

A LIFETIME JOURNEY
WITH
P. A. MACMAHON

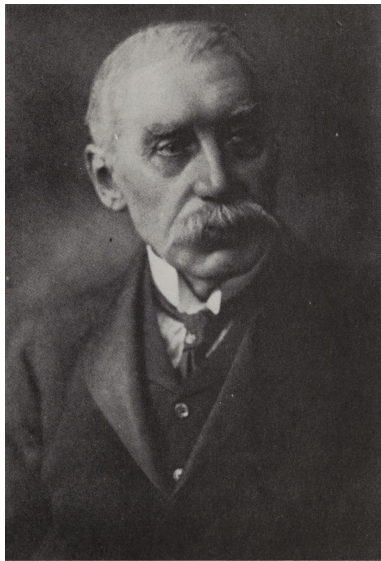


Figure: P. A. MacMahon ¹

¹Image as published in his Collected Papers, MIT Press.

AT A RECENT CONFERENCE HONORING BRUCE SAGAN,
RICHARD STANLEY GAVE A WONDERFUL TALK STARTING
WITH A THEOREM INDEPENDENTLY DISCOVERED BY
MACMAHON AND SCHUR:

Theorem

*THE NUMBER $f(n)$ OF PARTITIONS OF n IN WHICH NO
PART APPEARS EXACTLY ONCE EQUALS THE NUMBER OF
PARTITIONS OF n INTO PARTS $\not\equiv \pm 1 \pmod{6}$.*

EXAMPLE:

$$n = 8$$

SINGLETONS

$$4 + 4, 3 + 3 + 1 + 1, 2 + 2 + 2 + 2, 2 + 2 + 2 + 1 + 1, \\ 2 + 2 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

$$\# \pm 1 \pmod{6}$$

$$8, 6 + 2, 4 + 4, 4 + 2 + 2, 3 + 3 + 2, 2 + 2 + 2 + 2$$

THIS REMINDED ME THAT I HAD PUBLISHED A SHORT
PAPER IN 1976:
A GENERALIZATION OF A PARTITION THEOREM OF
MACMAHON
WHICH WAS BASED ON THIS SAME RESULT OF
MACMAHON.
AND THUS WAS BORN THE IDEA FOR THIS TALK.

MACMAHON WAS A PIONEER IN COMBINATORICS AND
WROTE A 2-VOLUME WORK:
“COMBINATORY ANALYSIS”
WHICH WAS THE GO-TO BOOK IN COMBINATORICS FOR
ALMOST HALF A CENTURY.
TODAY, RICHARD IS CLEARLY THE PREEMINENT
COMBINATORIALIST, AND HIS 2-VOLUME WORK
“ENUMERATIVE COMBINATORICS”
WILL SURELY BE THE GO-TO BOOK IN COMBINATORICS
FOR MUCH MORE THAN HALF A CENTURY.

RICHARD PERHAPS BEST DESCRIBED MACMAHON'S IMPORTANCE:

Before 1960 there are just a handful of mathematicians who did a substantial amount of enumerative combinatorics. The most important and influential of these is Percy Alexander MacMahon (1854-1929). He was a highly original pioneer, whose work was not properly appreciated during his lifetime except for his contributions to invariant theory and integer partitions. Much of the work in EAC in the 60's and 70's can trace its roots to MacMahon. Some salient examples are his anticipation of the theory of P -partitions, his development of the theory of plane partitions, and his work on permutation enumeration. It is also interesting that he gave the Rogers-Ramanujan identities a combinatorial interpretation.

SO HOW DID SOMEONE LIKE ME, WHO CONSIDERED
HIMSELF A NUMBER-THEORIST (AND A STUDENT OF HANS
RADEMACHER), WIND UP IN THE WORLD OF P. A.
MACMAHON?



Figure: Gian-Carlo Rota ²

²<https://mathshistory.st-andrews.ac.uk/Biographies/Rota/Rota.jpeg>

I SPENT MY 1970-1971 SABBATICAL AT M. I. T. AS A
VISITING PROFESSOR.

