyclic bar complex is defined in [9], but we have not been able to find the correct iterated integral map.

Since the dga $\Omega_G(X)$ carries an action of $\Omega_G = \Omega_G(\mathrm{pt}) = S(\mathbf{g}^*)^G$, we may form the cyclic bar complex over this ring instead of over $\mathbb{R}$; this is called $\mathsf{C}_{\Omega_G}(\Omega_G(X))$. For example, $\Omega_{\mathbb{T}} \cong \mathbb{R}[u]$, and $\Omega_{\mathbb{T}}(X) = \Omega(X)^{\mathbb{T}}[u]$. Similarly, we may form a Chen normalized version of this, which we denote $\mathsf{N}_{\Omega_G}(\Omega_G(X))$. Using the iterated integral map, the A_∞-algebra $\mathsf{N}_{\Omega_G}(\Omega_G(X))[u]$ is mapped to the A_∞-algebra of differential forms $\Omega(EG \times_G LX)[u]$, and we can prove the following equivariant version of Theorem A.

Theorem B. *If X is a G-manifold, the iterated integral map gives a homomorphism of A_∞-algebras*

$$\sigma : \mathsf{N}_{\Omega_G}(\Omega_G(X))[u] \to \Omega(EG \times_G LX)[u],$$

which induces an isomorphism on equivariant cohomology

$$H(\mathsf{N}_{\Omega_G}(\Omega_G(X))[u], b + uB) \cong H_{\mathbb{T}\times G}(LX)$$

if X is simply-connected. Here, the group $\mathbb{T} \times G$ acts on LX as follows: the circle group $\mathbb{T}$ rotates the loop and the group G acts on the manifold X.

We turn in Section 6 to the study of Bismut's equivariant extension of the Chern character, which requires the introduction of some new ideas. The first of these is that the equivariant Chern character is a power series in u^{-1}, so must live in some kind of periodic equivariant cohomology of LX. There is a difficulty with the usual periodic equivariant cohomology, defined by

$$\mathrm{HP}_{\mathbb{T}}(LX) = H(\Omega(LX)[u, u^{-1}], d + u\tilde{P}),$$

which is illustrated by a theorem of Goodwillie [10]: if X is simply-connected, then

$$\mathrm{HP}_{\mathbb{T}}(LX) = \mathbb{R}[u, u^{-1}].$$

The solution to this problem is to allow equivariant differential forms which are power series in u^{-1}, leading us to consider the cohomology theory

$$h_{\mathbb{T}}(LX) = H(\Omega(LX)[u, u^{-1}]\!], d + u\tilde{P})$$

introduced in [13], where a localization theorem analogous to the one for compact $\mathbb{T}$-spaces is proved. Some of our results do not apply to this theory, since it is based on a completed tensor product, so that the relevant spectral sequences do not converge. Thus, for the theory $h_{\mathbb{T}}(\cdot)$ applied to the space LX, our paper is restricted to constructing Bismut's equivariant Chern characters $h_{\mathbb{T}}(LX)$; we hope that the isomorphisms on cohomology of Theorems A and B can be extended to this context.

The other difficulty that we must deal with is that the equivariant Chern character cannot be an iterated integral. It is not difficult to show that any equivariant differential form on LX given by an iterated integral is invariant under the action of $\mathrm{Diff}[0, 1]$ on LX given by reparametrising loops, whereas the equivariant Chern character does not have this property. In order to get around this difficulty, we use Theorem B to construct a wider class of equivariant differential forms on LX as iterated integrals, by replacing the manifold X by the $\mathbb{T}$-manifold $X \times \mathbb{T}$, and pulling the resulting iterated integrals on $L(X \times \mathbb{T})$ back to LX by the map $\mu : LX \to L(X \times \mathbb{T})$ which sends γ to $(\gamma, -\operatorname{id}_{\mathbb{T}})$, $\operatorname{id}_{\mathbb{T}}$ being the identity loop in $L\mathbb{T}$. The corresponding map

$$\tilde{\sigma} = \mu^* \sigma : \mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T})) \to \Omega(LX)[u]$$

will be referred to as the **extended iterated integral map**. Theorem B does not apply in this situation, since $X \times \mathbb{T}$ is not simply-connected. However, we are able to prove that the extended iterated integral map descends to the quotient of $\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T})$ by the ideal generated by $(1, dt) - (1) \in \mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(\mathbb{T}))$, and that the induced map in cohomology is an isomorphism

$$\sigma_{\mathbb{T}} : H(NC_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))/((1, dt) - (1)), b_{\mathbb{T}} + uB) \cong H_{\mathbb{T}}(LX).$$

We use this result in Section 6 to construct the Chern character $\operatorname{Ch}(E, \nabla)$ as a formal sum of iterated integrals. If $p : X \to \operatorname{End}(\mathbb{C}^N)$ is an idempotent matrix of smooth functions on X (that is, $p^2 = p$), we can associate to p a vector bundle $E = \operatorname{im}(p)$. We form the associated Grassmannian connection $\nabla = p \cdot d \cdot p + (1 - p) \cdot d \cdot (1 - p)$ on the trivial bundle $\mathbb{C}^N$ and since this connection preserves sections of E it induces a connection ∇_E on E. Furthermore, by a result of Narasimhan and Ramanan, any vector bundle with connection may be realised in this way [16]. The matrix of one-forms defined by the connection ∇ is $A = (2p - 1)dp$, and its curvature is $F = (dp)^2$; let $\mathcal{A} \in \Omega_{\mathbb{T}}(X \times \mathbb{T})\hat{\otimes}_{\Omega_{\mathbb{T}}}\mathbb{C}[u, u^{-1}]\!]$ be the equivariant one-form $A - u^{-1}F\,dt$. Now introduce the cyclic chain

$$\operatorname{Tr}(p, \mathcal{A}, \ldots, \mathcal{A})_k = \sum_{i_0, \ldots, i_k} p_{i_0 i_1} \otimes \mathcal{A}_{i_1 i_2} \otimes \ldots \otimes \mathcal{A}_{i_k i_0}.$$

where the tensor product is taken over $\Omega_{\mathbb{T}}$.

Theorem C. *The cyclic chain* $\operatorname{ch}(p)$ *in* $\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))\hat{\otimes}_{\Omega_{\mathbb{T}}}\mathbb{C}[u, u^{-1}]\!]$*, given by*

$$\operatorname{ch}(p) = \sum_{k=0}^{\infty} \operatorname{Tr}(p, \mathcal{A}, \ldots, \mathcal{A})_k$$

is closed and its image under the extended iterated integral map $\tilde{\sigma}$ *is Bismut's equivariant Chern character form*

$$\operatorname{Ch}(E, \nabla_E) \in \Omega(LX)[u, u^{-1}]\!].$$

Finally we mention some of the standard conventions we will use throughout this article. When dealing with graded objects, we will always use the super-sign convention, so that interchanging an object of degree p with one of degree q will introduce a sign $(-1)^{pq}$; in particular, by the bracket $[a, b]$ of two operators, we mean the supercommutator

$$[a, b] = ab - (-1)^{|a||b|}ba.$$

We also use the convention that $\mathbb{R}[x]$ is a symmetric algebra if x has even degree, and an antisymmetric algebra if x has odd degree. If V_1 and V_2 are two $\mathbb{Z}_2$-graded vector spaces, we mean by $V_1 \otimes V_2$ the $\mathbb{Z}_2$-graded tensor product of V_1 and V_2. The canonical map S_{12} from $V_1 \otimes V_2$ to $V_2 \otimes V_1$ is defined by

$$S_{12}(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1.$$

Using the map S_{12}, we can associate to any permutation $\sigma \in S_n$ an isomorphism of vector spaces

$$S_\sigma : V_1 \otimes \ldots \otimes V_n \to V_{\sigma_1} \otimes \ldots \otimes V_{\sigma_n}.$$

The authors would like to thank J.-M. Bismut for some helpful discussions relating to the ideas in this paper. This work was assisted by a grant from the NSF to the first author, and a grant from the SERC to the second author.

1. Equivariant differential forms and A_∞-algebras

In this section, we will give a new definition of the complex of equivariant differential forms for a manifold with a circle action. Our reason for preferring this to the usual one of Cartan [4] is that it fits in more naturally with the theory of iterated integrals. We will give a formalism which makes it is easy to discuss different sorts of equivariant cohomology at the same time. We will also explain how the theory of A_∞-algebras enters naturally when we consider products on our version of the complex of equivariant differential forms.

Let Y be a smooth manifold equipped with a smooth circle action, that is a smooth one-parameter group of diffeomorphisms φ_t with period 1. Let T be the infinitesimal generator of this circle action, that is the vector field tangent to the $\mathbb{T}$-orbits. The $\mathbb{T}$-action on Y defines several operators on the space of differential forms on Y. The first of these is the contraction, or interior product, with the vector field T, which we will denote simply by ι:

$$\iota = \iota_T : \Omega^\bullet(Y) \to \Omega^{\bullet-1}(Y).$$

We also have the averaging operator A defined by

$$A(\omega) = \int_0^1 \varphi_t^* \omega dt.$$

Using the circle action, we can define a sequence of operators

$$\tilde{P}_k : \Omega(Y)^{\otimes k} \to \Omega(Y), \qquad 1 \le k < \infty$$

as follows. Let Δ_k be the k-simplex

$$\big\{(t_1, \dots t_k) \in \mathbb{R}^k \mid 0 \le t_1 \le \dots \le t_k \le 1\big\}$$

and consider the maps

$$\begin{array}{ccc} \Delta_k \times Y & \xrightarrow{\Phi_k} & Y^k \\ {\scriptstyle\pi}\big\downarrow & & \\ Y & & \end{array}$$

where $\Phi_k(t_1, \dots, t_k; x) = (\varphi_{t_1}(x), \dots, \varphi_{t_k}(x))$ and π is the projection. We embed the k-fold tensor product $\Omega(Y)^{\otimes k}$ into $\Omega(Y^k)$ as a dense subspace of $\Omega(Y^k)$ and then define P_k to be the composition

$$\Omega^\bullet(Y)^{\otimes k} \to \Omega^\bullet(Y^k) \xrightarrow{\Phi_k^*} \Omega^\bullet(\Delta_k \times Y) \xrightarrow{\pi_*} \Omega^{\bullet - k}(Y)$$

where π_* is integration along the fibres of π. In order to get the signs correct it is best to change this map by a sign and define

$$\tilde{P}_k(\omega_1, \dots, \omega_k) = (-1)^{(k-1)|\omega_1| + (k-2)|\omega_2| + \dots + |\omega_{k-1}| + k(k-1)/2} P_k(\omega_1, \dots, \omega_k).$$

Given a differential form ω on Y, let $\omega(t)$ be the differential form $\varphi_t^*(\omega)$ on Y. We can give a more explicit formula for $\tilde{P}_k$:

$$\tilde{P}_k(\omega_1, \dots, \omega_k) = \int_{\Delta_k} \iota\omega_1(t_1) \wedge \dots \wedge \iota\omega_k(t_k) \, dt_1 \dots dt_k.$$

The only difficult point in deriving the above formula is to get the signs correct. The operation of integration along the fibres π_* is defined by writing a differential form α on $\Delta_k \times Y$ as

$$\sum_I dt_I \wedge \alpha_I(t_1, \dots, t_k)$$

where I is a multi-index and $\alpha_I(t_1, \dots, t_k)$ is a differential form on $\Delta_k \times Y$ which involves none of the dt_i but does depend on $t_1, \dots, t_k$. The push-forward π_* is now defined to be the Lebesgue integral of the differential form $\alpha_{1\dots k}(t_1, \dots, t_k)$ with respect to the variables t_i. Note that π_* has degree $-k$.

If ω_i, $1 \le i \le k$, are differential forms on $\Omega(Y)$, then

$$\Phi_k^*(\omega_1 \wedge \dots \wedge \omega_k) = (\omega_1(t_1) + dt_1 \wedge \iota\omega_1(t_1)) \wedge \dots \wedge (\omega_k(t_k) + dt_k \wedge \iota\omega_k(t_k))$$

and if we bring all of the factors of dt_i to the left, we see that

$$\pi_*\Phi_k^*(\omega_1 \wedge \dots \wedge \omega_k) = \\ (-1)^{(k-1)|\omega_1|+(k-2)|\omega_2|+\dots+|\omega_{k-1}|+k(k-1)/2}\pi_*\left(dt_1 \dots dt_k \wedge \iota\omega_1(t_1) \dots \iota\omega_k(t_k)\right)$$

from which the relation between $\tilde{P}_k$ and P_k follows.

The following lemma is straightforward to prove.

Lemma 1.1. *The operator $\tilde{P} = \tilde{P}_1$ is equal to $\iota A = A\iota$, and satisfies the formulas*

$$d\tilde{P} + \tilde{P}d = \tilde{P}^2 = 0.$$

Next we describe the notion of a **differential graded Λ-module** (which was introduced by Kassel, under the name of a mixed complex). Let $\Lambda = \mathbb{R}[\varepsilon]$ be the exterior algebra in one generator of degree -1. Then a dg-Λ-module $(C^\bullet, b, B)$ is just a sequence of complex vector spaces C^p equipped with operators $b : C^p \longrightarrow C^{p+1}$, $B : C^p \longrightarrow C^{p-1}$ which satisfy the formulas

$$b^2 = [b, B] = B^2 = 0.$$

The operator B is the action of ε on the complex $C^\bullet$, while b is the differential. For example, the dg-Λ-module which will lead to equivariant cohomology is $\Omega(Y) = (\Omega(Y), d, \tilde{P})$ where Y is a smooth $\mathbb{T}$-manifold, d is the exterior derivative and $\tilde{P}$ is the operator on forms described above.

We now explain what we mean by the cohomology of a dg-Λ-module. Let $\mathbb{R}[u]$ be the polynomial algebra in one indeterminate u of degree 2 and let W be a graded module over $\mathbb{R}[u]$. Forming the chain complex $W \otimes C$ with boundary map $b + uB$, we define

$$H(C; W) = H(W \otimes C, b + uB).$$

The following lemma follows from standard homological algebra.

Lemma 1.2. *Let*

$$0 \longrightarrow W_1 \xrightarrow{i} W_2 \xrightarrow{j} W_3 \longrightarrow 0$$

be a short exact sequence of $\mathbb{R}[u]$-modules. Then there is a long exact sequence of cohomology groups

$$\dots \longrightarrow H^\bullet(C; W_1) \xrightarrow{i} H^\bullet(C; W_2) \xrightarrow{j} H^\bullet(C; W_3) \xrightarrow{\partial} H^{\bullet-1}(C; W_1) \longrightarrow \dots.$$

The next result expresses a basic invariance property of the cohomology of a dg-Λ-module. The proof may be found, for example, in [9].

