Congruences and Cranks for Partitions Bounded by Part Size and Number

Dyson's Rank Works for Infinitely Many Partition Congruences!

Brandt Kronholm

– student of George Andrews
and friend of Bruce Berndt.

Associate Professor
School of Mathematical and Statistical Sciences
University of Texas Rio Grande Valley

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$P(n,4) \equiv 0 \pmod 3$ *if and only if*
$n = 36k + 0, 1, 2, 3, 7, 9, 10, 12, 13, 15, 17, 19, 21, 22, 24, 25, 27, 29, 31, 32, 33, 34, 35.$

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A *supercrank* is a statistic on partitions that witnesses each and every instance of divisibility modulo a given prime.

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Theorem 3 (Eichhorn, K.)

For every $m > 0$, whenever $m|P(n,2)$, both the first part and the second part act as a supercrank for $P(n,2)$ modulo every m.

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Let $6j-1$ be prime. Then $\lambda_1 - \lambda_3$ is a supercrank for $P(n,3)$ modulo $6j-1$.

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– Almost an omnicrank.., Proof: Polyhedral Geometry.

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Theorem 5 (Eichhorn, K.)

Table: *For this table, $P(n,d) \equiv 0 \pmod{m}$ if and only if $n = m\,\mathrm{lcm}(d)k + t_1, t_2, \ldots, t_s$, and the statistic $\delta(\lambda) = \sum_i \delta_i \lambda_i$ is a supercrank witnessing these congruences when taken modulo m.*

d	m	$n = m\,\mathrm{lcm}(d)k + t$	supercrank $\delta(\lambda)$
3	2	$12k + 0, 1, 2, 5, 7, 10, 11$	λ_2
3	3	$18k + 0, 1, 2, 6, 12, 16, 17$	λ_1
3	4	$24k + 0, 1, 2, 7, 10, 12, 14, 17, 22, 23$	$\lambda_1 - \lambda_3$
3	6	$36k + 0, 1, 2, 12, 17, 19, 24, 34, 35$	$\lambda_1 + 3\lambda_3$
3	9	$54k + 0, 1, 2, 18, 36, 52, 53$	$\lambda_1 + 4\lambda_3, \lambda_1 - 2\lambda_3$
4	2	$24k + 0, 1, 2, 3, 6, 9, 13, 16, 19, 20, 21, 22, 23$	λ_2
4	3	$36k + 0, 1, 2, 3, 7, 9, 10, 12, 13, 15, 17, 19, 21, 22, 24, 25, 27, 29, 31, 32, 33, 34, 35$	$\lambda_1 - \lambda_4$
4	4	$48k + 0, 1, 2, 3, 20, 21, 22, 24, 25, 26, 43, 44, 45, 46, 47$	$\lambda_1 + \lambda_3 - \lambda_4$
4	7	$84k + 0, 1, 2, 3, 22, 37, 39, 41, 43, 45, 60, 79, 80, 81, 82, 83$	$\lambda_1 - \lambda_3 - 2\lambda_4$

Proof: We compute exact polynomial formulas from the generating functions.

An Update: Conjecture

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Conjecture 6 (Eichhorn, K.)

Other than Theorem 3, Theorem 4, and Theorem 5, there are no other d and m for which there are supercranks for $P(n, d)$ modulo m.

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Another thought:

Remark 7

When are supercranks possible?

The Interval Theorem

Theorem 8 (The Interval Theorem, K., (2007))

For any prime ℓ, any non-negative integer k, and any $2 \leq m \leq f(M)$, we have

$$p\left(\ell \operatorname{lcm}(m)k - v, m\right) \equiv 0 \pmod{\ell}$$

for $0 < v < \frac{m(m+1}{2}$, where $\operatorname{lcm}(m)$ is the least common multiple of the integers from 1 to m.

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Example of $p(n, 15) \pmod{13}$: 119 consecutive congruences

$$\begin{aligned}
p(4684680k - 1, 15) &\equiv 0 \pmod{13} \\
p(4684680k - 2, 15) &\equiv 0 \pmod{13} \\
&\vdots \\
p(4684680k - 117, 15) &\equiv 0 \pmod{13} \\
p(4684680k - 118, 15) &\equiv 0 \pmod{13} \\
p(4684680k - 119, 15) &\equiv 0 \pmod{13}
\end{aligned}$$

Example of $p(n, 14) \pmod{13}$: 104 consecutive arithmetic progressions

$$p(4684680k-1,14) \equiv 0 \pmod{13}$$
$$p(4684680k-2,14) \equiv 0 \pmod{13}$$
$$p(4684680k-3,14) \equiv 0 \pmod{13}$$
$$p(4684680k-4,14) \equiv 0 \pmod{13}$$
$$\vdots$$
$$p(4684680k-110,14) \equiv 0 \pmod{13}$$
$$p(4684680k-102,14) \equiv 0 \pmod{13}$$
$$p(4684680k-103,14) \equiv 0 \pmod{13}$$
$$p(4684680k-104,14) \equiv 0 \pmod{13}$$

The Interval Theorem

Example of $p(n, 13) \pmod{13}$: 90 consecutive arithmetic progressions

$$
\begin{aligned}
p(4684680k - 1, 13) &\equiv 0 \pmod{13}\\
p(4684680k - 2, 13) &\equiv 0 \pmod{13}\\
p(4684680k - 3, 13) &\equiv 0 \pmod{13}\\
p(4684680k - 4, 13) &\equiv 0 \pmod{13}\\
&\vdots\\
p(4684680k - 87, 13) &\equiv 0 \pmod{13}\\
p(4684680k - 88, 13) &\equiv 0 \pmod{13}\\
p(4684680k - 89, 13) &\equiv 0 \pmod{13}\\
p(4684680k - 90, 13) &\equiv 0 \pmod{13}
\end{aligned}
$$