ROTA'S SEMINAR WAS IN FULL SWING. REGULAR
ATTENDEES INCLUDED RICHARD, DENNIS WHITE, CURTIS
GREENE. IN ADDITION THERE WERE MANY VISITORS.
IN THE MIDDLE OF THE YEAR, ROTA INVITED ME TO EDIT
THE COLLECTED PAPERS OF MACMAHON FOR THE M. I. T.
PRESS.

WAS THIS A GOOD IDEA?

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MY FIRST THOUGHT WAS: MACMAHON'S BEEN DEAD FOR
40 YEARS. SURELY, HIS WORK IS OUT OF DATE.

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MY FIRST THOUGHT WAS: MACMAHON'S BEEN DEAD FOR
40 YEARS. SURELY, HIS WORK IS OUT OF DATE.

MY SECOND (AND MUCH BETTER) THOUGHT WAS: WHAT
A GREAT WAY TO LEARN THE MATHEMATICS INFLUENCED
BY MACMAHON. I WILL WRITE TO EVERYONE WHO HAS
CITED MACMAHON FOR ADVICE AND COMMENT. THIS
PROVIDED THE SUBSTANCE OF THE COMMENTARY I
PROVIDED FOR THE COLLECTED WORKS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

CAMBRIDGE, MASSACHUSETTS 02139

June 25, 1975

Dear George,

Thanks for the photograph of
P. A. MacMahon. It is now con-
spicuously displayed in my office.
I hope MacMahon's collected works
which you have edited will be pub-
lished soon. You probably have re-
ceived by now some of my latest
papers. Please continue to send me
your papers.

Sincerely,
Richard Stanley

MARTIN GARDNER
10 EUCLID AVENUE
HASTINGS-ON-HUDSON
NEW YORK 10706

Dear Mr. Andrews -

24 Dec /1971

I couldn't find a ^{reference to a} single McMahon ~~ref~~ paper not on your list.

I have devoted only two of my Scientific American columns to recreations presented by M.C.M. in his little book, New Math. Pastimes. The first column (on his color squares & color cubes) was in the March, 1961 issue, but is reprinted in expanded form, & corrected, in my book, New Mathematical Diversions from Scientific American, Simon & Schuster, 1966, Chapt. 16*. Note a list of references in the back. The most important is the German book, by Winter, which is devoted entirely to McMahon's 30 color-cubes. Not on the list is an article dealing with an S-cube puzzle which McMahon handled in England as the "Mayblox" puzzle.

"The Mayblox Problem," by Margaret A. Farrell,
Journal of Recreational Mathematics, Vol. 2,
Jan, 1969, pages 51-56.

My second column, ^(which was) on McMahon's color Δ 's, Oct, 1968, is enclosed (I had a spare set of ten sheets, the magazine does not provide offprints of my column). This will eventually go into a book collection, but not for many

(over)

* There is a paperback edition.

years, I should add that a firm in California is planning to make a set of the 30 color-cubes. I have done a booklet to go with the set, covering a great many ~~as~~ features not in my column, but when the booklet will be printed (if at all!) I can't say. The last I heard, the firm was in financial straits.

I wish I had more to say, but as you see, I am familiar only with the literature on recreations stemming from MacMahon's book.

I'm sure you know that McM's Introduct. to Combin. Analysis was reprinted in Famous Problems and Other Monographs, 1955, a ~~short~~ Chelsea paperback still in print.

I'm delighted to hear that his collected papers are on the way, and that they are in such capable hands. Thanks for the advance copy of the bibliography, a person who might know of some missed references (if there are any) is Prof. H.S.M. Coxeter, at Toronto Univ.

Best,

Martin Gardner

STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

COMPUTER SCIENCE DEPARTMENT

Telephone:
415-317-2300

December 14, 1971

Prof. George E. Andrews
Department of Mathematics
Pennsylvania State University
250 McAllister Building
University Park, Penn. 16802

Dear George,

Thanks for your letter regarding MacMahon.

I'm not sure I have "mastered" his books, but I did fight my way all the way through them and took notes, here and there modernizing what he did a little. I'm afraid my notes won't be of any use to you, however, since they are personal, cryptic, and un-Xeroxable, and I have no time right now to clean them up.