Proposition 1.3. *Let $f : (C_1, b_1, B_1) \longrightarrow (C_2, b_2, B_2)$ be a map of dg-Λ-modules. If f induces an isomorphism $H(C_1, b_1) \longrightarrow H(C_2, b_2)$, then for any coefficients W, $f : H(C_1; W) \longrightarrow H(C_2; W)$ is an isomorphism.*

When we apply this formalism to the dg-Λ-module $\Omega(Y) = (\Omega(Y), d, \tilde{P})$ determined by a smooth manifold Y equipped with a smooth $\mathbb{T}$ action, we get various versions of equivariant cohomology. We will often refer to elements of $W \otimes \Omega(Y)$ as **equivariant differential forms**. The most important examples are the cases where W is the polynomial ring $\mathbb{R}[u]$, the ring of finite Laurent polynomials $\mathbb{R}[u, u^{-1}]$, and similar modules, so that we will most commonly be dealing with equivariant differential forms which can be expressed as polynomials or, more generally, power series in u and u^{-1} with coefficients in $\Omega(Y)$. We will use the notation d_u for the operator $d + u\tilde{P}$ and refer to it as the **equivariant exterior derivative**. Equivariant cohomology itself is defined to be

$$H_{\mathbb{T}}(Y) = H(\Omega(Y); \mathbb{R}[u]).$$

There is also periodic equivariant cohomology

$$\mathrm{HP}_{\mathbb{T}}(Y) = H_{\mathbb{T}}(Y; \mathbb{R}[u, u^{-1}]) = H(\Omega(Y); \mathbb{R}[u, u^{-1}]).$$

We get yet another form of equivariant cohomology by taking coefficients in the $\mathbb{R}[u]$-module $\mathbb{R}[u, u^{-1}]/u\mathbb{R}[u]$

$$H_{\mathbb{T}}(Y; \mathbb{R}[u, u^{-1}]/u\mathbb{R}[u]) = H(\Omega(Y) \otimes \mathbb{R}[u, u^{-1}]/u\mathbb{R}[u])$$

which is the functor called $G_{\mathbb{T}}(Y)$ in [12]. The exact sequence of coefficients

$$0 \to u\mathbb{R}[u] \to \mathbb{R}[u, u^{-1}] \to \mathbb{R}[u, u^{-1}]/u\mathbb{R}[u] \to 0$$

gives the fundamental exact sequence in equivariant cohomology of [12, page 404]. Note also that if we regard $\mathbb{R}$ as the $\mathbb{R}[u]$-module $\mathbb{R}[u]/u\mathbb{R}[u]$, so that u acts as zero, we get

$$H(\Omega(Y); \mathbb{R}) = H(Y)$$

where $H(Y)$ is the de Rham cohomology of Y.

These definitions of equivariant cohomology groups remain valid if the manifold Y is infinite dimensional but the resulting cohomology theories do not seem to be adequate. For example, in the definition of $\Omega(Y)[u, u^{-1}]$, we have only allowed finite Laurent series of the form $\sum_{i=-N}^{N} a_i u^i$ but if Y is infinite dimensional we would like to be allowed expressions of the form $\sum_{i=-\infty}^{\infty} a_i u^i$.

Another symptom of the difficulty is the following result, due to Goodwillie [10]: if X is compact and simply-connected, then $H_{\mathbb{T}}(LX; \mathbb{R}[u, u^{-1}]) = \mathbb{R}[u, u^{-1}]$. In this paper, we will get around these difficulties by using the natural generalization of the cohomology theory introduced in [13]: if (C, b, B) is a dg-Λ-module, then define

$$h(C, b, B) = H(C[u, u^{-1}]\!], b + uB),$$

where $C[u, u^{-1}]\!]$ is the complex whose space of degree i is the set of all sums $\sum_{k=-\infty}^{N} u^k c_k$, for N some finite number, where $\deg c_k = i - 2k$. For example, if Y is an $\mathbb{T}$-manifold, we define

$$h_{\mathbb{T}}(Y) = h(\Omega(Y), d, \tilde{P}).$$

The functors $h(C, b, B)$ do not satisfy 1.3, since the space $C[u, u^{-1}]\!]$ is not an algebraic tensor product. As an example, consider the map $h_{\mathbb{T}}(\mathrm{pt}) \longrightarrow h_{\mathbb{T}}(E\mathbb{T})$ induced by the constant map $E\mathbb{T} \longrightarrow \mathrm{pt}$; whereas

$$h_{\mathbb{T}}(\mathrm{pt}) = \mathbb{R}[u, u^{-1}]\!],$$

we have by the results of [13]

$$h_{\mathbb{T}}(E\mathbb{T}) = 0.$$

In defining $h(\cdot)$, we should probably use series whose coefficients obey fixed decay conditions as $k \to -\infty$. However, the appropriate decay conditions to be used will probably only be clear after the study of natural examples such as those we present in this paper and its sequels.

We will now compare the cohomology of our complex of equivariant differential forms with that of Cartan. If Y is an $\mathbb{T}$-manifold, let $\Omega(Y)^{\mathbb{T}}$ be the space of invariant forms on Y; since $[d, \iota] = 0$ on $\Omega(Y)^{\mathbb{T}}$, $(\Omega(Y)^{\mathbb{T}}, d, \iota)$ is a dg-Λ-module. We will continue to write d_u for the operator $d + u\iota$ on the space $\Omega_{\mathbb{T}}(Y) = \Omega(Y)^{\mathbb{T}}[u]$. Let

$$i : \Omega(Y)^{\mathbb{T}} \longrightarrow \Omega(Y), \quad A : \Omega(Y) \longrightarrow \Omega(Y)^{\mathbb{T}}$$

be the inclusion and the averaging operators. It is straightforward to check that both are maps of dg-Λ-modules.

Proposition 1.5. *These maps A and i induce inverse isomorphisms between cohomology groups*

$$H(\Omega(Y)^{\mathbb{T}}; W) \cong H(\Omega(Y); W)$$

for any coefficients W.

Proof. By Lemma 1.3, it suffices to prove this for $W = \mathbb{R}$, in other words, to show that de Rham cohomology may be computed on $\mathbb{T}$-invariant forms. This is a standard result, which is proved by constructing a homotopy operator $s : \Omega(Y) \longrightarrow \Omega(Y)$ such that $[d, s] = 1 - iA$. Define $h_t : [0, t] \times Y \longrightarrow Y$ by setting $h_t(u, y) = \varphi_u(y)$ and define $s(\omega)$ as follows:

$$s(\omega) = \int_0^1 (\pi_t)_* h_t^*(\omega)\, dt$$

It is straightforward to check that s satisfies the above formulas. (In fact, since $[P, s] = 0$, it follows that $[d_u, s] = 1 - iA$, so that we do not need to invoke Lemma 1.3.) $\square$

Now we study the maps $\tilde{P}_k$; they satisfy certain relations, which amount to saying that $\sum_{k=0}^{\infty} \tilde{P}_k$ is a Hochschild cocycle on the dga $\Omega(Y)$ with coefficients in $\Omega(Y)$ itself.

Proposition 1.6. *If $\varepsilon_i = |\omega_1| + \cdots + |\omega_i| - i$, then*

$$\begin{aligned} d\tilde{P}_k(\omega_1, \ldots, \omega_k) = &- \sum_{i=1}^{n} (-1)^{\varepsilon_{i-1}} \tilde{P}_k(\omega_1, \ldots, d\omega_i, \ldots, \omega_k) \\ &- \omega_1 \tilde{P}_{k-1}(\omega_2, \ldots, \omega_k) \\ &- \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} \tilde{P}_{k-1}(\omega_1, \ldots, \omega_i\omega_{i+1}, \ldots, \omega_k) \\ &+ (-1)^{\varepsilon_{k-1}} \tilde{P}_{k-1}(\omega_1, \ldots, \omega_{k-1})\omega_k. \end{aligned}$$

Proof. If we apply the exterior differential d to the formula for $\tilde{P}_k$,

$$\tilde{P}_k(\omega_1, \ldots, \omega_k) = \int_{\Delta_k} \iota\omega_1(t_1) \wedge \ldots \wedge \iota\omega_k(t_k)\, dt_1 \ldots dt_k,$$

we obtain the sum

$$\sum_{i=1}^{k}(-1)^{\varepsilon_{i-1}}\int_{\Delta_k}\iota\omega_1(t_1)\dots([d,\iota]\omega_i)(t_i)\dots\iota\omega_k(t_k)\,dt_1\dots dt_k$$
$$-\sum_{i=1}^{k}(-1)^{\varepsilon_{i-1}}\int_{\Delta_k}\iota\omega_1(t_1)\dots\iota(d\omega_i)(t_i)\dots\iota\omega_k(t_k)\,dt_1\dots dt_k.$$

Using the fact that $([d,\iota]\omega)(t)=d\omega(t)/dt$, we see that the ith term of the first sum equals

$$(-1)^{\varepsilon_{i-1}}\int_{0\le t_1\le\dots\le t_{i-1}\le t_{i+1}\le\dots\le t_k\le 1}\iota\omega_1(t_1)\dots(\omega_i(\iota\omega_{i+1}))(t_{i+1})\dots\iota\omega_k(t_k)\,dt_1\dots\widehat{dt_i}\dots dt_k$$
$$-(-1)^{\varepsilon_{i-1}}\int_{0\le t_1\le\dots\le t_{i-1}\le t_{i+1}\le\dots\le t_k\le 1}\iota\omega_1(t_1)\dots((\iota\omega_{i-1})\omega_i)(t_{i-1})\dots\iota\omega_k(t_k)\,dt_1\dots\widehat{dt_i}\dots dt_k.$$

Using the fact that $-(-1)^{\varepsilon_i}\iota\omega_i\wedge\omega_{i+1}+(-1)^{\varepsilon_{i-1}}\omega_i\wedge\iota\omega_{i+1}=-(-1)^{\varepsilon_i}\iota(\omega_i\wedge\omega_{i+1})$, we see how the term $-(-1)^{\varepsilon_{i-1}}P_{k-1}(\omega_1,\dots,\omega_i\omega_{i+1},\dots,\omega_k)$ emerges from this sum, while the two terms which remain are equal to $-\omega_1\tilde{P}_{k-1}(\omega_2,\dots,\omega_k)$ and $(-1)^{\varepsilon_{k-1}}\tilde{P}_{k-1}(\omega_1,\dots,\omega_{k-1})\omega_k$. Putting all of this together, we obtain the desired formula. □

Note that in terms of our other definition of P_k, using the diagram

$$\begin{array}{ccc}\Delta_k\times Y & \xrightarrow{\Phi_k} & Y^k\\ \pi\big\downarrow & & \\ Y & & \end{array}$$

this result may be viewed as an application of Stokes's Theorem; however, it is easier to get the signs correct by the method that we followed above.

We will now describe the algebraic structure on the space $\Omega(Y)[u]$ given by the maps $\tilde{P}_k$, which is where A_∞-algebras arise in our work. For each $k>0$, let us define a map $\tilde{m}_k:\Omega(Y)[u]^{\otimes k}\longrightarrow\Omega(Y)[u]$ of degree $2-k$, as follows:

$$\tilde{m}_k(\omega_1,\dots,\omega_k)=\begin{cases}d\omega_1+u\tilde{P}_1(\omega_1), & k=1,\\ -(-1)^{|\omega_1|}\omega_1\wedge\omega_2+u\tilde{P}_2(\omega_1,\omega_2), & k=2,\\ u\tilde{P}_k(\omega_1,\dots,\omega_k), & \text{otherwise.}\end{cases}$$

Observe that if any ω_i is a function, the term $\tilde{P}_k(\omega_1,\dots,\omega_k)$ vanishes. The maps $\tilde{m}_1$ and $\tilde{m}_2$ may be thought of as deformations of the differential and product of the dga $\Omega(Y)$; we will explain the reason for the sign $-(-1)^{|\omega_1|}$ in the definition of $\tilde{m}_2$ later. The next theorem is one of the main results of this paper: it will enable us to express the sense in which the iterated integral map for the free loop space preserves products.

Theorem 1.7. *The maps $\tilde{m}_k:\Omega(Y)[u]^{\otimes k}\longrightarrow\Omega(Y)[u]$ satisfy the following sequence of formulas:*

$$(*_k)\qquad \sum_{i+j=k+1}\sum_{l=1}^{i}(-1)^{\varepsilon_{l-1}}\tilde{m}_i(\omega_1,\dots,\omega_{l-1},\tilde{m}_j(\omega_l,\dots,\omega_{l+j-1}),\omega_{l+j},\dots,\omega_k)=0.$$

Proof. It is easy to check from the formula for $\tilde{P}_k$ that

$$\iota\tilde{P}_k(\omega_1,\dots,\omega_k)=0;$$

it follows immediately that

$$\tilde{P}_i(\omega_1, \dots, \omega_{l-1}, \tilde{P}_j(\omega_l, \dots, \omega_{l+j-1}), \omega_{l+j}, \dots, \omega_k) = 0.$$

Using this and 1.6, a calculation shows that the above formulas are satisfied. □

Recall the definition of an A_∞-algebra: this structure, discovered by Stasheff [17], is a non-associative generalization of a dga (see also [9] for an account motivated by the constructions of this paper). Let A be a graded vector space, and for each $k > 0$, let $\tilde{m}_k : A^{\otimes k} \longrightarrow A$ be a multilinear map of degree $2 - k$. The maps $\tilde{m}_k$ form an A_∞-structure if they satisfy the formulas that we have just verified in the case $A = \Omega(Y)[u]$:

$$\sum_{i+j=k+1} \sum_{l=1}^{i} (-1)^{\varepsilon_{l-1}} \tilde{m}_i(a_1, \dots, a_{l-1}, \tilde{m}_j(a_l, \dots, a_{l+j-1}), a_{l+j}, \dots, a_k) = 0.$$

We will also assume that our A_∞-algebra has an identity 1 of degree 0, satisfying the formulas

$$\begin{aligned} &\tilde{m}_2(a, 1) = \tilde{m}_2(1, a) = a, \\ &\tilde{m}_i(a_1, \dots, 1, \dots, a_i) = 0, \quad i \neq 2. \end{aligned}$$

It is often more useful to work with the maps m_k, which are defined as follows:

$$m_k(a_1, \dots, a_k) = (-1)^{(k-1)|a_1| + (k-2)|a_2| \dots + |a_{k-1}| + k(k-1)/2} \tilde{m}_k(a_1, \dots, a_k).$$

The maps m_k provide A with a differential $d = m_1 : A \longrightarrow A$ and a product $m_2 : A \otimes A \longrightarrow A$. The differential is a derivation with respect to the product, but the product is not associative. Nevertheless the product is homotopy associative and m_3 provides the homotopy; the operators m_k provide a whole coherent family of higher homotopies. Indeed, for $k \leq 3$, Formula $(*_k)$ becomes as follows:

(1) the operator $d = m_1 = \tilde{m}_1$ is a differential on A, that is,

$$m_1(m_1(a)) = 0;$$

(2) (**Leibniz's rule**) the bilinear map $m_2(a_1, a_2) = -(-1)^{|a_1|} \tilde{m}_2(a_1, a_2)$ defines a product on A and m_1 is a derivation with respect to this product:

$$m_1(m_2(a_1, a_2)) - m_2(m_1(a_1), a_2) - (-1)^{|a_1|} m_2(a_1, m_1(a_2)) = 0;$$

(3) (**homotopy associativity**) the operators m_1, m_2 and m_3 satisfy the following weakened version of associativity:

$$\begin{aligned} &- m_1(m_3(a, b, c)) + m_2(m_2(a, b), c) - (-1)^{|a|} m_2(a, m_2(b, c)) \\ &\qquad - m_3(m_1(a), b, c) - (-1)^{|a|} m_3(a, m_1(b), c) - (-1)^{|a|+|b|} m_3(a, b, m_1(c)) = 0 \end{aligned}$$

In the case of equivariant differential forms $\Omega(Y)[u]$, we see that m_k may be rewritten in terms of the maps P_k as follows:

$$m_k(\omega_1, \dots, \omega_k) = \begin{cases} d\omega_1 + uP_1(\omega_1), & k = 1, \\ \omega_1 \wedge \omega_2 + uP_2(\omega_1, \omega_2), & k = 2, \\ uP_k(\omega_1, \dots, \omega_k), & \text{otherwise.} \end{cases}$$

It follows from the homotopy associativity of m_2 that the product in cohomology $\mathrm{H}_{\mathbb{T}}(Y) = \mathrm{H}(\Omega(Y)[u]; \tilde{m}_1)$ induced by m_2 is associative; however, the existence of a A_∞-structure is much stronger than the existence of an associative product at the level of cohomology. Fortunately, this product on $\mathrm{H}_{\mathbb{T}}(Y)$ is equal to the usual one.