The Interval Theorem

Example of $p(n, 12) \pmod{13}$: 77 consecutive arithmetic progressions

$$\begin{aligned}
p(360360k - 1, 12) &\equiv 0 \pmod{13} \\
p(360360k - 2, 12) &\equiv 0 \pmod{13} \\
p(360360k - 3, 12) &\equiv 0 \pmod{13} \\
p(360360k - 4, 12) &\equiv 0 \pmod{13} \\
&\vdots \\
p(360360k - 74, 12) &\equiv 0 \pmod{13} \\
p(360360k - 75, 12) &\equiv 0 \pmod{13} \\
p(360360k - 76, 12) &\equiv 0 \pmod{13} \\
p(360360k - 77, 12) &\equiv 0 \pmod{13}
\end{aligned}$$

The Interval Theorem

Example of $p(n, 11) \pmod{13}$: 65 consecutive arithmetic progressions

$$
\begin{aligned}
p(360360k - 1, 11) &\equiv 0 \pmod{13}\\
p(360360k - 2, 11) &\equiv 0 \pmod{13}\\
p(360360k - 3, 11) &\equiv 0 \pmod{13}\\
p(360360k - 4, 11) &\equiv 0 \pmod{13}\\
&\vdots\\
p(360360k - 62, 11) &\equiv 0 \pmod{13}\\
p(360360k - 63, 11) &\equiv 0 \pmod{13}\\
p(360360k - 64, 11) &\equiv 0 \pmod{13}\\
p(360360k - 65, 11) &\equiv 0 \pmod{13}
\end{aligned}
$$

The Interval Theorem

Example of $p(n, 10) \pmod{13}$: 54 consecutive arithmetic progressions

$$p(32760k - 1, 10) \equiv 0 \pmod{13}$$
$$p(32760k - 2, 10) \equiv 0 \pmod{13}$$
$$p(32760k - 3, 10) \equiv 0 \pmod{13}$$
$$p(32760k - 4, 10) \equiv 0 \pmod{13}$$
$$\vdots$$
$$p(32760k - 51, 10) \equiv 0 \pmod{13}$$
$$p(32760k - 52, 10) \equiv 0 \pmod{13}$$
$$p(32760k - 53, 10) \equiv 0 \pmod{13}$$
$$p(32760k - 54, 10) \equiv 0 \pmod{13}$$

Example of $p(n, 9) \pmod{13}$: 44 consecutive arithmetic progressions

$$p(32760k - 1, 9) \equiv 0 \pmod{13}$$
$$p(32760k - 2, 9) \equiv 0 \pmod{13}$$
$$p(32760k - 3, 9) \equiv 0 \pmod{13}$$
$$p(32760k - 4, 9) \equiv 0 \pmod{13}$$
$$\vdots$$
$$p(32760k - 41, 9) \equiv 0 \pmod{13}$$
$$p(32760k - 42, 9) \equiv 0 \pmod{13}$$
$$p(32760k - 43, 9) \equiv 0 \pmod{13}$$
$$p(32760k - 44, 9) \equiv 0 \pmod{13}$$

The Interval Theorem

Example of $p(n,8) \pmod{13}$: 35 consecutive arithmetic progressions

$$
\begin{aligned}
p(10920k-1,8) &\equiv 0 \pmod{13}\\
p(10920k-2,8) &\equiv 0 \pmod{13}\\
p(10920k-3,8) &\equiv 0 \pmod{13}\\
p(10920k-4,8) &\equiv 0 \pmod{13}\\
&\vdots\\
p(10920k-32,8) &\equiv 0 \pmod{13}\\
p(10920k-33,8) &\equiv 0 \pmod{13}\\
p(10920k-34,8) &\equiv 0 \pmod{13}\\
p(10920k-35,8) &\equiv 0 \pmod{13}
\end{aligned}
$$

The Interval Theorem

Example of $p(n, 7) \pmod{13}$: 27 consecutive arithmetic progressions

$$
\begin{aligned}
p(5460k-1,7) &\equiv 0 \pmod{13}\\
p(5460k-2,7) &\equiv 0 \pmod{13}\\
p(5460k-3,7) &\equiv 0 \pmod{13}\\
p(5460k-4,7) &\equiv 0 \pmod{13}\\
&\vdots\\
p(5460k-24,7) &\equiv 0 \pmod{13}\\
p(5460k-25,7) &\equiv 0 \pmod{13}\\
p(5460k-26,7) &\equiv 0 \pmod{13}\\
p(5460k-27,7) &\equiv 0 \pmod{13}
\end{aligned}
$$

Example of $p(n,6) \pmod{13}$: 20 consecutive arithmetic progressions

$$
\begin{aligned}
p(780k-1,6) &\equiv 0 \pmod{13}\\
p(780k-2,6) &\equiv 0 \pmod{13}\\
p(780k-3,6) &\equiv 0 \pmod{13}\\
p(780k-4,6) &\equiv 0 \pmod{13}\\
&\vdots\\
p(780k-17,6) &\equiv 0 \pmod{13}\\
p(780k-18,6) &\equiv 0 \pmod{13}\\
p(780k-19,6) &\equiv 0 \pmod{13}\\
p(780k-20,6) &\equiv 0 \pmod{13}
\end{aligned}
$$