As for other works by MacMahon, I'm really amazed by what you already have dug up. He is mentioned in Foggendorf's bio-bibliographies, which I suppose you have checked.

I enclose two reprints. The major thing (my joint work with Bender which gives a constructive proof of many of MacMahon's plane partition formulae) is scheduled to appear in JOT-A, July 1972. I understand Basil Gordon has solved some of MacMahon's conjectures that we restate in that article.

D. Foata [Proc. AMS 19 (1968), 236-240] gave a direct proof of the index theorem MacMahon published in Am. J. Math 1913. In Foata's thesis he presented a new proof and generalization of MacMahon's Master Theorem. You should write to Foata who is spending the year at Gainesville.

After reading PAM's two volumes I have noticed several times a rediscovery of something he did. But I can't dig these up, nor can you probably make use of them.

The only other suggestion I have for you is to ask Martin Gardner (10 Euclid Ave., Hastings-on-Hudson, N.Y.). He is an incredibly good source of information and he once did a column on MacMahon's colored cubes, so I bet he has some helpful hints.

I'm anxious to see the collected works in print.

Sincerely,

Don Knuth
Donald E. Knuth
Professor

DEK:hl

1

Introduction to
The Collected Papers of P. A. MacMahon
Volume 2

-by

Gian-Carlo Rota

The history of mathematics, like all history generally, is replete with injustice. All publics, even the intellectual public of mathematicians, exhibit towards the past the forgetful, oversimplifying, hero-worshipping attitude that we have come to identify with man behavior.

Great advances in science are pinned on a few extraordinary white-haired individuals who alone, by the magic power of genius denied to ordinary mortals - and then safely getting in off the hook - are made responsible for Progress.

The public allows details, and audibly adopts Bergson's dictum that every man is responsible for at most one idea. To reveal that behind every great man one can find a doctrine of lesser known names who paved the way and obtained most of the results for which He is known, is to commit a crime of lese majesty. Whence does one owe Apollonius to Euclid, Cavalieri to Leibniz, Euler to Bolshchikov, Hardy to Hilbert, MacMahon to Poincaré should stand ready for the nominal author of a discredited majority.

UNIVERSITY OF CAPE TOWN

(WITH WHICH IS INCORPORATED THE SOUTH AFRICAN COLLEGE)



Dept. of Astronomy
9 February 1972

Dr. G. E. Andrews
Department of Mathematics
230 McAllister Building
University Park, Penna. 16802

Dear Dr. Andrews:

In response to your letter of 28 January I enclose some Xerox copies of the early references to MacMahon's work in astronomy, as gleaned from the Monthly Notices of the Royal Astronomical Society. I did not copy out his presidential addresses, since they did not seem appropriate to your present purpose, but I include his obituary notice which does contain some information that might be useful to you.

Apparently MacMahon only wrote the one astronomical paper, which was criticized by Eddington but was, in the light of long hindsight, very nearly correct even in detail. Certainly his suggestion was a good one and the objections that Eddington raised proved, finally, to be incomplete and less well informed than one might expect from so famous a personage. I think it's fair to say that Eddington's comments held back the application of MacMahon's proposed technique for quite a long time; the first successful investigator to make use of it, A. R. Whitford, was unaware of either the suggestion or its criticism when he started the work. Both had faded into history by then.

I did not bring reprints with me to Cape Town, but I can give you references to my own work concerning lunar occultations, all of which is recent and should be available to you:

"Discovery and Measurement of Double Stars by Lunar Occultations,"
Astrophysics and Space Science 11, 28 (1971).

"Photoelectric Measurements of Lunar Occultations I: The Process"
Astronomical Journal 75, 575 (1970).

same, II: A. J. 75, 583 (1970); same, IV: A. J. 75, 963 (1970).

"The Angular Diameter of Lambda Aquarii", Astrophysical Journal Letters 160, L181 (1970).

I hope this is of some help.