Proposition 1.8. *The product in* $\mathrm{H}_{\mathbb{T}}(Y)$ *induced by* m_2 *agrees with the usual product.*

Proof. The inclusion

$$(\Omega_{\mathbb{T}}(Y); d + u\iota) \longrightarrow (\Omega(Y)[u]; \tilde{m}_1)$$

induces an isomorphism in cohomology by 1.5, so it suffices to show that the product

$$m_2(\omega_1, \omega_2) = \omega_1 \wedge \omega_2 - (-1)^{|\omega_1|} u \tilde{P}_2(\omega_1, \omega_2) = \omega_1 \wedge \omega_2 - \frac{(-1)^{|\omega_1|}}{2} \iota\omega_1 \wedge \iota\omega_2$$

on $\Omega_{\mathbb{T}}(Y)$ induces the same product in cohomology as the wedge product.

In order to prove this, we will construct a homotopy between the two maps $(\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2$ and $(\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2 - (-1)^{|\omega_1|} u \tilde{P}_2(\omega_1, \omega_2)$; that is, we will construct a map

$$\theta : \Omega_{\mathbb{T}}(Y)^{\otimes 2} \longrightarrow \Omega_{\mathbb{T}}(Y)$$

of degree -1 such that

$$d_u \theta(\omega_1, \omega_2) - \theta(d_u \omega_1, \omega_2) - (-1)^{|\omega_1|} \theta(\omega_1, d_u \omega_2) = -\frac{(-1)^{|\omega_1|} u}{2} \iota\omega_1 \wedge \iota\omega_2.$$

Our formula for θ is

$$\theta(\omega_1, \omega_2) = -\frac{(-1)^{|\omega_1|}}{2} \frac{\partial \omega_1}{\partial u} \wedge \iota\omega_2,$$

and it is easy to check the above identity. $\square$

2. Iterated integrals

Let X be a finite dimensional smooth manifold and let LX be the free loop space of X, that is the space of all smooth maps from $\mathbb{T}$ to X. This loop space may be given the structure of an infinite dimensional manifold modeled on a Frechet space in a natural way, and the circle group acts smoothly by rotating loops: $(\varphi_t \gamma)(s) = \gamma(s + t)$. In this section we describe the basic theory of iterated integrals as it applies to LX; many of the calculations are similar to those for the maps $\tilde{P}_k$ which define the A_∞-structure on $\Omega(LX)[u]$ of the last section.

If ω is a one-form on X, then $\gamma \longrightarrow \int_\gamma \omega$ is a function on LX. Iterated integrals generalise this idea. If ω is a differential form on X, let $\omega(t)$ be the differential form $e_t^*(\omega)$ on LX where e_t is the map $LX \longrightarrow X$ given by evaluating loops at time t. Given forms $\omega_0, \dots, \omega_k$, the iterated integral

$$\sigma(\omega_0, \dots, \omega_k)$$

is a form on LX of total degree $|\omega_0| + \cdots + |\omega_k| - k$, defined by the formula

$$\sigma(\omega_0, \dots, \omega_k) = \int_{\Delta_k} \omega_0(0) \wedge \iota\omega_1(t_1) \wedge \ldots \wedge \iota\omega_k(t_k)\, dt_1 \ldots dt_k.$$

In terms of the maps $\tilde{P}_k : \Omega(LX)^{\otimes k} \longrightarrow \Omega(LX)$ of the last section, we may also write

$$\sigma(\omega_0, \dots, \omega_k) = \omega_0(0)\, \tilde{P}_k(\omega_1(0), \dots, \omega_k(0)).$$

Up to a sign, the iterated integral map can also be described as follows. Consider the diagram

$$\begin{array}{ccc} \Delta_k \times LX & \xrightarrow{e_k} & X^{k+1} \\ {\scriptstyle\pi}\big\downarrow & & \\ LX & & \end{array}$$

where $e_k(t_1, \dots, t_k; \gamma) = (\gamma(0), \gamma(t_1), \dots, \gamma(t_k))$ and π is the projection. We embed the $k+1$-fold tensor product $\Omega(X)^{\otimes k+1}$ as a dense subspace of $\Omega(X^{k+1})$ and then consider the composition

$$\Omega^\bullet(X)^{\otimes k+1} \subset \Omega^\bullet(X^{k+1}) \xrightarrow{e_k^*} \Omega^\bullet(\Delta_k \times LX) \xrightarrow{\pi_*} \Omega^{\bullet - k}(LX)$$

where π_* is integration along the fibres of π; in fact

$$\sigma(\omega_0, \dots, \omega_k) = (-1)^{k|\omega_0| + (k-1)|\omega_1| + \dots + 2|\omega_{k-2}| + |\omega_{k-1}| + k(k-1)/2} \pi_* e_k^*(\omega_0, \dots, \omega_k).$$

We now build a model for the differential forms on LX from these iterated integrals. We begin by briefly recalling the definition of the **cyclic bar complex** of $\Omega(X)$. Let $\mathsf{C}(\Omega(X))$ be the direct sum

$$\sum_{k=0}^{\infty} \Omega(X) \otimes s\Omega(X)^{\otimes k}.$$

Here s is the suspension functor on graded vector spaces, that is the functor which simply reduces degree by 1. Recall the definitions of three operators on this complex, the exterior differential b_0, the Hochschild boundary operator b_1 and Connes's coboundary operator B. The operator b_0 is the extension of the exterior derivative d to $\mathsf{C}(\Omega(X))$, and is given by the formula

$$b_0(\omega_0, \dots, \omega_k) = -\sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}} (\omega_0, \dots, \omega_{i-1}, d\omega_i, \omega_{i+1}, \dots, \omega_k).$$

where $\varepsilon_i = |\omega_0| + \dots + |\omega_i| - i$. The formula for the operator b_1 is

$$\begin{aligned} b_1(\omega_0, \dots, \omega_k) = &-\sum_{i=0}^{k-1} (-1)^{\varepsilon_i} (\omega_0, \dots, \omega_{i-1}, \omega_i \omega_{i+1}, \omega_{i+2}, \dots, \omega_k) \\ &+ (-1)^{(|\omega_k| - 1)\varepsilon_{k-1}} (\omega_k \omega_0, \omega_1, \dots, \omega_{k-1}), \end{aligned}$$

and the formula for B is

$$\begin{aligned} B(\omega_0, \dots, \omega_k) = &\sum_{i=0}^{k} (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k - \varepsilon_{i-1})} (1, \omega_i, \dots, \omega_k, \omega_0, \dots, \omega_{i-1}) \\ &- \sum_{i=0}^{k} (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k - \varepsilon_{i-1})} (\omega_i, \dots, \omega_k, \omega_0, \dots, \omega_{i-1}, 1) \end{aligned}$$

These sign conventions take into account the fact that the differential forms ω_i for $i > 0$ occur with an implicit suspension which reduces their degree to $|\omega_i| - 1$. In practice, only the first line of the definition of B will enter into our formulas, because we will usually work with a normalized quotient of the cyclic bar complex in which a chain of the form $(a_0, \dots, a_{i-1}, 1, 1, a_i, \dots, a_k)$, $i > 0$, is identified to zero.

Let $b = b_0 + b_1$ be the total boundary operator on $\mathsf{C}(\Omega(X))$. It is a standard calculation that $(\mathsf{C}(\Omega(X)), b, B)$ is a dg-Λ-module; the associated homology theories are known as cyclic homology theories. (Cyclic homology properly speaking is the theory with coefficients $W = \mathbb{R}[u, u^{-1}]/u\mathbb{R}[u]$, but we will not be making use of this particular theory.) Note that σ defines a map

$$\mathsf{C}(\Omega(X)) \to \Omega(LX)$$

which we will now prove is a map of dg-Λ-modules. The following theorem expresses the relation between cyclic homology and equivariant cohomology that was first observed in Jones [12].

Theorem 2.1. *The iterated integral map σ induces a map of dg-Λ-modules*

$$(\mathsf{C}(\Omega(X)), b, B) \to (\Omega(LX), d, \tilde{P}).$$

Proof. The formula $\sigma b = d\sigma$ follows from 1.6, given our formula for the iterated integral in terms of the map $\tilde{P}_k$ on LX; it can also be checked directly by repeating the method used to prove 1.6.

To prove that $\tilde{P}\sigma = \sigma B$, we must compute $\tilde{P}\sigma(\omega_0, \dots, \omega_k) = A\iota\sigma(\omega_0, \dots, \omega_k)$; as mentioned before, the second line in the definition of B drops out of the calculation, since $\sigma(\omega_i, \dots, \omega_k, \omega_0, \dots, \omega_{i-1}, 1) = 0$. The formula for $A\iota\sigma(\omega_0, \dots, \omega_k)$ is

$$\int_{[0,1]\times\Delta_k} \iota\omega_0(s)\, \iota\omega_1(s+t_1) \dots \iota\omega_k(s+t_k)\, ds\, dt_1 \dots dt_k,$$

where the differential form $\omega(t)$ is regarded as a periodic function of t. We may divide the region of integration $[0,1] \times \Delta_k$ into $k+1$ $(k+1)$-simplices

$$R_i = \big\{(s, t_1, \dots, t_k) \in [0,1] \times \Delta_k \bigm| 0 \le s + t_i \le \dots \le s + t_k \le s \le s + t_1 \le \dots \le s + t_{i-1} \le 1\big\},$$

where we take $s_i + t$ modulo 1. If we make an obvious change of coordinates, we see that

$$\int_{R_i} \iota\omega_0(s) \wedge \iota\omega_1(s+t_1) \wedge \dots \wedge \iota\omega_k(s+t_k) = (-1)^{\varepsilon_{i-1}(\varepsilon_k - \varepsilon_{k-i})} \sigma(1, \omega_i, \dots, \omega_k, \omega_0, \dots, \omega_{i-1}).$$

So the region R_i contributes the i-th term in the formula for σB and we see that $\tilde{P}\sigma = \iota A\sigma = \sigma B$. $\square$

It is natural to ask whether every differential form on LX is an iterated integral. We will now show that iterated integrals descend to a certain quotient of the free loop space. Observe that the group $\mathrm{Diff}[0,1]$ of diffeomorphisms of the unit interval acts on the free loop space LM by reparametrization: for $f \in \mathrm{Diff}[0,1]$ and $\gamma \in LM$,

$$(f \cdot \gamma)(t) = \gamma(f(t)).$$

The Lie algebra $\mathrm{diff}[0,1]$ of vector fields on the unit interval which vanish at the endpoints is an infinitesimal version of the Lie group $\mathrm{Diff}[0,1]$, and we may consider the Lie pair $(\mathrm{diff}[0,1], \mathrm{Diff}[0,1])$. In general, a Lie pair is a pair $(\mathbf{g}, G)$ consisting of a Lie algebra $\mathbf{g}$ and Lie group G, with the requirement that there is an exponential map from $\mathbf{g}$ to G, and that G acts on $\mathbf{g}$ in a manner compatible with the exponential map. (All of this trouble arises from the fact that $\mathrm{diff}[0,1]$ is not quite the Lie algebra of $\mathrm{Diff}[0,1]$, since the exponential map is not surjective at the identity in $\mathrm{Diff}[0,1]$.) If a Lie pair acts on a manifold M, in the sense that G acts on M by diffeomorphisms and $\mathbf{g}$ by vector fields, with suitable compatibility conditions, we say that a differential form is **basic** if it is invariant under the action of G and vanishes if contracted with a vector field coming from the action of $\mathbf{g}$ (this condition is called horizontality).

Proposition 2.2. *Iterated integrals are basic with respect to the action of the Lie pair* $(\mathrm{diff}[0,1], \mathrm{Diff}[0,1])$ *on LM.*

Proof. We start by showing that iterated integrals are invariant under the action of $\mathrm{Diff}[0,1]$. Any diffeomorphism of the unit interval fixing the endpoints is order preserving so we get an induced diffeomorphism of Δ_k defined by $f(t_1, \dots, t_k) = (f(t_1), \dots, f(t_k))$. If $F \in \mathrm{Diff}(\Delta_k) \times \mathrm{Diff}(LX)$ is the diffeomorphism $f^{-1} \times f$, then the following diagram commutes:

$$\begin{array}{ccc}
\Delta_k \times LX & \xrightarrow{e_k} & X^{k+1} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle 1} \\
\Delta_k \times LX & \xrightarrow{e_k} & X^{k+1}
\end{array}$$

As above, let $\pi : \Delta_k \times LX \to LX$ be the projection. Then $\pi F = f\pi$ and since both F and f are diffeomorphisms it follows that on differential forms $\pi_* F^* = f^* \pi_*$. It is now clear that $f^* \pi_* e_k^* = \pi_* F^* e_k^* = \pi_* e_k^*$ and therefore $f^* \sigma = \sigma f^*$.