Example of $p(n,5) \pmod{13}$: 14 consecutive arithmetic progressions

$$\begin{aligned}
p(780k-1,5) &\equiv 0 \pmod{13}\\
p(780k-2,5) &\equiv 0 \pmod{13}\\
p(780k-3,5) &\equiv 0 \pmod{13}\\
p(780k-4,5) &\equiv 0 \pmod{13}\\
&\vdots\\
p(780k-11,5) &\equiv 0 \pmod{13}\\
p(780k-12,5) &\equiv 0 \pmod{13}\\
p(780k-13,5) &\equiv 0 \pmod{13}\\
p(780k-14,5) &\equiv 0 \pmod{13}
\end{aligned}$$

The Interval Theorem

Example of $p(n,4) \pmod{13}$: 9 consecutive arithmetic progressions

$$\begin{aligned}
p(156k-1,4) &\equiv 0 \pmod{13}\\
p(156k-2,4) &\equiv 0 \pmod{13}\\
p(156k-3,4) &\equiv 0 \pmod{13}\\
p(156k-4,4) &\equiv 0 \pmod{13}\\
p(156k-5,4) &\equiv 0 \pmod{13}\\
p(156k-6,4) &\equiv 0 \pmod{13}\\
p(156k-7,4) &\equiv 0 \pmod{13}\\
p(156k-8,4) &\equiv 0 \pmod{13}\\
p(156k-9,4) &\equiv 0 \pmod{13}
\end{aligned}$$

The Interval Theorem

Example of $p(n,3) \pmod{13}$: 5 consecutive arithmetic progressions

$$\begin{aligned}
p(78k-1,3) &\equiv 0 \pmod{13} \\
p(78k-2,3) &\equiv 0 \pmod{13} \\
p(78k-3,3) &\equiv 0 \pmod{13} \\
p(78k-4,3) &\equiv 0 \pmod{13} \\
p(78k-5,3) &\equiv 0 \pmod{13}
\end{aligned}$$

The Interval Theorem

Example of $p(n,3) \pmod{13}$: 5 consecutive arithmetic progressions

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$$p(78k-4,3) \equiv 0 \pmod{13}$$
$$p(78k-5,3) \equiv 0 \pmod{13}$$

Example of $p(n,2) \pmod{13}$: 2 consecutive arithmetic progressions

$$p(26k-1,2) \equiv 0 \pmod{13}$$
$$p(26k-2,2) \equiv 0 \pmod{13}$$

Theorem 8 (**The Interval Theorem**, K., (2007))

For any prime ℓ*, any non-negative integer* k*, and any* $2 \leq m \leq f(M)$*, we have*

$$p(\ell \operatorname{lcm}(m)k - v, m) \equiv 0 \pmod{\ell}$$

for $0 < v < \frac{m(m+1)}{2}$*, where* $\operatorname{lcm}(m)$ *is the least common multiple of the integers from 1 to m.*

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Update: For a given number of parts m such that $2 \leq m \leq \ell + 1$ and all primes ℓ, we always have *at least* two cranks witnessing the congruences in the Interval Theorem.

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Theorem 9 (Eichhorn, K., Larsen,(2022))

For a given number of parts m such that $2 \leq m \leq \ell + 1$, and all primes ℓ,

1. *The statistic "the number of parts less than $\ell + 1$" is a crank witnessing each and every congruence described by the Interval Theorem.*
2. *The statistic "the number of parts larger than 1" is a crank witnessing each and every congruence described by the Interval Theorem.*

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- We can show that as the prime ℓ grows, so does the number of cranks.

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Methods of Proof for Theorem 9

Definition 10 (MB stats: Partition statistics based on the multiplicities of certain parts in a partition.)

Let λ be a partition of n into parts from the set $[m]$. We write λ in "multiplicity notation," so that $\lambda = (1^{e_1}, 2^{e_2}, \ldots, m^{e_m})$ is the partition with exactly e_i parts of size i for each $i \in [m]$. We define a *multiplicity-based statistic* or *MB statistic* $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{Z}^m$ to be a function $\tau : \mathcal{P}(n, m) \to \mathbb{Z}$ such that

$$\tau(\lambda) = \sum_{i=1}^{m} \tau_i e_i.$$

The function $\tau(\lambda)$ is simply a linear combination of the multiplicities of the parts of λ.

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Lemma 11 (A difference of generating functions reduces to a polynomial.)

Given a prime ℓ, set $\zeta = \exp(2\pi i/\ell)$. For any MB statistic, if

$$\frac{1-q^D}{\prod_{j=1}^{m}\left(1-\zeta^{\tau_j} q^j\right)}$$

reduces to a polynomial in q of degree $d < D$, then for $0 < v < D - d$ and $k \geq 1$,

1. *$p(Dk - v, m) \equiv 0 \pmod{\ell}$, and*
2. *$\tau(\lambda)$ is a crank witnessing the congruences of The Interval Theorem for $2 \leq m \leq \ell + 1$.*

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reduces to a polynomial in q of degree $d < D$, then for $0 < v < D - d$ and $k \geq 1$,

1. $p(Dk - v, m) \equiv 0 \pmod{\ell}$, *and*
2. $\tau(\lambda)$ *is a crank witnessing the congruences of The Interval Theorem for* $2 \leq m \leq \ell + 1$.

Note: The difference of two power series reduces to a polynomial *without* modular arithmetic.

SASTRA Ramanujan Prize and Conference- December 2022

I chose to present the MB-crank material at the 2022 SASTRA Ramanujan Prize & International Conference on Number Theory.

Receiving a gift from Alladi after my presentation.

SASTRA Ramanujan Prize and Conference- December 2022

Receiving a gift from Alladi after my presentation.
Note: Ruixiang Zhang was the 2022 Ramanujan Prize winner.