Sincerely,

R Edward Nathan
R. Edward Nathan

PARTITION ANALYSIS AS CONCEIVED BY MACMAHON
 RELIES ON THE LINEAR OPERATOR Ω APPLIED TO
 MULTIPLE LAURENT SERIES.

$$\Omega_{\geq} \sum_{n_1, \dots, n_k = -\infty}^{\infty} A_{n_1, n_2, \dots, n_k} \lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_k^{n_k} = \sum_{n_1, \dots, n_k \geq 0} A_{n_1, n_2, \dots, n_k}$$

IN ACTUAL APPLICATION, RESULTS OF VALUE ARE OF THE FOLLOWING NATURE:

$$\Omega \frac{1}{(1-x\lambda)(1-y/\lambda)} = \frac{1}{(1-x)(1-xy)}$$

PROOF:

$$\begin{aligned}\Omega \frac{1}{(1-x\lambda)(1-y/\lambda)} &= \Omega \sum_{r,s \geq 0} x^r y^s \lambda^{r-s} \\ &= \sum_{s \geq 0, r \geq s} x^r y^s \\ &= \sum_{s,r \geq 0} x^{r+s} y^s \\ &= \frac{1}{(1-x)(1-xy)}. \quad \square\end{aligned}$$

IN COMBINATORY ANALYSIS (VOL. 2, P. 102) MACMAHON
LISTS MANY SUCH RESULTS, E.G.:

$$\Omega_{\geq} \frac{1}{(1 - \lambda x)(1 - \lambda y)(1 - z/\lambda)} = \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - yz)}$$

$$\Omega_{\geq} \frac{1}{(1 - \lambda^2 x)(1 - y/\lambda)} = \frac{1 + xy}{(1 - x)(1 - xy^2)}$$

$$\Omega_{\geq} \frac{1}{(1 - \lambda^2 x)(1 - y/\lambda)(1 - z/\lambda)} = \frac{1 + xy + xz + xyz}{(1 - x)(1 - xy^2)(1 - xz^2)}$$

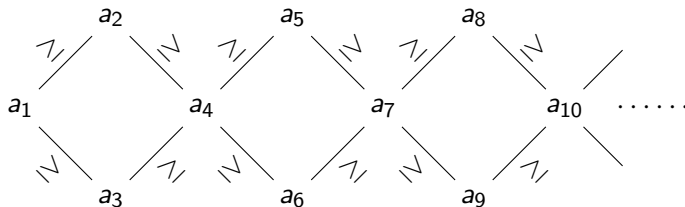
⋮

SINCE 1998, THERE HAS BEEN A CONTINUING SERIES OF
PAPERS (ALMOST ALL JOINT WITH PETER PAULE) UNDER
THE TITLE
MACMAHON'S PARTITION ANALYSIS: I-XV

THE SUBTITLES WERE:

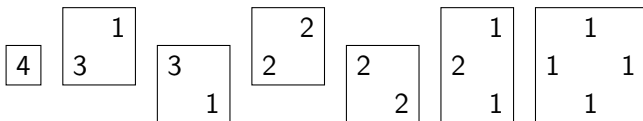
THE LECTURE HALL PARTITION THEOREM
FUNDAMENTAL THEOREM
THE OMEGA PACKAGE
HYPERGEOMETRIC MULTISUMS
BIJECTIONS, RECURSIONS, AND MAGIC SQUARES
A NEW REDUCTION ALGORITHM
CONSTRAINED COMPOSITIONS
PLANE PARTITION DIAMONDS
 k -GON PARTITIONS
PLANE PARTITIONS WITH DIAGONALS
BROKEN DIAMONDS AND MODULAR FORMS
PLANE PARTITIONS
SCHMIDT-TYPE PARTITIONS AND MODULAR FORMS
PARTITIONS WITH n COPIES OF n
PARITY

THE EARLY PAPER FOCUSED MOST ON PARTITIONS ON
DIFFERENT KINDS OF GRAPHS:
PLANE PARTITION DIAMONDS



THE
GENERATING FUNCTION FOR THESE PARTITIONS IS

$$\prod_{n=1}^{\infty} \frac{1 + q^{3n-1}}{1 - q^n} = 1 + q + 3q^2 + 4q^3 + 7q^4 + \dots$$