To see that iterated integrals are horizontal, we observe that an element of diff$[0,1]$, that is, a vector field $h(t)d/dt$ on $[0,1]$ vanishing at the end-points, corresponds to the vector field $X = h(t)\dot{\gamma}(t)$ on LM. It is easy to calculate that

$$\begin{aligned}\iota(X)\sigma(a_0,\dots,a_k) &= \int_{\Delta_k} h(0)\iota\gamma_s^* a_0 \wedge \iota\gamma_{t_1}^* a_1 \wedge \dots \wedge \iota\iota\gamma_{t_i}^* a_i \wedge \gamma_{t_k}^* a_k \, dt_1 \dots dt_k \\ &+ \sum_{i=1}^k (-1)^{|a_0|+\dots+|a_i-1|-i+1} \int_{\Delta_k} h(t_i)\gamma_s^* a_0 \wedge \iota\gamma_{t_1}^* a_1 \wedge \dots \wedge \iota\iota\gamma_{t_i}^* a_i \wedge \gamma_{t_k}^* a_k \, dt_1 \dots dt_k;\end{aligned}$$

the first term vanishes since $h(0) = 0$, while the second vanishes because $\iota\iota = 0$. $\square$

The action of Diff$[0,1]$ is not free, so in no sense can the iterated integrals be considered as differential forms on a quotient of LM. However, let us pretend otherwise for a moment. We call the (non-existent) quotient the space of unparametrized loops, and denote it by $\mathcal{L}M$; it is the base of a principal fibration

$$\begin{array}{ccc} \mathrm{Diff}[0,1] & \longrightarrow & LM \\ & & \big\downarrow \\ & & \mathcal{L}M \end{array}$$

It would be very pleasant if the iterated integrals were dense in the space of differential forms on $\mathcal{L}M$. In fact, even this is not quite true, since iterated integrals of degree zero are invariant under a further operation on the loop space, that of cancellation: if an unparametrized loop contains a segment in which we follow a path out and then immediately back, then by cancellation we mean the removal of this segment of the path. In any case, we will not delve any further into this question.

Finally we describe Chen's normalization and its relation with the iterated integral map. For more details, in particular the proofs of 2.3 and 2.4 below, see either Chen's original paper [6], or [9]. The purpose of Chen's normalization of the cyclic bar complex is to remove chains of negative degree. Let Ω be a dga. If f is an element of Ω^0, we define operators $S_i(f)$ on $\mathsf{C}(\Omega)$ by the formula

$$S_i(f)(\omega_0,\dots,\omega_k) = (\omega_0,\dots,\omega_{i-1},f,\omega_i,\dots,\omega_k).$$

We now define $\mathsf{D}(\Omega)$ to be the subspace of $\mathsf{C}(\Omega)$ generated by the images of the operators $S_i(f)$ and $R_i(f) = [b, S_i(f)]$. The following result shows that the Chen normalization leaves many of the structures that we have considered on $\mathsf{C}(\Omega)$ in place.

Lemma 2.3. *The differentials b and B map $\mathsf{D}(\Omega(X))$ to itself.*

Thus, $(\mathsf{D}(\Omega), b, B)$ is a sub-dg-Λ-module of $(\mathsf{C}(\Omega), b, B)$. The Chen normalised cyclic bar complex of Ω is the quotient complex $\mathsf{N}(\Omega) = \mathsf{C}(\Omega)/\,\mathsf{D}(\Omega)$; by the above lemma, it is itself a dg-Λ-module.

Proposition 2.4. *If Ω is connected, that is, if $H^0(\Omega) = \mathbb{R}$, then the complex $(\mathsf{D}(\Omega), b)$ is acyclic. In particular, if X is a connected manifold, then $(\mathsf{D}(\Omega(X)), b)$ is acyclic.*

The following result is Lemma 4.3.1 of Chen's paper [5, p. 237], and relates the abstract notion of the Chen normalization with the geometry of the iterated integral.

Proposition 2.5. *If X is connected, then the complex $\mathsf{D}(\Omega(X))$ is the kernel of the iterated integral map σ.*

Proof. We will only give an outline of the proof here, referring the reader to the original paper for further details. Let Ω be a sub-dga of $\Omega(X)$ with the same cohomology, and such that $\Omega^0 = \mathbb{R}$, and let $\mathsf{C}(\Omega, \Omega(X))$ be the subcomplex of $\mathsf{C}(\Omega(X))$ defined by

$$\sum_{k=0}^{\infty} \Omega(X) \otimes s\Omega^{\otimes k}.$$

Since $\mathsf{C}(\Omega, \Omega(X)) + \mathsf{D}(\Omega(X)) = \mathsf{C}(\Omega(X))$ and $\mathsf{C}(\Omega, \Omega(X)) \cap \mathsf{D}(\Omega(X))$ equals $\mathsf{E}(\Omega(X))$, the span of the set of cochains for which $\omega_i = 1$ for some $i > 0$, and since the iterated integral map vanishes on $\mathsf{D}(\Omega(X))$, it suffices to show that

$$\sigma : \mathsf{C}(\Omega, \Omega(X))/\,\mathsf{E}(\Omega(X)) \longrightarrow \Omega(LX)$$

is an injection. The main idea of the proof is to reduce it to the case in which $k = 1$ by means of the following lemma, which is proved in [5].

Pick a basepoint $x \in X$, and let $L_\varepsilon X \subset LX$ be the space of loops which equal x in the region $[0, \varepsilon] \cup [1 - \varepsilon, 1]$. There is a product map λ_x from $L_\varepsilon X \times L_\varepsilon X$ to $L_\varepsilon X$, defined by composition of loops (this is familiar from homotopy theory, of course, since $L_\varepsilon X$ is an H-space); λ_x is defined by the formula

$$\lambda_x(\beta, \gamma)_t = (\beta \times \gamma)_t = \begin{cases} \beta_{2t} & t \leq \frac{1}{2} \\ \gamma_{2t-1} & t \geq \frac{1}{2}. \end{cases}$$

Lemma 2.6. *If $(\omega_0, \dots, \omega_k) \in \Omega(X)^{\otimes k+1}$ is a k-chain on $\Omega(X)$, then*

$$\lambda_x^* \sigma(\omega_0, \dots, \omega_k) = \sum_{i=0}^{k} \omega_0(x) \sigma(1, \omega_1, \dots, \omega_i) \times \sigma(1, \omega_{i+1}, \dots, \omega_k).$$

Using this lemma, we see that the theorem follows from the observation that the iterated integral map

$$\omega \in \Omega \mapsto \sigma(1, \omega) \in \Omega(LX)$$

has kernel spanned by $(1, 1)$. $\square$

Finally, let us prove a technical lemma which will be useful later in this paper.

Lemma 2.7. *If $f : A_1 \longrightarrow A_2$ is a homomorphism of* dga*s which induces an isomorphism on cohomology, then the extension of f to a map of complexes*

$$f : (\mathsf{C}(A_1), b) \longrightarrow (\mathsf{C}(A_2), b)$$

induces an isomorphism in cohomology.

Proof. If A is a dga, consider the bar filtration on $\mathsf{C}(A)$ given by

$$F^{-k}\,\mathsf{C}(A) = \operatorname{span}\{(\omega_0, \dots, \omega_i) \mid i \leq k\}.$$

The E_1-term of the spectral sequence associated to this filtration is isomorphic to $\mathsf{C}(H(A))$; more precisely, $E_1^{-p,q}$ is equal to span of the collection of chains in $\mathsf{C}(H(A))$ of the form

$$([\omega_0], \dots, [\omega_p]) \quad \text{where } [\omega_i] \in H(A) \text{ and } \sum |\omega_i| = q.$$

Since the spectral sequence converges to $H(\mathsf{C}(A))$, the lemma follows easily by a standard comparison theorem. $\square$

3. Hochschild homology and the cohomology of LX

The purpose of this section is to prove the following result of Chen, which is part of Theorem A of the Introduction. The proof is an adaptation of an argument of Adams [1].

Theorem 3.1. *If X is simply-connected, then the map of complexes*

$$\sigma : (\mathsf{C}(\Omega(X)), b) \longrightarrow (\Omega(LX), d)$$

is an isomorphism on cohomology.

Proof. Recall that the bar resolution $B(M, \Omega, N)$ of a dga Ω, where M is a left differential Ω-module and N is a right differential Ω-module, is the direct sum

$$\mathsf{B}(M, \Omega, N) = \sum_{k=0}^{\infty} M \otimes (s\Omega)^{\otimes k} \otimes N.$$

There are two differentials b_0 and b_1, of degree 1 on $\mathsf{B}(M, \Omega, N)$; b_0 is defined to be the extension of the differentials on M, N and Ω as in Section 2,

$$\begin{aligned} -b_0(m, \omega_1, \dots, \omega_k, n) = {} & (dm, \omega_1, \dots, \omega_k, n) \\ & + \sum_{i=1}^{k} (-1)^{\varepsilon_{i-1}} (m, \omega_1, \dots, \omega_{i-1}, d\omega_i, \omega_{i+1}, \dots, \omega_k, n) \\ & + (-1)^{\varepsilon_k} (m, \omega_1, \dots, \omega_k, dn), \end{aligned}$$

where $\varepsilon_i = |m| + |\omega_1| + \dots + |\omega_i| - i$, and b_1 is defined by

$$\begin{aligned} -b_1(m, \omega_1, \dots, \omega_k, n) = {} & (m\omega_1, \omega_2, \dots, \omega_k, n) \\ & + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} (m, \omega_1, \dots, \omega_{i-1}, \omega_i\omega_{i+1}, \omega_{i+2}, \dots, \omega_k, n) \\ & + (-1)^{\varepsilon_k} (m, \omega_1, \dots, \omega_{k-1}, \omega_k n). \end{aligned}$$

It is easy to check that $[b_0, b_1] = 0$, and we define the total differential to be $b = b_0 + b_1$.

Let us introduce the Chen normalized version of $\mathsf{B}(M, \Omega, N)$. If $f \in \Omega^0$, let $S_i(f)$, $1 \leq i \leq k$, be the operator on $M \otimes (s\Omega)^{\otimes k} \otimes N$ defined by the formula

$$S_i(f)(m, \omega_1, \dots, \omega_k, n) = (m, \omega_1, \dots, \omega_{i-1}, f, \omega_i, \dots, \omega_k, n).$$

We define $\mathsf{D}(M, \Omega, N)$ to be the subspace of $\mathsf{B}(M, \Omega, N)$ generated by the images of the operators $S_i(f)$ and $R_i(f) = [b, S_i(f)]$; as in the case of the cyclic bar complex, this complex is closed under b, and may be shown to be acyclic if Ω is connected. We define the Chen normalized bar complex as follows:

$$\mathsf{N}(M, \Omega, N) = \mathsf{B}(M, \Omega, N) / \mathsf{D}(M, \Omega, N).$$

Note that $\mathsf{N}(M, \Omega, N)$ is zero in negative total degrees. There is a natural filtration of $\mathsf{N}(M, \Omega, N)$ defined by

$$(m, \omega_1, \dots, \omega_k, n) \in F^s \, \mathsf{N}(M, \Omega, N) \quad \text{if } \deg m + \deg n \geq s,$$

and the total differential $b_0 + b_1$ preserves this filtration. We would like to compute the E_2-term of the resulting spectral sequence. To state the result, pick an augmentation $\varepsilon : \Omega \longrightarrow \mathbb{R}$, which defines an Ω-bimodule structure on $\mathbb{R}$. The proof of the following lemma is straightforward, and is left to the reader.

Lemma 3.2.
(1) $E_1^{p,q} \cong \sum_{i+j=p} M^i \otimes H^q(\mathsf{N}(\mathbb{R},\Omega,\mathbb{R})) \otimes N^j$
(2) $E_2^{p,q} \cong \sum_{i+j=p} H^i(M) \otimes H^q(\mathsf{N}(\mathbb{R},\Omega,\mathbb{R})) \otimes H^j(N)$

If X is a manifold with basepoint x, the dga $\Omega(X)$ has the augmentation $\varepsilon(\omega)=\omega(x)$. Let L_xX be the space of all smooth loops in X based at x, and let PX be the space of all smooth paths in X. Just as in Section 2 we can define iterated integral maps for these spaces which, by the same proof as for 2.1, give maps of complexes

$$\begin{aligned}\sigma &: (\mathsf{N}(\Omega(X),\Omega(X),\Omega(X)),b) \to (\Omega(PX),d)\\ \sigma &: (\mathsf{N}(\mathbb{R},\Omega(X),\mathbb{R}),b) \to (\Omega(L_xX),d)\end{aligned}$$

such that the following diagram of complexes commutes:

$$(*)\qquad \begin{array}{ccccc} \mathsf{N}(\Omega(X),\Omega(X),\Omega(X)) & \longrightarrow & \mathsf{N}(\Omega(X)) & \longrightarrow & \mathsf{N}(\mathbb{R},\Omega(X),\mathbb{R}) \\ {\scriptstyle\sigma}\big\downarrow & & {\scriptstyle\sigma}\big\downarrow & & {\scriptstyle\sigma}\big\downarrow \\ \Omega(PX) & \longrightarrow & \Omega(LX) & \longrightarrow & \Omega(L_xX) \end{array}$$

Here, the maps in the first row are defined by sending

$$(\omega_0,\dots,\omega_{k+1}) \in \mathsf{N}(\Omega(X),\Omega(X),\Omega(X))$$

to

$$(-1)^{|\omega_{k+1}|(|\omega_0|+\cdots+|\omega_k|-k)}(\omega_{k+1}\omega_0,\omega_1,\dots,\omega_k) \in \mathsf{N}(\Omega(X)),$$

and $(\omega_0,\dots,\omega_k) \in \mathsf{N}(\Omega(X))$ to $\omega_0(x)(\omega_1,\dots,\omega_k) \in \mathsf{N}(\mathbb{R},\Omega(X),\mathbb{R})$. The maps in the second row are induced by the inclusions $L_xX \hookrightarrow LX \hookrightarrow PX$. It is easy to check that the diagram commutes, and that these horizontal maps are maps of complexes.

Lemma 3.3. *The map $\sigma : \mathsf{N}(\Omega(X),\Omega(X),\Omega(X)) \to \Omega(PX)$ is an isomorphism on cohomology.*

Proof. Define $\alpha : \Omega(X) \to \mathsf{N}(\Omega(X),\Omega(X),\Omega(X))$ by the formula

$$\alpha\omega = (\omega,1) \in \mathsf{N}(\Omega(X),\Omega(X),\Omega(X)),$$

and let $\beta = e_0^* : \Omega(X) \to \Omega(PX)$ be the map induced by evaluation at $t=0$; note that $\sigma\alpha=\beta$. To see that α is an isomorphism on cohomology, we make use of the contraction s on $\mathsf{N}(\Omega(X),\Omega(X),\Omega(X))$ defined by the formula

$$s(\omega_0,\dots,\omega_k) = (\omega_0,\dots,\omega_k,1).$$

It is easy to see that $b_0s+sb_0=0$, while $b_1s+sb_1 = 1-\alpha\eta$, where

$$\eta : \mathsf{N}(\Omega(X),\Omega(X),\Omega(X)) \to \Omega(X)$$

is the map which vanishes on $\Omega(X)\otimes(s\Omega(X))^{\otimes k}\otimes\Omega(X))$ for $k>0$ and satisfies $\eta(\omega_0,\omega_1) = \omega_0\omega_1$ for $k=0$. Since $\eta\alpha=1$ on $\Omega(X)$, we see that η is a homotopy inverse to $\alpha : \Omega(X) \to \mathsf{N}(\Omega(X),\Omega(X),\Omega(X))$.