Receiving a gift from Alladi after my presentation.
Note: Ruixiang Zhang was the 2022 Ramanujan Prize winner. (not me)

Group photo.

SASTRA Ramanujan Prize and Conference- December 2022

L-R: K., Will Sawin, Michael Schlosser, Krishna Alladi

Visiting Ramanujan's High School.

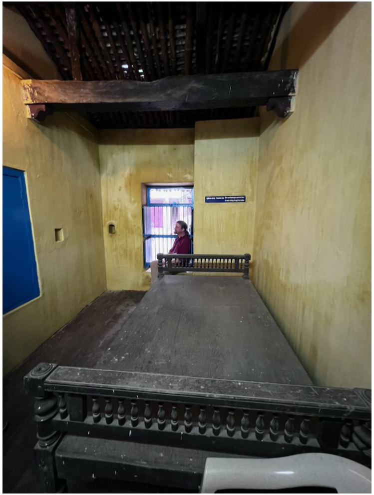
A photo op with some of the students at SASTRA!

SASTRA Ramanujan Prize and Conference- December 2022

Will Sawin (2021), Shai Evra (2020), and Krishna Alladi - post ceremony press conference.

Buying ’70s-’80s Bollywood Soundtracks on vinyl

Buying ’70s-’80s Bollywood Soundtracks on vinyl - while in India!



Sitting in Ramanujan's window.

The Interval Theorem: Open Question - Missing Cranks!

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Recall:

Theorem 9 (Eichhorn, K., Larsen,(2022))

For a given number of parts m such that $2 \leq m \leq \ell + 1$, and all primes ℓ,

1. *The statistic "the number of parts less than $\ell + 1$" is a crank witnessing each and every congruence described by the Interval Theorem.*
2. *The statistic "the number of parts larger than 1" is a crank witnessing each and every congruence described by the Interval Theorem.*

Example of $p(n, 15) \pmod{13}$: 119 consecutive congruences

$$p(4684680k - 1, 15) \equiv 0 \pmod{13}$$

$$\vdots$$

$$p(4684680k - 119, 15) \equiv 0 \pmod{13}$$

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Theorem 9 says there are cranks for $p(n, m) \pmod{13}$ for $m = 2, 3 \ldots, 14$ - but not for $m = 15$ where we have another infinite family of congruences. **Question:** Is there a crank that witness *all* the congruences given by the Interval Theorem?

The Interval Theorem: Another Update.

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- And...if we insist on each partition having a largest part of size m, then our statistic becomes "largest part minus the number of parts".

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The Interval Theorem: Another Update.

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- AKA, Dyson's Rank witnesses the infinite family of congruences in the Interval Theorem!
- Since $p(n, m) = P(n + m, m)$, we have the following:

Proposition 12 (Eichhorn, K., Larsen, (2022))

For any prime ℓ, any nonnegative integer k, and any $2 \leq m \leq \ell$, we have

$$P\left(\ell \operatorname{lcm}(m)k + m - v, m\right) \equiv 0 \pmod{\ell}$$

for $0 < v < \frac{m(m+1)}{2}$. Moreover, Dyson's rank modulo ℓ witnesses these congruences.

A Refinement of the Interval Theorem: $p(n, m, N)$

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Theorem 13 (Eichhorn, Engle, K., (2022))

Let ℓ be an odd prime and suppose $2 \leq m \leq \ell + 1$, and $1 \leq s \leq m$. Then for $k, j \geq 1$,

$$p(\ell \operatorname{lcm}(m)k - v, m, \ell \operatorname{lcm}(m-1)j - s) \equiv 0 \pmod{\ell}$$

for $\frac{s(s-1)}{2} < v < ms - \frac{s(s-1)}{2}$.

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Example 14 (The Gaussian polynomial $\left[{238+5 \atop 5}\right] = \sum_{n=0}^{1190} p(n, 5, 238) q^n$)

Set $\ell = 5$, $m = 5$, and $j = 4$. For $s = 2$, we have congruences in seven $(1 < v < 9)$ consecutive arithmetic progressions.

$$\begin{aligned}
p(292, 5, 238) &\equiv p(592, 5, 238) \equiv p(892, 5, 238) \equiv 0 \pmod{5} \\
p(293, 5, 238) &\equiv p(593, 5, 238) \equiv p(893, 5, 238) \equiv 0 \pmod{5} \\
p(294, 5, 238) &\equiv p(594, 5, 238) \equiv p(894, 5, 238) \equiv 0 \pmod{5} \\
&\vdots \\
p(298, 5, 238) &\equiv p(598, 5, 238) \equiv p(898, 5, 238) \equiv 0 \pmod{5}.
\end{aligned}$$

A Refinement of the Refinement of the Interval Theorem: $p(n, m, (a, b])$

We add a second level of refinement by treating $p(n, m, (a, b])$, the number of partitions of n with at most m parts, largest part greater than a but at most b. Notice that

$$p(n, m, (a, b]) = p(n, m, b) - p(n, m, a). \tag{1}$$

Corollary 15 (Eichhorn, Engle, K., 2022)

Let ℓ be an odd prime and suppose $2 \leq m \leq \ell + 1$, $1 \leq s \leq m$, and $j \geq 1$. Then for all $k \geq 1$,

$$p\left(\ell \operatorname{lcm}(m)k - v, m, \left(\ell \operatorname{lcm}(m-1)(j-1) - s, \ell \operatorname{lcm}(m-1)j - s\right]\right) \equiv 0 \pmod{\ell}$$

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Big Idea: Given a congruence $p(n, m) \equiv 0 \pmod{\ell}$, we can "chop up" the set of partitions counted by $p(n, m)$ into much,

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We add a second level of refinement by treating $p(n, m, (a, b])$, the number of partitions of n with at most m parts, largest part greater than a but at most b. Notice that

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Big Idea: Given a congruence $p(n, m) \equiv 0 \pmod{\ell}$, we can "chop up" the set of partitions counted by $p(n, m)$ into much, much smaller sets, in several different ways, each smaller set itself contains a number of partitions that is also congruent to zero modulo ℓ.