On the other hand, it is clear that e_0 is a smooth homotopy equivalence, and so induces an isomorphism on cohomology. Since the horizontal arrows in the diagram

$$\begin{tikzcd} \Omega(X) \arrow[r, "\alpha"] \arrow[d, equal] & \mathsf{N}(\Omega(X),\Omega(X),\Omega(X)) \arrow[d, "\sigma"] \\ \Omega(X) \arrow[r, "\beta"] & \Omega(PX) \end{tikzcd}$$

are isomorphisms on cohomology, it follows that the right-hand vertical arrow is as well. □

The next step in the proof is to show that the third column of $(*)$ is an isomorphism on cohomology. To do this, we use the following idea of Adams. Let $e : PX \to X\times X$ be the map $e(\gamma) = (\gamma(0), \gamma(1))$. The induced homomorphism on differential forms $e^* : \Omega(X \times X) \to \Omega(PX)$ makes $\Omega(PX)$ into a module over $\Omega(X \times X)$. If we filter $\Omega(PX)$ by

$$F^s(\Omega(PX)) = \sum_{r\geq s} \Omega^r(X \times X)\Omega(PX),$$

the associated spectral sequence is just the Serre spectral sequence, and since X is simply-connected, we may check that its E_2-term is $E_2^{p,q} = H^p(X \times X, \mathbb{R}) \otimes H^q(L_xX, \mathbb{R})$. The iterated integral map $\sigma : \mathsf{N}(\Omega(X),\Omega(X),\Omega(X)) \to \Omega(PX)$ preserves the filtrations defined on $\mathsf{N}(\Omega(X),\Omega(X),\Omega(X))$ and $\Omega(PX)$ and so induces a map of spectral sequences which on the $E_2^{p,q}$-term equals

$$1 \otimes \sigma : H^p(\Omega(X) \otimes \Omega(X)) \otimes H^q(\mathsf{N}(\mathbb{R}, \Omega(X), \mathbb{R})) \to H^p(X \times X) \otimes H^q(L_xX).$$

On the E_∞-terms, the map of spectral sequences is $\sigma : H(\mathsf{N}(\Omega(X),\Omega(X),\Omega(X))) \to H(PX)$, and this map is an isomorphism by 3.3. The map of spectral sequences is also an isomorphism on the terms $E_2^{p,0}$. We now use the Zeeman comparison theorem for spectral sequences [14], which applies since the spectral sequences come from finite filtrations and thus converge. This shows that the map of spectral sequences is an isomorphism on the terms $E_2^{0,q}$, in other words, that $\sigma : H(\mathsf{N}(\mathbb{R}, \Omega(X), \mathbb{R})) \to H(L_xX)$ is an isomorphism.

The proof that the middle column is an isomorphism on cohomology is similar: we filter the Hochschild complex $\mathsf{N}(\Omega(X))$ as follows

$$(\omega_0, \dots, \omega_k) \in F^s\, \mathsf{N}(\Omega(X)) \quad \text{if and only if } \deg \omega_0 \geq s.$$

The spectral sequence defined by this filtration has E_2-term $H^p(\Omega(X)) \otimes H^q(\mathsf{N}(\mathbb{R}, \Omega(X), \mathbb{R}))$. If $e : LX \to X$ is the map of evaluation at $t = 0$, $e(\gamma) = \gamma(0)$, the induced homomorphism e^* : $\Omega(X) \to \Omega(LX)$ makes $\Omega(LX)$ into a module over $\Omega(X)$. If we filter $\Omega(LX)$ by

$$F^s\Omega(LX) = \sum_{r\geq s} \Omega^r(X)\Omega(LX),$$

the associated spectral sequence is the Serre spectral sequence with E_2-term $H^p(X) \otimes H^q(L_xX)$, which converges to $H^{p+q}(LX)$. Once more the iterated integral map respects the filtration and so defines a map of spectral sequences which on the $E_2^{p,q}$-term is given by

$$1 \otimes \sigma : H^p(\Omega(X)) \otimes H^q(\mathsf{N}(\mathbb{R}, \Omega(X), \mathbb{R})) \to H^p(X) \otimes H^q(L_xX).$$

By the above results, this is an isomorphism on the E_2-terms, and applying the Zeeman comparison theorem once more, we deduce that $\sigma : \mathsf{N}(\Omega(X)) \to \Omega(LX)$ is an isomorphism on cohomology, completing the proof of 3.1. □

4. Products

In 2.1, we gave formulas for the operators d and $\tilde{P}$ on $\Omega(LX)$ at the level of iterated integrals. In this section we give formulas at the level of iterated integrals for the exterior product and the multilinear maps $\tilde{P}_k$ on $\Omega(LX)$. We start with the exterior product, for which the corresponding product on iterated integrals is well-known: it is the shuffle product.

If $(a_1, \dots, a_p)$ and $(b_1, \dots, b_q)$ are two ordered sets, then a **shuffle** χ of $(a_1, \dots, a_p)$ and $(b_1, \dots, b_q)$ is a permutation of the ordered set $(a_1, \dots, a_p, b_1, \dots, b_q)$ with the property that $\chi(a_i)$ occurs before $\chi(a_j)$, and $\chi(b_i)$ occurs before $\chi(b_j)$, if $i < j$. Given $\alpha = (\alpha_0, \dots, \alpha_p)$ and $\beta = (\beta_0, \dots, \beta_q)$ in $\mathsf{N}(\Omega(X))$, their shuffle product is defined by the formula

$$S(\alpha, \beta) = (-1)^{|\beta_0|(|\alpha_1|+\cdots+|\alpha_p|-p)} \sum_{\chi} \alpha_0\beta_0 \otimes S_\chi(s\alpha_1 \otimes \ldots \otimes s\alpha_p \otimes s\beta_1 \otimes \ldots \otimes s\beta_q),$$

where χ runs through all shuffles, and S_χ is the canonical isomorphism from $(s\Omega(X))^{\otimes(p+q)}$ to itself defined by the permutation χ. It is a standard computation to check that the shuffle product is graded commutative, and that the total boundary operator $b = b_0 + b_1$ is a derivation.

Proposition 4.1. *Under the iterated integral map σ, the shuffle product on $\mathsf{N}(\Omega(X))$ is carried into the wedge product on $\Omega(LX)$.*

Proof. The formula for $\sigma(\alpha) \wedge \sigma(\beta)$ is

$$\int_{\Delta_p \times \Delta_q} \alpha_0(0)\iota\alpha_1(t_1)\ldots\iota\alpha_p(t_p)\beta_0(0)\iota\beta_1(s_1)\ldots\beta_q(s_q)$$
$$= (-1)^{|\beta_0|(|\alpha_1|+\cdots+|\alpha_p|-p)} \int_{\Delta_p \times \Delta_q} \alpha_0(0)\beta_0(0)\iota\alpha_1(t_1)\ldots\iota\alpha_p(t_p)\iota\beta_1(s_1)\ldots\beta_q(s_q)$$

Now let χ be a shuffle of the ordered sets $(t_1, \dots, t_p) \in \Delta_p$, $(s_1, \dots, s_q) \in \Delta_q$, and let R_χ be the subset of $\Delta_p \times \Delta_q$ consisting of those points $(t_1, \dots, t_p, s_1, \dots, s_q)$ such that the $(p+q)$-tuple $\chi(t_1, \dots, t_p, s_1, \dots, s_q)$ is monotonically increasing. It is clear that each set R_χ is a $(p+q)$-simplex and that $\Delta_p \times \Delta_q$ is the union of the R_χ; this is the shuffle product triangulation of $\Delta_p \times \Delta_q$. It is straightforward to check that

$$\int_{R_\chi} \alpha_0(0)\iota\alpha_1(t_1)\ldots\iota\alpha_p(t_p)\beta_0(0)\iota\beta_1(s_1)\ldots\iota\beta_q(s_q)$$
$$= \sigma(\alpha_0\beta_0 \otimes S_\chi(s\alpha_1 \otimes \ldots \otimes s\alpha_p \otimes s\beta_1 \otimes \ldots \otimes s\beta_q)),$$

and the proposition follows easily. □

Now we look for the operators on $\mathsf{N}(\Omega(X))$ which correspond under the iterated integral map to the operators $\tilde{P}_n$ on $\Omega(LX)$ defined in Section 1. If $B_n : \mathsf{N}(\Omega(X))^{\otimes n} \to \mathsf{N}(\Omega(X))$ is the operator corresponding to $\tilde{P}_n$, then $B_1 = B$ will be Connes's operator, the $\mathbb{R}[u]$-bilinear operator $S(a, b) + uB_2(a, b)$ will define a product on $\mathsf{N}(\Omega(X))[u]$, the operator uB_3 will give a chain homotopy between the two different ways of associating three cyclic chains a, b and c in a triple product, and so on. This is a special case of the fact that the cyclic bar complex of any commutative dga has a natural A_∞-structure, as will be shown in [9]. Here we will confine ourselves to the case of $\mathsf{N}(\Omega(X))$ and use several short cuts which are possible in this special case.

In order to define the higher maps B_n, we need a little combinatorial machinery. Given numbers $i_1, \dots, i_n$, let $C(i_1, \dots, i_n)$ be the set $\{(1, 0), \dots, (1, i_1), \dots, (n, 0), \dots, (n, i_n)\}$, ordered lexicographically, that is $(k_1, l_1) < (k_2, l_2)$ if and only if $k_1 < k_2$ or $k_1 = k_2$ and $l_1 < l_2$. A **cyclic shuffle** χ is a permutation of the set $C(i_1, \dots, i_n)$ which satisfies the following two conditions:

(1) $\sigma(i, 0) < \sigma(j, 0)$ if $i < j$, and

(2) for each $1 \le m \le n$, there is a number $0 \le j_m \le i_m$ such that

$$\sigma(m, j_m) < \dots < \sigma(m, i_m) < \sigma(m, 0) < \dots \sigma(m, j_m - 1).$$

We will denote the set of cyclic shuffles by $S(i_1, \dots, i_n)$.

If we imagine the set $C(i_1, \dots, i_n)$ arranged as a grid in the plane, so the p-th column is made up of the points $(p, 0), (p, 1), \dots, (p, i_p)$, then a cyclic shuffle is given by first applying a cyclic permutation to each column and then shuffling the columns together in such a way that $(p, 0)$ occurs before $(q, 0)$ if $p < q$.

Another way to understand the notion of a cyclic shuffle, and one motivation for its introduction, is as follows. If

$$(t_1, \dots, t_n; s_{(1,0)}, \dots, s_{(1,i_1)}; \dots; s_{(n,0)}, \dots, s_{(n,i_n)}) \in \Delta_n \times \Delta_{i_1} \times \dots \times \Delta_{i_n},$$

is a point in a product of simplices, we form the $(n + i_1 + \dots + i_n)$-tuple of numbers

$$(t_1, t_1 + s_{(1,i_1)}, \dots, t_1 + s_{(1,i_1)}, \dots, t_n, t_n + s_{(n,0)}, \dots, t_n + s_{(n,i_n)}),$$

where each of the real numbers in this expression is taken modulo 1. The permutation needed to reorder these numbers in $[0, 1]$ into increasing order is a cyclic shuffle. Furthermore if we define R_χ to be the subset of the product of simplices such that the cyclic shuffle χ puts the above set of points in increasing order then the set R_χ is a simplex and the collection of simplices R_χ gives a triangulation of the product of simplices.

Given elements $\alpha_k = (a_{(k,0)}, \dots, a_{(k,i_k)})$, $1 \le k \le n$, in $\mathsf{N}(\Omega(X))$, we define the operator $B_n(\alpha_1, \dots, \alpha_n) \in \mathsf{N}(\Omega(X))$ by the formula

$$B_n(\alpha_1, \dots, \alpha_n) = \sum_{\chi \in S(i_0, \dots, i_n)} 1 \otimes S_\chi(s\alpha_{(1,0)} \otimes \dots \otimes s\alpha_{(1,i_1)} \otimes \dots \otimes s\alpha_{(n,0)} \otimes \dots \otimes s\alpha_{(n,i_n)}).$$

Let us check that these operators B_n correspond to the operators $\tilde{P}_n$ under the iterated integral map.

Proposition 4.2. $\sigma B_n = \tilde{P}_n \sigma$

Proof. If we combine the above triangulation of a product of simplices indexed by cyclic shuffles and the method used in Section 2 to prove that $\sigma B = \tilde{P}\sigma$, the proof is straightforward. □

Imitating the construction of Section 1, we define multilinear products m_n on $\mathsf{N}(\Omega(X))[u]$ by

$$m_n(a_1, \dots, a_n) = \begin{cases} (b + uB)a_1, & n = 1, \\ -(-1)^{|a_1|} S(a_1, a_2) + uB_2(a_1, a_2), & n = 2, \\ uB_n(a_1, \dots, a_n), & \text{otherwise}, \end{cases}$$

where S is the shuffle product. The following result completes the proof of Theorem A of the introduction.

Proposition 4.3. *The operators m_n define an A_∞-structure on $\mathsf{N}(\Omega(X))[u]$.*

Proof. This follows from the corresponding statement 1.8 about the operators m_n on $\Omega(LX)[u]$ defined in terms of the $\tilde{P}_n$, and the fact that σ is injective on $\mathsf{N}(\Omega(X))$, which was proved in 2.5. □

In fact, the operators B_n define an A$_\infty$-structure on the normalized cyclic bar complex $\mathsf{N}(\Omega)[u]$ for any commutative dga, but the proof in [9], though straightforward, is rather long. We have avoided this extra work by using the fact that σ is injective on the Chen normalised cyclic bar complex.

5. Iterated integrals for manifolds with a compact group action

In this section, we will present an equivariant version of the theory of iterated integrals, which applies when a connected compact group G acts on the manifold Y. This will be used in the next section in the special case in which G is the circle $\mathbb{T}$ and Y is the manifold $X \times \mathbb{T}$.

We need a little bit of Chern-Weil theory, in the form due to Cartan [4]. Recall that a classifying bundle for a Lie group G is any principal bundle $EG \longrightarrow BG$ with structure group G such that EG is contractible. There is usually no classifying space EG of G which is a finite-dimensional manifold, but if G is connected and compact, we may realise EG as a Hilbert manifold, such that the principal bundle $EG \longrightarrow BG$ possesses a canonical connection invariant under the action of G. Under these conditions, BG is simply-connected: for example, if G is the circle $\mathbb{T}$, then $B\mathbb{T}$ may be realized as the infinite dimensional projective space $\mathbb{C}P^\infty$.

Let $\Omega_G(Y)$ be the graded algebra $(\Omega(Y) \otimes S(\mathbf{g}^*))^G$, where the degree of the vector space $\mathbf{g}^*$ is 2. On the graded algebra $\Omega(Y) \otimes S(\mathbf{g}^*)$, we may define a derivation d_G by the following formula, using a basis X_i of $\mathbf{g}$ with dual basis ξ^i of $\mathbf{g}^*$:

$$d_G = d + \sum_i \iota(X_i)\xi^i,$$

where $\iota(X_i)$ is the contraction with the vector field induced by $X_i \in \mathbf{g}$ on the manifold X. This derivation commutes with the action of G, so that it maps $\Omega_G(Y) = (\Omega(Y) \otimes S(\mathbf{g}^*))^G$ to itself. The square of d_G is equal to

$$d_G^2 = \sum_i L(X_i)\xi^i,$$

where $L(X_i)$ is the Lie derivative with respect to the vector field X_i. Since d_G^2 vanishes on $\Omega_G(Y)$, we see that $\Omega_G(Y)$ is a dga with differential d_G.

The Hochschild complex $\mathsf{C}(\Omega_G(Y))$ of $\Omega_G(Y)$ comes equipped with Connes's B-operator and with the Hochschild boundary operator $b_G = b_1 + b_0$, where b_0 is the extension of d_G to $\mathsf{C}(\Omega_G(Y))$ given by the familiar formula

$$b_0(\omega_0, \dots, \omega_k) = -\sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}} (\omega_0, \dots, d_G\omega_i, \dots, \omega_k).$$

If Y is a smooth G-manifold, then the equivariant cohomology $H_G(Y)$ of Y is the cohomology of the homotopy quotient $EG \times_G Y$, which is a fibre bundle with base BG and fibre Y. The Chern-Weil map defined by means of the canonical connection on EG gives a homomorphism of dgas

$$\Omega_G(Y) \longrightarrow \Omega(EG \times_G Y)$$

and if G is a compact group, this induces an isomorphism on cohomology (see [4] or [15]).

The dga of equivariant differential forms is an algebra over $\Omega_G = \Omega_G(\mathrm{pt})$, which may of course be identified with the commutative algebra $S(\mathbf{g}^*)^G$. (It is a well known theorem that this algebra is a polynomial algebra with dimension equal to the rank of G, but we will not need this.) Thus we can form the cyclic bar complex of $\Omega_G(Y)$ over the algebra Ω_G, by taking all of the tensor products over Ω_G. We will denote this complex by $\mathsf{C}_{\Omega_G}(\Omega_G(Y))$:

$$\mathsf{C}_{\Omega_G}(\Omega_G(Y)) = \sum_{k=0}^{\infty} \overbrace{\Omega_G(Y) \otimes_{\Omega_G} \dots \otimes_{\Omega_G} \Omega_G(Y)}^{k+1 \text{ times}}.$$

We will now define the **equivariant iterated integral map**

$$\sigma_G : \mathsf{C}_{\Omega_G}(\Omega_G(Y)) \to \Omega_G(LY),$$

which is an equivariant generalization of the ordinary iterated integral map. First of all, we observe that there is a natural embedding

$$\sum_{k=0}^{\infty} \overbrace{\Omega_G(Y) \otimes_{\Omega_G} \ldots \otimes_{\Omega_G} \Omega_G(Y)}^{k+1 \text{ times}} \hookrightarrow \sum_{k=0}^{\infty} \Big(S(\mathbf{g}^*) \otimes \Omega(Y)^{\otimes(k+1)} \Big)^G.$$

If we compose this with iterated integral map

$$1 \otimes \sigma : \sum_{k=0}^{\infty} \big(S(\mathbf{g}^*) \otimes \Omega(Y)^{\otimes(k+1)} \big)^G \to \big(S(\mathbf{g}^*) \otimes \Omega(LY) \big)^G = \Omega_G(Y),$$

we obtain the equivariant iterated integral map. It is easily checked that σ_G is a map of complexes

$$\sigma_G : (\mathsf{C}_{\Omega_G}(\Omega_G(Y)), b_G) \to (\Omega_G(LY), d_G).$$

The Chen normalized bar complex $\mathsf{N}_{\Omega_G}(\Omega_G(Y))$ is the quotient of $\mathsf{C}_{\Omega_G}(\Omega_G(Y))$ defined by formulas similar to those used to define $\mathsf{N}(\Omega)$ in the case of a dga Ω over $\mathbb{R}$. We define the complex of degenerate chains $\mathsf{D}_{\Omega_G}(\Omega_G(Y))$ as the image of $\mathsf{C}_{\Omega_G}(\Omega_G(Y))$ of the maps $S_i(f)$, $f \in {\Omega_G}^0(Y) = C^\infty(Y)^G$, defined by

$$S_i(f)(\omega_0, \ldots, \omega_k) = (\omega_0, \ldots, \omega_{i-1}, f, \omega_i, \ldots, \omega_k),$$

and $R_i(f) = [b_G, S_i(f)]$. As before, the complex $\mathsf{D}_{\Omega_G}(\Omega_G(Y))$ is acyclic if Y is connected. It is also easy to see that the equivariant iterated integral map vanishes on the kernel of the Chen normalization, so that we obtain the following lemma.

Lemma 5.1. *The equivariant iterated integral map is a strict homomorphism of A_∞-algebras*

$$\sigma_G : \mathsf{N}_{\Omega_G}(\Omega_G(Y))[u] \to \Omega_G(LY)[u],$$

in which the differentials are respectively $b_G + uB$ and $d_G + uP$.

We will now show that σ_G induces an isomorphism on cohomology if Y is simply-connected. To do this, we will construct a commuting square

$$\begin{array}{ccc}
\mathsf{N}_{\Omega_G}(\Omega_G(Y)) & \xrightarrow{\sigma_G} & \Omega_G(Y) \\
\downarrow & & \downarrow \\
\mathsf{N}_{\Omega_G}(\Omega(EG \times_G Y)) & \longrightarrow & \Omega(EG \times_G LY)
\end{array}$$

in which the vertical and bottom arrows induce isomorphisms in cohomology. The two vertical arrows are defined by means of the Chern-Weil map, and thus always induce isomorphisms in cohomology, even if Y is not simply-connected. Thus, it only remains to fill in the bottom line.

Since $EG \times_G Y$ is a Hilbert manifold, we may apply the results of Sections 2 and 3, in the setting of Hilbert manifolds, to obtain the following lemma.

Lemma 5.2. *The iterated integral map defines a strict homomorphism of A_∞-algebras*

$$\mathsf{N}(\Omega(EG\times_G Y))[u] \to \Omega(L(EG\times_G Y))[u],$$

which induces an isomorphism on cohomology if $EG\times_G Y$ is simply-connected.

Observe that there is a natural map

$$EG\times_G LY \to L(EG\times_G Y) \cong L(EG)\times_{LG} LY,$$

defined by taking the inclusion of EG into $L(EG)$ as constant loops. The composition

$$\mathsf{C}(\Omega(EG\times_G Y)) \to \Omega(L(EG\times_G Y)) \to \Omega(EG\times_G LY)$$

descends to a map

$$\mathsf{C}_{\Omega_G}(\Omega(EG\times_G Y)) \to \Omega(EG\times_G LY).$$

To see this, observe that a typical chain in the kernel of the map

$$\mathsf{C}(\Omega(EG\times_G Y)) \to \mathsf{C}_{\Omega_G}(\Omega(EG\times_G Y))$$

has the form

$$C = (a_0,\dots,a_i m, a_{i+1},\dots,a_k) - (a_0,\dots,a_i, m a_{i+1},\dots,a_k),$$

with $a_i\in\Omega(EG\times_G Y)$ and $m\in\Omega_G$. Under the composition of the iterated integral map

$$\mathsf{C}(\Omega(EG\times_G Y)) \to \Omega(L(EG\times_G Y))$$

with the restriction map $\Omega(L(EG\times_G Y)) \to \Omega(EG\times_G LY)$, the chain C maps to zero, as required. We will prove that the composition

$$(\mathsf{N}_{\Omega_G}(\Omega_G(Y)), b_G) \to (\mathsf{N}_{\Omega_G}(\Omega(EG\times_G Y)), b) \to (\Omega(EG\times_G LY), d)$$

is an isomorphism in cohomology, if Y is simply-connected, by comparing the Serre-type spectral sequences attached to these complexes.

The graded algebra Ω_G acts on the complex $\mathsf{N}_{\Omega_G}(\Omega_G(Y))$ by multiplication, and we filter $\mathsf{N}_{\Omega_G}(\Omega_G(Y))$ as follows:

$$F^s\,\mathsf{N}_{\Omega_G}(\Omega_G(Y)) = \sum_{r\geq s} \Omega_G^r\cdot \mathsf{N}_{\Omega_G}(\Omega_G(Y)).$$

It is easy to see that the complex $E_0 = \operatorname{gr}\mathsf{N}_{\Omega_G}(\Omega_G(Y))$ may be identified with $\Omega_G\otimes\mathsf{N}(\Omega(Y))$ with differential $1\otimes b$, and hence that both the E_1 and E_2-terms of the spectral sequence equal $\Omega_G\otimes H(\mathsf{N}(\Omega(Y)))$. The corresponding spectral sequence for $\Omega(EG\times_G LY)$ is defined by means of the filtration

$$F^s\Omega(EG\times_G LY) = \sum_{r\geq s}\Omega^r(BG)\cdot\Omega(EG\times_G LY),$$

associated to the fibration

$$\begin{array}{ccc} LY & \longrightarrow & EG\times_G LY \\ & & \big\downarrow \\ & & BG \end{array}$$

Since BG is simply-connected, this Serre spectral sequence has E_2-term equal to $H(BG)\otimes H(LY) \cong \Omega_G\otimes H(LY)$. Thus, the map of spectral sequences induces at the E_2-level the map

$$\Omega_G\otimes H(\mathsf{N}(\Omega(Y))) \xrightarrow{1\otimes\sigma} \Omega_G\otimes H(LY),$$

which by 3.1 is an isomorphism since Y is simply-connected. We have proved the following result.

Proposition 5.3. *If Y is a simply-connected G-manifold, the map of complexes*

$$\sigma_G : (\mathsf{N}_{\Omega_G}(\Omega_G(Y)), b_G) \longrightarrow (\Omega_G(LY), d_G)$$

is an isomorphism on cohomology.

This proposition has the following corollary. The product of groups $\mathbb{T} \times G$ acts on LY, since the action of $\mathbb{T}$ on LY by rotation of the loop commutes with the action of G. If W is an $\mathbb{R}[u]$-module, we will write $H_{\mathbb{T}\times G}(LY; W)$ for the cohomology of the complex

$$(\Omega_G(LY)[u], d_G + uP).$$

Corollary 5.4. *If Y is a simply-connected G-manifold, the induced map*

$$\sigma_G : H(\mathsf{N}_{\Omega_G}(\Omega_G(Y)) \otimes W, b_G + uB) \longrightarrow H_{\mathbb{T}\times G}(LY; W)$$

is an isomorphism.

The example to which we would like to apply 5.3 is as follows. Let X be a simply-connected manifold, and let Y be the manifold $X \times \mathbb{T}$, with circle action given by rotation of $\mathbb{T}$. Unfortunately, Y is not simply-connected, so the above theorem does not apply directly; however, we can prove something almost as strong.

Let us start by investigating how the iterated integral map fails to be an isomorphism for the circle $\mathbb{T}$. Since the inclusion of invariant differential forms $\Omega(\mathbb{T})^{\mathbb{T}} = \mathbb{R}[dt] \hookrightarrow \Omega(\mathbb{T})$ is an isomorphism on cohomology, it follows by 2.7 that $\mathsf{N}(\Omega(\mathbb{T}))$ has the same cohomology as $\mathsf{N}(\mathbb{R}[dt])$. The complex $\mathsf{N}(\mathbb{R}[dt])$ is easy to write down explicitly: it is spanned by chains $(1, dt, \dots, dt)$ in degree zero and $(dt, \dots, dt)$ in degree one. Under the iterated integral map, the chain $(1, dt, \dots, dt)$ with k occurrences of dt maps to

$$\sigma(1, dt, \dots, dt) = \frac{(\sigma(1, dt))^k}{k!} = \frac{n^k}{k!},$$

where n is the homomorphism $n : L\mathbb{T} \longrightarrow \pi_1(\mathbb{T})$ which labels the component. Likewise, the chain $(dt, \dots, dt)$ with $k+1$ occurrences of dt maps to

$$\sigma(dt, \dots, dt) = e_0^*(dt) \wedge \frac{n^k}{k!},$$

where by $e_0^*(dt)$ we mean the one-form obtained by pulling back $dt \in \Omega(\mathbb{T})$ to $L\mathbb{T}$ by the evaluation map at time $t = 0$. Since the Hochschild boundary vanishes on $\mathsf{N}(\Omega(\mathbb{T})^{\mathbb{T}})$, we obtain the following result.

Proposition 5.5. *The Hochschild homology of the* dga $\Omega(\mathbb{T})$ *is isomorphic to the algebra* $\mathbb{R}[n, dt]$, *where* $\deg n = 0$ *and* $\deg dt = 1$.

The loop space $L\mathbb{T}$ has homotopy type $\pi_1(\mathbb{T}) \times \mathbb{T}$, since each component is homotopy equivalent to $\mathbb{T}$. It follows that $H^\bullet(L\mathbb{T})$ may be identified with $H^\bullet(\pi_1(\mathbb{T})) \otimes H^\bullet(\mathbb{T}) = C(\pi_1(\mathbb{T})) \otimes H^\bullet(\mathbb{T})$, where $C(\pi_1(\mathbb{T}))$ is the space of continuous real-valued functions on $\pi_1(\mathbb{T})$. Thus, when we take cohomology, the iterated integral map $\mathsf{N}(\Omega(\mathbb{T})) \longrightarrow \Omega(L\mathbb{T})$ becomes the inclusion

$$\mathbb{R}[n] \otimes H^\bullet(\mathbb{T}) \hookrightarrow C(\pi_1(\mathbb{T})) \otimes H^\bullet(\mathbb{T}),$$

where $\mathbb{R}[n]$ is embedded in $C(\pi_1(\mathbb{T}))$ by sending its generator n to the function $n \in C(\pi_1(\mathbb{T}))$. Our theorem identifying these cohomology spaces breaks down, but it remains true if we restrict to the union of a finite number of components of $L\mathbb{T}$. If $S \subset \pi_1(\mathbb{T}) \cong \mathbb{Z}$ is a finite subset of $\pi_1(\mathbb{T})$, let $\mathbb{R}[n]_S$

be the ideal of polynomials which vanish when restricted to S thought of as a subset of $\mathbb{R}$, and let $L_S\mathbb{T}$ be the union of components

$$\begin{array}{ccc} L_S\mathbb{T} & \longrightarrow & L\mathbb{T} \\ \Big\downarrow & & \Big\downarrow \\ S & \longrightarrow & \pi_1(\mathbb{T}) \end{array}$$

The algebra $\mathbb{R}[n]$ acts on $\mathsf{N}(\Omega(\mathbb{T}))$ by letting its generator n act as shuffle product with $(1, dt)$; if C is a complex over $\mathbb{R}[n]$, we will denote the quotient $C/\mathbb{R}[n]_S C$ simply by $C/\mathbb{R}[n]_S$.