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We add a second level of refinement by treating $p(n, m, (a, b])$, the number of partitions of n with at most m parts, largest part greater than a but at most b. Notice that

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Example 16

Set $\ell = 5$, $m = 5$, $k = 1$, and $t = 6$.
By the Interval Theorem, $p(294, 5) \equiv 0 \pmod 5$.
Corollary 15 reveals many different sets $(a, b]$ for which $p(294, 5, (a, b]) \equiv 0 \pmod 5$.
Varying the parameter $s = 2, 3, 4$ (changing $(a, b]$, the minimum and maximum bounds on N) gives us three distinct sums equal to $p(294, 5)$ where each summand $p(294, 5, (a, b]) = p(294, 5, 60I_j - s)$ is also a multiple of 5.

- $s = 2$

$$\begin{aligned} p(294,5) &= \sum_{j\geq 1} p(294, 5, 60I_j - 2) \\ &= p(294,5,(-2,58]\,) + p(294,5,(58,118]\,) + p(294,5,(118,178]\,) \\ &+ p(294,5,(178,238]\,) + p(294,5,(238,298]\,) \\ &= 0 + 1{,}069{,}755 + 1{,}432{,}910 + 342{,}485 + 23{,}160 \\ &= 2{,}868{,}310 \equiv 0 \pmod 5. \end{aligned}$$

Example 17

- $s = 3$

$$p(294, 5) = \sum_{j \geq 1} p(294, 5, 60I_j - 3)$$
$$= p(294, 5, (-3, 57]\,) + p(294, 5, (57, 117]\,) + p(294, 5, (117, 177]\,)$$
$$+ p(294, 5, (177, 237]\,) + p(294, 5, (237, 297]\,)$$
$$= 0 + 1{,}034{,}725 + 1{,}455{,}640 + 353{,}210 + 24{,}735$$
$$= 2{,}868{,}310 \equiv 0 \pmod 5.$$

- $s = 4$

$$p(294, 5) = \sum_{j \geq 1} p(294, 5, 60I_j - 4)$$
$$= p(294, 5, (-4, 56]\,) + p(294, 5, (56, 116]\,) + p(294, 5, (116, 176]\,)$$
$$+ p(294, 5, (176, 236]\,) + p(294, 5, (236, 296]\,)$$
$$= 0 + 999{,}650 + 1{,}478{,}115 + 364{,}160 + 26{,}385$$
$$= 2{,}868{,}310 \equiv 0 \pmod 5.$$

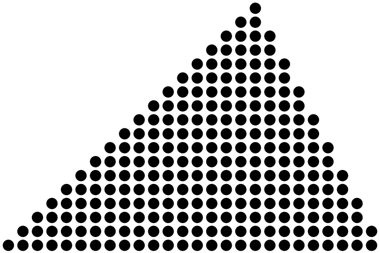
$$p(51, 3) \equiv 0 \pmod 3$$

A Refinement of the Refinement of the Interval Theorem: $p(n, m, (a, b])$

$$p(51, 3) \equiv 0 \pmod 3$$

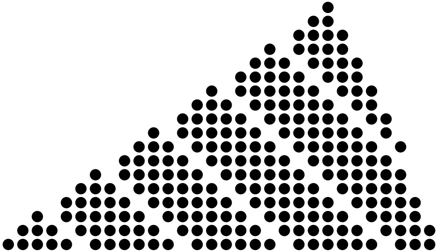
Some integer lattice point combinatorics.

$|\mathcal{P}(51,3)| = p(51,3) = 243$



$\mathcal{P}(51,3)$

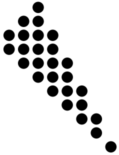
This is the set of partitions of 51 into at most three parts – AKA $\mathcal{P}(51,3)$.



$\mathcal{P}(51,3)$

We break it apart.

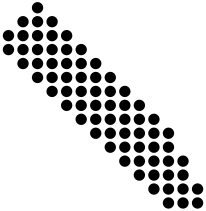
$|\mathcal{P}(51, 3, (17, 22])| = 27$



$\mathcal{P}(51, 3, (17, 22])$

We break it apart into the following subsets:

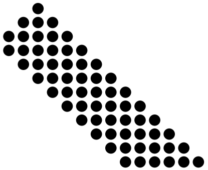
$|\mathcal{P}(51, 3, (23, 28])| = 72$



$\mathcal{P}(51, 3, (23, 28])$

We break it apart into the following subsets:

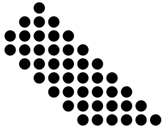
$|\mathcal{P}(51, 3, (29, 34])| = 63$



$\mathcal{P}(51, 3, (29, 34])$

We break it apart into the following subsets:

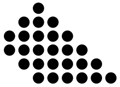
$|\mathcal{P}(51,3,(35,40])| = 45$



$\mathcal{P}(51,3,(35,40])$

We break it apart into the following subsets:

$|\mathcal{P}(51,3,(41,46])| = 27$



$\mathcal{P}(51,3,(41,46])$

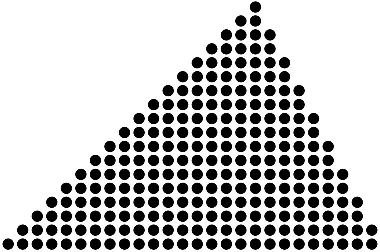
We break it apart into the following subsets:

$|\mathcal{P}(51, 3, (47, 52])| = 9$

$\mathcal{P}(51, 3, (47, 52])$

We break it apart into the following subsets:

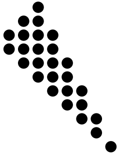
$|\mathcal{P}(51,3)| = p(51,3) = 243$



$\mathcal{P}(51,3) = \cdots$

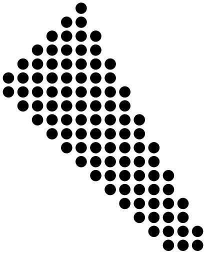
The union of which is $\mathcal{P}(51,3)$.

$p(51, 3, 22) = 27$



$$\mathcal{P}(51, 3, 22) = \mathcal{P}(51, 3, (17, 22])$$

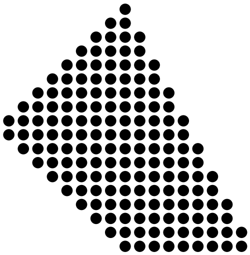
$$= p(51, 3, 22) = 27.$$



$$\mathcal{P}(51, 3, 28) = \mathcal{P}(51, 3, (17, 22]) \cup \mathcal{P}(51, 3, (23, 28])$$

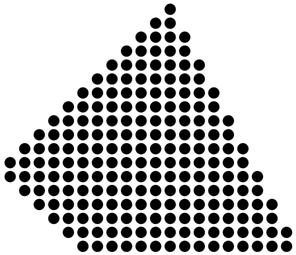
$p(51, 3, 28) = 99.$

$p(51, 3, 34) = 162$



$$\mathcal{P}(51, 3, 34) = \mathcal{P}(51, 3, (17, 22]) \cup \mathcal{P}(51, 3, (23, 28]) \cup \mathcal{P}(51, 3, (29, 34])$$

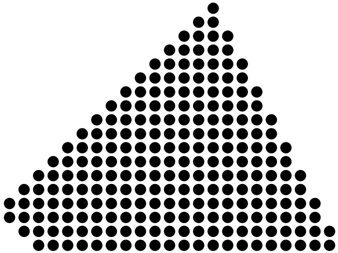
$p(51, 3, 34) = 162.$



$$\mathcal{P}(51,3,40) = \mathcal{P}(51,3,(17,22]) \cup \mathcal{P}(51,3,(23,28]) \cup \mathcal{P}(51,3,(29,34]) \cup \mathcal{P}(51,3,(35,40])$$

$p(51,3,40) = 207.$

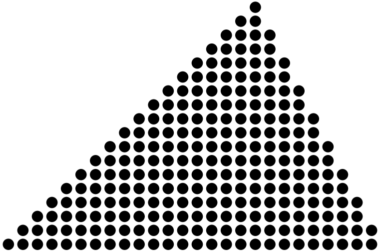
$p(51, 3, 46) = 234$



$$\mathcal{P}(51, 3, 46) = \mathcal{P}(51, 3, (17, 22]) \cup \mathcal{P}(51, 3, (23, 28]) \cup \mathcal{P}(51, 3, (29, 34]) \\ \cup \mathcal{P}(51, 3, (35, 40]) \cup \mathcal{P}(51, 3, (41, 46])$$

$p(51, 3, 46) = 234.$

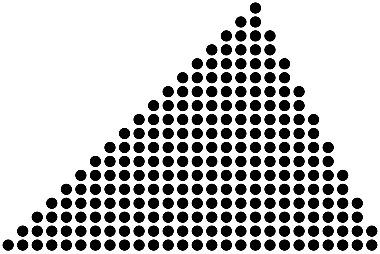
$p(51,3,52) = 243 = p(51,3)$



$$\mathcal{P}(51,3,52) = \mathcal{P}(51,3,(17,22]) \cup \mathcal{P}(51,3,(23,28]) \cup \mathcal{P}(51,3,(29,34])$$
$$\cup\, \mathcal{P}(51,3,(35,40]) \cup \mathcal{P}(51,3,(41,46]) \cup \mathcal{P}(51,3,(47,52])$$

$p(51,3,52) = 243.$

$p(51,3,52) = 243 = p(51,3)$



$$\mathcal{P}(51,3) = \mathcal{P}(51,3,(17,22]) \cup \mathcal{P}(51,3,(23,28]) \cup \mathcal{P}(51,3,(29,34])$$
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$p(51,3,52) = p(51,3) = 243.$

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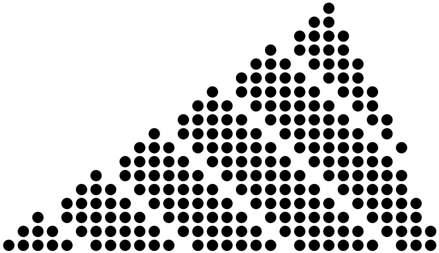

$$\mathcal{P}(51, 3) = \mathcal{P}(51, 3, (17, 22]) \cup \mathcal{P}(51, 3, (23, 28]) \cup \mathcal{P}(51, 3, (29, 34])$$
$$\cup\, \mathcal{P}(51, 3, (35, 40]) \cup \mathcal{P}(51, 3, (41, 46]) \cup \mathcal{P}(51, 3, (47, 52])$$