Proposition 5.6. *The cohomology of the quotient of the Chen normalized bar complex of the* dga $\Omega(\mathbb{T})$ *by the ideal* $R[n]_S$ *is isomorphic to the cohomology of* $L_S\mathbb{T}$*:*

$$\sigma : H(\mathsf{N}(\Omega(\mathbb{T}))/\mathbb{R}[n]_S, b_{\mathbb{T}}) \cong H(L_S\mathbb{T}, d_{\mathbb{T}}).$$

Proof. The first statement follows from our explicit calculation of $\mathsf{N}(\Omega(\mathbb{T})^{\mathbb{T}})$, since the map

$$\mathsf{N}(\Omega(\mathbb{T}))/\mathbb{R}[n]_S \cong H^\bullet(\mathbb{T}) \otimes C(S) \hookrightarrow \mathsf{N}(\Omega(\mathbb{T}))/\mathbb{R}[n]_S$$

induces an isomorphism on cohomology. □

Chen has proved general results about the Hochschild homology of the dga $\Omega(X)$ when X is not simply-connected in [7].

The above result has an immediate extension to the case of $X\times\mathbb{T}$, by means of the Eilenberg-Zilber Theorem. We recall that this theorem shows that the shuffle product map

$$\mathsf{C}(\Omega(X)) \otimes \mathsf{C}(\Omega(\mathbb{T})) \to \mathsf{C}(\Omega(X \times \mathbb{T}))$$

induces an isomorphism on cohomology. Replacing the cyclic bar complexes by their Chen normalizations, we see that

$$\mathsf{N}(\Omega(X)) \otimes \mathsf{N}(\Omega(\mathbb{T})) \to \mathsf{N}(\Omega(X \times \mathbb{T}))$$

induces an isomorphism on cohomology. On restricting to a finite set $S \subset \pi_1(\mathbb{T})$, the iterated integral map

$$\sigma_G : \mathsf{N}(\Omega(X \times \mathbb{T}))/\mathbb{R}[n]_S \to \Omega(L_S(X \times \mathbb{T}))$$

induces an isomorphism on cohomology. Going back over the proof of 5.3 and its corollary 5.4, we see that there is no difficulty obtaining the following result.

Proposition 5.7. *If* X *is simply-connected and* $S \subset \pi_1(\mathbb{T})$ *is finite, then the map of complexes*

$$\sigma_{\mathbb{T}} : (\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))/\mathbb{R}[n]_S, b_{\mathbb{T}}) \to (\Omega_{\mathbb{T}}(L_S(X \times \mathbb{T}))), d_{\mathbb{T}})$$

is an isomorphism on cohomology.

The manifold $L(X \times \mathbb{T})$ carries two circle actions, the first being the action of rotating loops and the second being given by the action of $\mathbb{T}$ on $X \times \mathbb{T}$. The corresponding operators ι on $\Omega(L(X \times \mathbb{T}))$ are denoted respectively ι_1 and ι_2, and by P_1 we mean the average of ι_1 under the first circle action. The complex of 5.3 which calculates the $\mathbb{T} \times \mathbb{T}$-equivariant cohomology of $L(X \times \mathbb{T})$ is $(\Omega(L(X \times \mathbb{T}))^{\mathbb{T}}[u_1, u_2], d + u_1 P_1 + u_2 \iota_2)$, where the space of differential forms $\Omega(L(X \times \mathbb{T}))^{\mathbb{T}}$ are those which are invariant under the second circle action; note that this definition is asymmetric in the two circle actions, since we only assume the differential forms to be invariant under the second circle action. This complex is a module over the algebra $\Omega_{\mathbb{T}\times\mathbb{T}} = S(\mathbf{t}^* \times \mathbf{t}^*) = \mathbb{R}[u_1, u_2]$, where the generator u_1 corresponds to the first circle action and u_2 to the second. If W is a module for the graded algebra $\mathbb{R}[u_1, u_2]$, we can form the complex

$$(\Omega(L(X \times \mathbb{T}))^{\mathbb{T}} \otimes W, d + u_1 P_1 + u_2 \iota_2),$$

whose cohomology we denote by $H_{\mathbb{T}\times\mathbb{T}}(L(X \times \mathbb{T}); W)$. Similarly, the algebra $\mathbb{R}[u_1, u_2]$ acts on the complex

$$(\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))[u_1], d_{\mathbb{T}} + u_1 B)$$

if we identify the generator of the algebra $\Omega_{\mathbb{T}}$ which acts on $\Omega_{\mathbb{T}}(X \times \mathbb{T})$ with u_2. Thus, we can form the tensor product of this complex with W over the algebra $\mathbb{R}[u_1, u_2]$. It is straightforward to see that the iterated integral map $\sigma_{\mathbb{T}}$ defines a map of complexes

$$(\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))[u_1] \otimes_{\mathbb{R}[u_1,u_2]} W, d_{\mathbb{T}} + u_1 B) \to (\Omega(L(X \times \mathbb{T}))^{\mathbb{T}} \otimes W, d + u_1 P_1 + u_2 \iota_2).$$

Combining this map with our results about restriction to a finite number of components of $L\mathbb{T}$, and using the fact that the operator B commutes with the action of $\mathbb{R}[n]$, we obtain the following result.

Proposition 5.8. *The iterated integral map defines a homomorphism of complexes from*

$$\big(\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))[u_1]/\mathbb{R}[n]_S \otimes_{\mathbb{R}[u_1,u_2]} W, d_{\mathbb{T}} + u_1 B\big) \\ \to (\Omega(L_S(X \times \mathbb{T}))^{\mathbb{T}} \otimes W, d + u_1 P_1 + u_2 \iota_2),$$

which is an isomorphism on cohomology if X is simply-connected.

If ρ is a homomorphism from $\mathbb{T}$ to $\mathbb{T} \times \mathbb{T}$, we can think of $L(X \times \mathbb{T})$ as a manifold with the circle action given by composing the torus action on $L(X \times \mathbb{T})$ with ρ; we denote this $\mathbb{T}$-manifold by $L^\rho(X \times \mathbb{T})$. Such a homomorphism is labelled by a pair of integers $(m, n) \in \mathbb{Z}^2$, and is given by the formula

$$\rho_t^{(m,n)}(\gamma, \lambda)(s) = (\gamma(s + mt), \lambda(s + mt) + nt),$$

where $\gamma \in LX$ and $\lambda \in L\mathbb{T}$. Observe that negative of the identity map $\mathrm{id}_{\mathbb{T}} \in L\mathbb{T}$ satisfies

$$\rho_t^{(m,m)}(\gamma, -\mathrm{id}_{\mathbb{T}})(s) = (\gamma(s + mt), -\mathrm{id}_{\mathbb{T}}).$$

The differential of the map ρ is a linear map $d\rho : \mathbf{t} \to \mathbf{t} \times \mathbf{t}$, whose adjoint defines a linear map $(d\rho)^* : \mathbf{t}^* \times \mathbf{t}^* \to \mathbf{t}^*$, and hence a homomorphism of algebras

$$(d\rho)^* : S(\mathbf{t}^* \times \mathbf{t}^*) \to S(\mathbf{t}^*).$$

In the case of the homorphism $\rho^{(m,n)}$, this map sends u_1 to mu, and u_2 to nu. If W is a $\mathbb{R}[u]$-module, let us denote W, thought of as a $\mathbb{R}[u_1, u_2]$-module by means of the homomorphism $(d\rho)^*$ by W^ρ. Then it is clear that the natural map

$$H_{\mathbb{T}\times\mathbb{T}}(L(X \times \mathbb{T}); W^\rho) \to H_{\mathbb{T}}(L^\rho(X \times \mathbb{T}); W)$$

is an isomorphism.

The homomorphism ρ which will interest us in the next section is the one corresponding to $(m, n) = (1, 1)$, in which case the effect of the homomorphism is to identify both u_1 and u_2 with u. We will denote by $L^\rho(X \times \mathbb{T})$ the loop space of $X \times \mathbb{T}$ with the circle action determined by the homomorphism ρ; its equivariant cohomology $H_{\mathbb{T}}(L^\rho(X \times \mathbb{T}); W)$ is a $\mathbb{R}[u]$-module. Note that the embedding μ of LX into $L^{(1,1)}(X \times \mathbb{T})$ given by sending the loop γ to $(\gamma, -\mathrm{id}_{\mathbb{T}})$ is a $\mathbb{T}$-equivariant homotopy equivalence. Thus, we see that the **extended iterated integral** map defined by composition of the iterated map

$$\sigma_{\mathbb{T}} : \mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))[u_1] \to \Omega(L(X \times \mathbb{T}))^{\mathbb{T}}[u_1, u_2]$$

with the pullback by μ

$$\mu^* : \Omega(L^{(1,1)}(X \times \mathbb{T}))^{\mathbb{T}}[u_1, u_2] \to \Omega(LX)[u_1, u_2]$$

defines an isomorphism in cohomology between the complexes

$$\big(\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))[u_1]/((1, dt) - (1))) \otimes_{\mathbb{R}[u_1,u_2]} W^{\rho}, b_{\mathbb{T}} + uB\big) \\ \to (\Omega(LX)[u] \otimes_{\mathbb{R}[u]} W, d + uP_1 + u\iota_2),$$

where we use the fact that the ideal in $\mathbb{R}[n]$ of polynomials vanishing at $n = 1$ is generated by the element $(1, dt) - (1) \in \mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(\mathbb{T}))$. We will denote the extended iterated integral map by $\tilde{\sigma}$.

It will be useful in the next section to have a more explicit formula for this extended iterated integral map. If we follow what happens to a chain under the above sequence of maps, we see that it amounts to the same thing as evaluating it at $u_1 = u_2 = u$, thus obtaining an element of $\mathsf{N}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))$, applying the iterated integral map to obtain a differential form on $L(X \times \mathbb{T})$, and then pulling back to LX by μ. In other words, the iterated integral of a cochain $(\omega_0(u_2) + dt\, \rho_0(u_2), \dots, \omega_k(u_2) + dt\, \rho_k(u_2))$ is equal to

$$\tilde{\sigma}(\omega_0(u_2) + dt\, \xi_0(u_2), \dots, \omega_k(u_2) + dt\, \xi_k(u_2)) = \\ \int_{\Delta_k} \omega_0(0) \wedge (\iota\omega_1(t_1) - \xi_1(t_1)) \wedge \dots \wedge (\iota\omega_k(t_k) - \xi_k(t_k))\big|_{u_1=u_2=u}.$$

To close this section, we mention a related result on the relationship between cyclic homology and equivariant cohomology. Observe that by Chern-Weil theory and 2.7, the induced map of Chen normalized cyclic bar complexes

$$\mathsf{N}(\Omega_G(Y))[u] \to \mathsf{N}(\Omega(EG \times_G Y))[u]$$

is a strict homomorphism of A_∞-algebras which induces an isomorphism on cohomology if $EG \times_G Y$ is simply-connected. Let φ denote the action of the circle $\mathbb{T}$ on the loop group LG, and let $\mathbb{T} \times_\varphi LG$ denote the semi-direct product of these two groups. It is clear that the group $\mathbb{T} \times_\varphi LG$ acts on LY, and the above results give us a model for the equivariant cohomology of LY with respect to this action. Indeed, from the identification

$$L(EG \times_G Y) = L(EG) \times_{LG} LY = E(LG) \times_{LG} LY,$$

we see that

$$H_{\mathbb{T} \times_\varphi LG}(LY) = H_{\mathbb{T}}(L(EG \times_G Y)).$$

So for any $\mathbb{C}[u]$-module W, we define the $\mathbb{T} \times_\varphi LG$-equivariant cohomology with coefficients in W to be $H_{\mathbb{T}}(L(EG \times_G Y); W)$. With this definition, the following result follows by combining 5.1 and the above observation.

Proposition 5.9. *The strict homomorphism of A_∞-algebras*

$$\mathsf{N}(\Omega_G(Y))[u] \to \Omega(L(EG \times_G Y))[u]$$

induces an isomorphism on cohomology if $EG \times_G Y$ is simply-connected. Thus, for any coefficients W, we have an isomorphism

$$H(\mathsf{N}(\Omega_G(Y)); W) \cong H_{\mathbb{T} \times_\varphi LG}(LY; W).$$

6. The equivariant Chern character

In this section, we will construct examples of cyclic chains that give interesting equivariant differential forms on loop space upon applying the (extended) iterated integral map. Since we will be defining classes related to the Chern character, we work over $\mathbb{C}$ instead of $\mathbb{R}$. We use the completed cyclic homology introduced in Section 2, which we recall is obtained from a dg-Λ-module (C, b, B) by forming the completed tensor product $C\hat{\otimes}\mathbb{C}[u, u^{-1}]\!]$, with differential $b + uB$; in the case of the dg-Λ-module $(\Omega(LX), d, P)$, the resulting complex is denoted by $\Omega(LX)[u, u^{-1}]\!]$. The other example that we need is the cyclic bar complex $\mathsf{C}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))$ of the last section which is a module over $\Omega_{\mathbb{T}} = \mathbb{R}[u]$; we will denote by $\tilde{\mathsf{C}}(\Omega(X))$ the complex

$$\mathsf{C}_{\Omega_{\mathbb{T}}}(\Omega_{\mathbb{T}}(X \times \mathbb{T}))\hat{\otimes}_{\Omega_{\mathbb{T}}}\mathbb{C}[u, u^{-1}]\!],$$

with boundary $b_{\mathbb{T}} + uB$, and similarly for its subcomplex of degenerate chains $\tilde{\mathsf{D}}(\Omega(X))$ and the Chen normalized complex $\tilde{\mathsf{N}}(\Omega(X)) = \tilde{\mathsf{C}}(\Omega(X))/\tilde{\mathsf{D}}(\Omega(X))$. Thus, the extended iterated integral map defines a map of complexes

$$\tilde{\sigma} : \tilde{\mathsf{N}}(\Omega(X)) \to \Omega(LX)[u, u^{-1}]\!].$$

Let $p \in C^\infty(X, \mathrm{End}(\mathbb{C}^N))$ be a projection; in other words, p is a smooth map from X to the space of projections in $\mathrm{End}(\mathbb{C}^N)$. In this section, we will construct an element $\mathrm{ch}(p) \in \tilde{\mathsf{C}}(\Omega(X))$ of total degree 0 such that

$$(b_{\mathbb{T}} + uB)\,\mathrm{ch}(p) \in \tilde{\mathsf{D}}(\Omega(X));$$

thus, $\mathrm{ch}(p)$ defines a closed element of $\tilde{\mathsf{N}}(\Omega(X))$ of total degree 0. We call this chain the equivariant Chern character because, as we will see, under the (extended) iterated integral map, it maps to the equivariant Chern character of the bundle $E = \operatorname{im} p$ constructed by Bismut [3].

A projection $p \in C^\infty(X, \mathrm{End}(\mathbb{C}^N))$ defines a splitting of the trivial complex vector bundle $\mathbb{C}^N$ into orthogonal bundles E (the image of p) and $E^\perp$ (the image of $p^\perp = 1-p$). There is a connection form on the trivial bundle $\mathbb{C}^N$ corresponding to the decomposition of $\mathbb{C}^N$ into $E \oplus E^\perp$, equal to

$$d + A = p \cdot d \cdot p + p^\perp \cdot d \cdot p^\perp;$$

a simple calculation shows that $A = p(dp) + p^\perp(dp^\perp)$. The curvature of this connection equals

$$\begin{aligned} F = (d + A)^2 &= p \cdot d \cdot p \cdot d \cdot p + p^\perp \cdot d \cdot p^\perp \cdot d \cdot p^\perp \\ &= p(dp)^2 + p^\perp(dp^\perp)^2 = (dp)^2. \end{aligned}$$

Let $\mathcal{A}$ be the $\mathrm{End}(\mathbb{C}^N)$-valued equivariant one-form $A - u^{-1}F\,dt$ on $X \times \mathbb{T}$. The following result shows that $d_u + \mathcal{A}$ is a flat equivariant connection on $X \times \mathbb{T}$, for which p is a covariant constant section.

Lemma 6.1. *The matrices of differential forms $\mathcal{A}$ and p satisfy the following formulas:*

$$\begin{aligned} d_u p + [\mathcal{A}, p] &= 0 \\ d_u\mathcal{A} + \mathcal{A}^2 &= 0 \end{aligned}$$

Proof. To prove the first formula, we calculate as follows:

$$d_u p + [\mathcal{A}, p] = (dp + [A, p]) - u^{-1}[F, p]\,dt = 0.$$

The second formula is proved by an equally simple calculation:

$$d_u\mathcal{A} + \mathcal{A}^2 = (dA + A^2 - F) - u^{-1}(dF + [A, F])\, dt.$$

By the Bianchi identities, these two terms are zero. □

We are now ready to state the formula for the character $\mathrm{ch}(p)$ of the idempotent p. We will perform the construction in a more general context. If Y is an $\mathbb{T}$-manifold and $E \to Y$ is a vector bundle over Y with compatible $\mathbb{T}$-action, **an equivariant connection** on E is an operator

$$\nabla_u : \Omega(Y, E)[u, u^{-1}\rrbracket \to \Omega(Y, E)[u, u^{-1}\rrbracket$$

of degree 1 such that for any $\omega \in \Omega(Y)[u, u^{-1}\rrbracket$ and $s \in \Omega(Y, E)[u, u^{-1}\rrbracket$, Leibniz's rule holds, that is,

$$\nabla_u(\omega)s = (d_u\omega)s + (-1)^{|\omega|}\omega(\nabla_u s).$$

Let $d_u + \mathcal{A}$ be a flat equivariant connection on the bundle $\mathbb{C}^N \to Y$, so that $\mathcal{A} \in \Omega^1(Y)[u, u^{-1}\rrbracket$ and $(d_u + \mathcal{A})^2 = 0$, and let p be a matrix of smooth functions $p \in C^\infty(X) \otimes \mathrm{End}(\mathbb{C}^N)$ such that $d_u p + [\mathcal{A}, p] = 0$. If ω_i are $N \times N$-matrices of differential forms, we denote by $\mathrm{Tr}(\omega_0, \dots, \omega_k)_k$ the element of the cyclic bar complex

$$\mathrm{Tr}(\omega_0, \dots, \omega_k)_k = \sum_{i_0 \dots i_k = 1}^{N} (\omega_0)_{i_0 i_1}(\omega_1)_{i_1 i_2} \dots (\omega_k)_{i_k i_0}.$$

Definition 6.2. *The character* $\mathrm{ch}(p, \mathcal{A})$ *is the following zero-degree element of the normalized cyclic bar complex* $\tilde{\mathsf{N}}(\Omega(X))$*:*

$$\sum_{k=0}^{\infty} \mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_k.$$

Proposition 6.3. *The cyclic chain* $\mathrm{ch}(p, \mathcal{A})$ *is closed.*

Proof. To show that $\sum_{k=0}^{\infty} \mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_k$ is closed, we start by showing that

$$b_0 \,\mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_{k-1} + b_1 \,\mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_k = 0.$$

This follows from the equivariant flatness, since

$$b_1 \,\mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_k = \mathrm{Tr}([\mathcal{A}, p], \mathcal{A}, \dots, \mathcal{A})_{k-1} - \sum_{i=1}^{k} \mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A}^2, \dots, \mathcal{A})_{k-1}$$

$$b_0 \,\mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_{k-1} = \mathrm{Tr}(d_u p, \mathcal{A}, \dots, \mathcal{A})_{k-1} - \sum_{i=1}^{k-1} \mathrm{Tr}(p, \mathcal{A}, \dots, d_u\mathcal{A}, \dots, \mathcal{A})_{k-1}.$$

Furthermore, it is clear that $B \,\mathrm{Tr}(p, \mathcal{A}, \dots, \mathcal{A})_k \in \tilde{\mathsf{D}}(\Omega(X))$, since p is a zero-form. Putting this all together gives the proof. □

We now specialize to the setting of 6.2 in which $Y = X \times \mathbb{T}$ and p is an idempotent on X. If E is a complex vector bundle on X with connection ∇, Narasimhan and Ramanan have proved [16] that there is an embedding of the bundle E in the trivial bundle $\mathbb{C}^N$ which induces an isomorphism of connections between ∇ and the Grassmannian connection, obtained by restricting d to the image of E. Thus, given a complex vector bundle E with connection ∇, we may associate to it a closed cyclic chain $\mathrm{ch}(p, \mathcal{A}) \in \tilde{\mathsf{N}}(\Omega(X))$. In the next theorem, we identify the image of $\mathrm{ch}(p, \mathcal{A})$ under the extended iterated integral map with the equivariant Chern character $\mathrm{Ch}(E, \nabla)$ of the bundle (E, ∇) constructed by Bismut [3].

Definition 6.4. *The equivariant Chern character of the complex vector bundle E with connection ∇ is the equivariant differential form*

$$\mathrm{Ch}(E,\nabla) = \mathrm{Tr}(\varphi(1)) \in \Omega^0(LX)[u,u^{-1}]\!],$$

where $\varphi(t) \in \Omega(LX) \otimes \mathrm{Hom}(E_{\gamma(0)}, E_{\gamma(t)})$ is the solution to the ordinary differential equation

$$\nabla_{\partial/\partial t}\varphi(t) = u^{-1}\varphi(t)\,F(t),$$

with initial condition $\varphi(0) = 1$.

Observe that if we restrict the differential form $\mathrm{Ch}(E,\nabla)$ to the fixed point set X of the circle action on LX, we obtain the differential form $\mathrm{Tr}(e^{F/u})$, or the ordinary Chern character of the bundle E. Thus, the equivariant Chern character is an equivariant extension of the Chern character away from the fixed point set.

Theorem 6.5. *The following formula relates the Chern character in cyclic homology of a bundle (E,∇) with its equivariant Chern character:*

$$\tilde{\sigma}\,\mathrm{ch}(p,\mathcal{A}) = \mathrm{Ch}(E,\nabla).$$

Proof. The proof is based on the perturbation expansion for a path ordered exponential, familiar from quantum theory. If E is a vector space, and $a(t) : [0,1] \longrightarrow \mathrm{End}(E)$, let $\varphi(t) \in \mathrm{End}(E)$ be the solution of the ordinary differential equation

$$\frac{d\varphi(t)}{dt} = \varphi(t)a(t),$$

with initial condition $\varphi(0) = 1$. Then $\varphi(1)$ is given by the formula

$$\varphi(1) = \sum_{k=0}^{\infty} \int_{\Delta_k} a(t_1)\dots a(t_k)\,dt_1\dots dt_k.$$

Let p be an idempotent which realises the bundle (E,∇). If we apply the perturbation expansion to the matrix $a(t) = \iota A(t) + u^{-1}F(t)$ on $\mathbb{C}^N$, which is chosen in such a way that $\mathrm{Ch}(E,\nabla) = \mathrm{Tr}(p\cdot\varphi(1))$, we see that

$$\mathrm{ch}(p) = \sum_{k=0}^{\infty} \mathrm{Tr}\Big(p\cdot\int_{\Delta_k} a(t_1)\dots a(t_k)\,dt_1\dots dt_k\Big).$$

Everything now follows from the fact that

$$\begin{aligned}\Big(p\cdot\int_{\Delta_k} a(t_1)\dots a(t_k)\,dt_1\dots dt_k\Big)_{ij} &= \sum_{i_1\dots i_k=1}^{N} \tilde{\sigma}(p_{ii_1}, \mathcal{A}_{i_1i_2}, \dots, \mathcal{A}_{i_kj})_k \\ &\in \Omega(LM)\otimes\mathrm{End}(\mathbb{C}^N),\end{aligned}$$

as is easily seen from the explicit formula for $\tilde{\sigma}$ at the end of Section 5. □

We will give one more example of the construction of 6.3, in which $Y = T^*X \times \mathbb{T}$ and the the rank N of the bundle on Y is 1. Let α be the canonical one-form on T^*X, which is written in local coordinates as $x_i\,d\xi^i$, and let $\omega = d\alpha$ be the standard symplectic form on T^*X. If we set $\mathcal{A} = \alpha - u^{-1}\omega\,dt$ and let p be the constant section 1, then we may form the closed chain

$$\mathrm{ch}(1, \alpha - u^{-1}\omega\,dt) = \sum_{k=0}^{\infty}(1, \alpha - u^{-1}\omega\,dt, \dots, \alpha - u^{-1}\omega\,dt)_k.$$

Since the extended iterated integral of the chain $(1, \alpha\,dt)$ on $L(T^*X) \cong T^*(LX)$ is the canonical one-form α_{LX} of $T^*(LX)$, it follows that the extended iterated integral of $\mathrm{ch}(1, \alpha - u^{-1}\omega\,dt)$ is just

$$\sum_{k=0}^{\infty} \frac{(\iota\alpha_{LX} + u^{-1}\omega_{LX})^k}{k!} = \exp\bigl(\iota\alpha_{LX} + u^{-1}\omega_{LX}\bigr).$$

The above construction is closely related to the equivariant differential form on LX, where X is a Riemannian manifold, that is constructed in [2]. In this setting, we may construct a section s of the fibration $T^*(LX) \to LX$ by sending γ_t to $(\gamma_t, \dot\gamma_t)$, where the tangent vector $\dot\gamma_t$ is identified with a cotangent vector by means of the metric on X.

Proposition 6.6. *The pull back of the differential form $\tilde\sigma\,\mathrm{ch}(1, \alpha - u^{-1}\omega\,dt)$ by $s : LX \to T^*(LX)$ equals $e^{-E+\omega_{LX}/u}$, where E is the energy function $E(\gamma) = \int_{\mathbb{T}} |\dot\gamma_t|^2$ on LX and ω_{LX} is the presymplectic form*

$$\omega_{LX}(X_t, Y_t) = \int_{\mathbb{T}} (X_t, \nabla_{\dot\gamma_t} Y_t).$$

Here, ∇ is the Levi-Civita connection on X.

References

1. J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. USA **42** (1956), 409–412.
2. M. F. Atiyah, *Circular symmetry and stationary phase approximation*, Colloque en l'honneur de Laurent Schwartz, Astérisque **131** (1985), 43–59.
3. J.-M. Bismut, *Index theorem and equivariant cohomology on the loop space*, Commun. Math. Phys. **98** (1985), 213–237.
4. H. Cartan, *La transgression dans un groupe de Lie et dans un espace fibré principe*, Colloque de toplogie (Espaces Fibrés), CBRM, Bruxelles, 1950, pp. 57–71.
5. K. T. Chen, *Iterated integrals of differential forms and loop space homology*, Ann. Math. **97** (1973), 217–246.
6. K. T. Chen, *Reduced bar constructions on de Rham complexes*, Algebra, topology and category theory, Edited by A. Heller and M. Tierney, Academic Press, New York, 1976.
7. K. T. Chen, *Extension of C^∞ function algebra by integrals and Malcev completion of π_1*, Adv. Math. **23** (1977), 181–210.
8. E. Getzler, *Cyclic homology and the path integral of the Dirac operator*, In preparation.
9. E. Getzler and J. D. S. Jones, *A_∞-algebras and the cyclic bar complex*, Illinois Jour. Math. (1989).
10. T. Goodwillie, *Cyclic homology, derivations and the free loopspace*, Topology **24** (1985), 187–215.
11. C. E. Hood and J. D. S. Jones, *Some algebraic properties of cyclic homology groups*, K-theory **1**, 361–384.
12. J. D. S. Jones, *Cyclic homology and equivariant homology*, Invent. Math. **87** (1987), 403–423.
13. J. D. S. Jones and S. Petrack, *Le théorème des points fixes en cohomologie equivariante en dimension infinie*, C. R. Acad. Sci. Paris **306** (1988), 75–78.
14. J. McLeary, *User's guide to spectral sequences*, Publish or Perish, Wilmington, 1985.

15. V. Mathai and D. G. Quillen, *Superconnections, equivariant differential forms and the Thom class*, Topology **25** (1986), 85–110.
16. M. S. Narasimhan and S. Ramanan, *Existence of universal connections*, Amer. J. Math **83** (1961), 563–572.
17. J. D. Stasheff, *Homotopy associativity of H-spaces, II*, Trans. Amer. Math. Soc. **108** (1963), 293–312.

Harvard University, Cambridge MA 02138, USA

University of Warwick, Coventry CV4 7AL, England

Université de Paris VII, 75230 Paris Cedex 05, France