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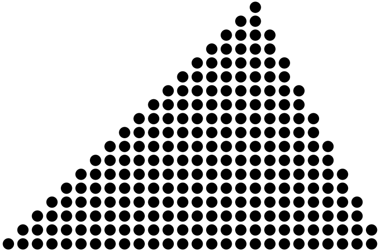
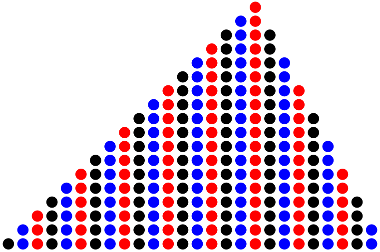


Color Coded by Crank Value



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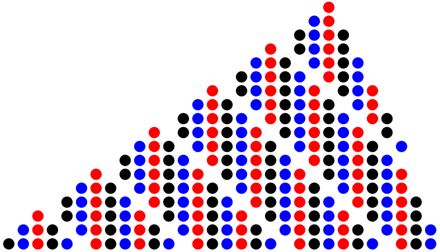

Polyhedral Geometry

Polyhedral Geometry

Polyhedral Geometry - AKA Integer Lattices

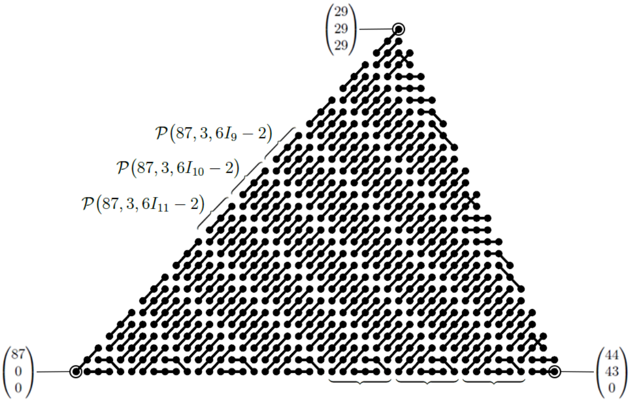

Figure: $\mathcal{P}(87,3)=\bigcup_{j=6}^{15}\mathcal{P}(87,3,6I_j-2)$, with each set of the form $\mathcal{P}(87,3,6I_j-2)$ covered by translations of triplets. We indicate a few individual partitions and sets $\mathcal{P}(87,3,6I_j-2)$. The set $\mathcal{P}(87,3,6I_8-2)=\mathcal{P}(87,3,(40,46])$ is in the first regime and consists of 42 triplets, while the sets $\mathcal{P}(87,3,6I_9-2)=\mathcal{P}(87,3,(46,52])$ and $\mathcal{P}(87,3,6I_{10}-2)=\mathcal{P}(87,3,(52,58])$ are in the second regime and consist of 39 and 33 triplets, respectively.

Cranks for $p(n, m, M)$ and $p(n, m, (a, b])$:

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Theorem 13 (Eichhorn, Engle, K., (2022))

Let ℓ be an odd prime and suppose $2 \leq m \leq \ell + 1$, and $1 \leq s \leq m$. Then for $k, j \geq 1$,

$$p(\ell \text{lcm}(m)k - v, m, \ell \text{lcm}(m-1)j - s) \equiv 0 \pmod{\ell}$$

for $\frac{s(s-1)}{2} < v < ms - \frac{s(s-1)}{2}$.

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Corollary 15 (Eichhorn, Engle, K., 2022)

Let ℓ be an odd prime and suppose $2 \le m \le \ell + 1$, $1 \le s \le m$, and $j \ge 1$. Then for all $k \ge 1$,

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Theorem 18 (Eichhorn, Engle, K., (2022))

For $\ell = 3$, the second part of the partition is a crank witnessing the congruences of Theorem 13 and Corollary 15 when $m \in \{2, 3\}$. For $\ell = 3$ and $m = 4$, if $n \leq 2N$, the second part of the partition is a crank witnessing the congruences of Theorem 13 and Corollary 15, whereas if $n > 2N$, the third part of the partition is a crank witnessing those congruences.

Methods of Proof:

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$$\begin{aligned} p(18k-t,3,6j-s) &= (6-t)\binom{3k+1}{2} + t\binom{3k}{2} - 3(s-t+4)\binom{3k-j+1}{2} \\ &- 3(6-(s-t+4))\binom{3k-j}{2} + 3(2s-t+2)\binom{3k-2j+1}{2} \\ &+ 3(6-(2s-t+2))\binom{3k-2j}{2} - (t-3s)\binom{3k-3j+1}{2} \\ &- (6-(t-3s))\binom{3k-3j}{2}. \end{aligned}$$

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- Enter Joselyne Aniceto, UTRGV, PhD 2025 (expected)

Andrews, Kronholm, Aniceto:

Andrews, Kronholm, Aniceto: AKA

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Andrews, Kronholm, Aniceto: AKA the “AKA Legacy”.

Andrews, Kronholm, Aniceto: AKA the “AKA Legacy”.

Figure: L-R: George Andrews, Brandt Kronholm, Joselyne Aniceto

Andrews, Kronholm, Aniceto: AKA the "AKA Legacy".

Figure: L-R: George Andrews, Brandt Kronholm, Joselyne Aniceto

Joelyne Aniceto - UTRGV 2025

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May, 2022: CBMS Conference at UTRGV

Figure: CBMS Conference: Ramanujan's Ranks, Mock Theta Functions, and Beyond.

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A more comprehensive Theorem on Congruences for $p(n, m, N)$.

Theorem 19 (Aniceto, K.)

Let ℓ be an odd prime and suppose $2 \leq m \leq \ell$, $s, t \in \mathbb{Z}$, and $j, k \in \mathbb{Z}_+$. Set $n = \ell \text{lcm}(m)k - t$ and $N = \ell \text{lcm}(m-1)j - s$. The congruence

$$p(n, m, N) \equiv 0 \pmod{\ell} \tag{2}$$

holds for

- (i) *for all $t \in \left(0, \frac{m^2+m}{2}\right)$ such that $n = \ell \text{lcm}(m)k - t \leq N$ and*
- (ii) *for a given $h \in \{1, \ldots, m-1\}$ for all $t \in \bigcap_{i=0}^{h} \left(is - \frac{i(i+1)}{2}, \frac{(m-i)(m-i+1)}{2} + is\right)$*

$$\left\lfloor \frac{hN + \frac{h(h+1)}{2}}{\ell \text{lcm}(m)} \right\rfloor < k \leq \left\lfloor \frac{(h+1)N + \frac{(h+1)(h+2)}{2}}{\ell \text{lcm}(m)} \right\rfloor. \tag{3}$$

Moreover, when $\gamma = \ell \text{lcm}(m) - \frac{m^2+m}{2}$ and m are even, we extend the interval in (i) to include $t \in \left(0, \frac{m^2+m}{2}\right) \cup (\gamma/2 - 1, \gamma/2 + 1)$ and for a given h, when $\delta = \ell \text{lcm}(max(m-h, h)) - \frac{(m-h)(m-h+1)+h(h+1)}{2}$ and m are both even integers, we extend the values t in the intervals being intersected in (ii) to

$$t \in \bigcap_{i=0}^{h} \left(\left(is - \frac{i(i+1)}{2}, \frac{(m-i)(m-i+1)}{2} + is \right) \cup \left(\delta/2 - 1 + is - \frac{i(i+1)}{2}, \delta/2 + 1 + is - \frac{i(i+1)}{2} \right) \right) \tag{4}$$

Additionally, since Gaussian Polynomials are reciprocal polynomials, the congruence

$$p(mN - n, m, N) \equiv 0 \pmod{\ell} \text{ also holds.} \tag{5}$$

Joselyne Aniceto: Revisiting Hansraj Gupta's "A Technique in Partitions"

Theorem 20 (Aniceto, K.)

Let $\{a, b, c\}$ be a set of three relatively prime numbers, with one of them being an even integer. Setting $n = abck + \frac{2abc-a-b-c}{2}$, then for $k \geq 0$ we have

$$p(n, \{a, b, c\}) \equiv 0 \pmod{abc/2}. \tag{6}$$

Theorem 20 (Aniceto, K.)

Let $\{a, b, c\}$ be a set of three relatively prime numbers, with one of them being an even integer. Setting $n = abck + \frac{2abc-a-b-c}{2}$, then for $k \geq 0$ we have

$$p(n, \{a, b, c\}) \equiv 0 \, (\text{mod } abc/2) \, . \tag{6}$$

We prove that a pair of cranks witness the congruences for an infinite special case of Theorem 20.

Theorem 21 (Aniceto, K.)

Let $\{1, 2, \ell\}$ be a set of part sizes containing the odd number $\ell \geq 3$. For $n = 2\ell k + \frac{3\ell-3}{2}$ and $k \geq 0$, we have

$$p(n, \{1, 2, \ell\}) \equiv 0 \, (\text{ mod } \ell \,) \, . \tag{7}$$

Whenever $\ell \equiv 1 \pmod 4$, the statistic "four times the number of 2s plus the number of ℓs", is a crank witnessing the congruence in (7). Whenever $\ell \equiv 3 \pmod 4$, the statistic "twice the number of 1s plus the number of ℓs", is a crank witnessing the congruence in (7).

Theorem 20 (Aniceto, K.)

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Let $\{1, 2, \ell\}$ be a set of part sizes containing the odd number $\ell \geq 3$. For $n = 2\ell k + \frac{3\ell-3}{2}$ and $k \geq 0$, we have

$$p\left(n, \{1, 2, \ell\}\right) \equiv 0 \left(\text{ mod } \ \ell \ \right). \tag{7}$$

Whenever $\ell \equiv 1 \pmod 4$, the statistic "four times the number of 2s plus the number of ℓs", is a crank witnessing the congruence in (7). Whenever $\ell \equiv 3 \pmod 4$, the statistic "twice the number of 1s plus the number of ℓs", is a crank witnessing the congruence in (7).

Methods of proof:

Theorem 20 (Aniceto, K.)

Let $\{a, b, c\}$ be a set of three relatively prime numbers, with one of them being an even integer. Setting $n = abck + \frac{2abc-a-b-c}{2}$, then for $k \geq 0$ we have

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One highlight is that we establish exact formulas for certain elements of an object from polyhedral geometry called the h^*-vector.

Jena Gregory: Congruences for Sums and Differences of Partitions

Theorem 22 (Gregory, K.)

Let ℓ be an odd prime. Set $n = \ell lcm(m)k + r$ and $n' = \ell lcm(m)(k+1) - r - \left(\frac{m^2+m}{2}\right)$. Then for $-\left(\frac{m^2+m}{2}\right) + 1 \leq r \leq \ell lcm(m) - \left(\frac{m^2+m}{2}\right)$ and $k \geq 0$ we have

$$p(n,m) \pm p(n',m) \equiv 0 \pmod{\ell}. \tag{8}$$

If $\ell - m$ is odd, we have the sum and if $\ell - m$ is even, we have the difference.

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This research is a portion of Jena Gregory's dissertation. (PhD ’26 UTRGV)

Thank you! brandt.kronholm@utrgv.edu

Alice and Freeman Dyson.