IN 1999, FRANK SCHMIDT PROPOSED THE FOLLOWING
PROBLEM IN THE AMERICAN MATH. MONTHLY:
LET $p(n)$ DENOTE THE NUMBER OF PARTITIONS OF n , AND
LET $f(n)$ DENOTE THE NUMBER OF PARTITIONS
 $\lambda_1 + \lambda_2 + \lambda_3 + \cdots$ SATISFYING $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ AND
 $n = \lambda_1 + \lambda_3 + \lambda_5 + \cdots$. THEN
FOR $n \geq 1$
$$p(n) = f(n).$$

FOR EXAMPLE, $p(6) = 11$:

6, $5 + 1$, $4 + 2$, $4 + 1 + 1$, $3 + 3$, $3 + 2 + 1$,
 $3 + 1 + 1 + 1$, $2 + 2 + 2$, $2 + 2 + 1 + 1$,
 $2 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1$,

$f(6) = 11$

$6, 6 + 5, 6 + 4, 6 + 3, 6 + 2, 6 + 1$
 $5 + 4 + 1, 5 + 3 + 1, 5 + 2 + 1, 4 + 3 + 2,$
 $4 + 3 + 2 + 1$

THE SOLUTION PUBLISHED WAS BY PETER MORK. THE MONTHLY ALSO NOTED THAT EIGHT OTHERS SOLVED THE PROBLEM: D. BECKWITH, R.J. CHAPMAN, G. LORD, C.C ROUSSEAU, P, SIMEONOV, J. H. STEELMAN, AND F. SCHMIDT (THE PROPOSER).

MORK'S SOLUTION IS QUITE CLEVER. IT IS A BIJECTION
BETWEEN THE TWO SETS. HERE IS THE IDEA:

$$7 + 4 + 4 + 3 + 1$$

X	X					
	X	X				
		X	X			

MORK'S SOLUTION IS QUITE CLEVER. IT IS A BIJECTION BETWEEN THE TWO SETS. HERE IS THE IDEA:

$$7 + 4 + 4 + 3 + 1$$

X	X					
	X	X				
		X	X			

NOW COMPUTE THE HOOKS AT THE SQUARES MARKED X:

$$11 + 9 + 5 + 4 + 3 + 1$$

BY CONSTRUCTION, WE KNOW A PRIORI THAT

$$7 + 4 + 4 + 3 + 1 = 11 + 5 + 3.$$

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.
1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

X

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.

1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

3 IS OF ODD INDEX; SO ITS SQUARE IS ON THE DIAGONAL.

X	X

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.

1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

3 IS OF ODD INDEX; SO ITS SQUARE IS ON THE DIAGONAL.

4 IS OF EVEN INDEX; SO OFF DIAGONAL.

X	
X	X

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.

1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

3 IS OF ODD INDEX; SO ITS SQUARE IS ON THE DIAGONAL.

4 IS OF EVEN INDEX; SO OFF DIAGONAL.

5 IS OF ODD INDEX; SO ON DIAGONAL.

X	X	
	X	X

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.

1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

3 IS OF ODD INDEX; SO ITS SQUARE IS ON THE DIAGONAL.

4 IS OF EVEN INDEX; SO OFF DIAGONAL.

5 IS OF ODD INDEX; SO ON DIAGONAL.

9 IS OF EVEN INDEX; SO OFF DIAGONAL.

X					
X	X				
	X	X			

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.

1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

3 IS OF ODD INDEX; SO ITS SQUARE IS ON THE DIAGONAL.

4 IS OF EVEN INDEX; SO OFF DIAGONAL.

5 IS OF ODD INDEX; SO ON DIAGONAL.

9 IS OF EVEN INDEX; SO OFF DIAGONAL.

11 IS OF ODD INDEX; SO ON DIAGONAL.

X	X					
	X	X				
		X	X			

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START
WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.

1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE
DIAGONAL.

3 IS OF ODD INDEX; SO ITS SQUARE IS ON THE DIAGONAL.

4 IS OF EVEN INDEX; SO OFF DIAGONAL.

5 IS OF ODD INDEX; SO ON DIAGONAL.

9 IS OF EVEN INDEX; SO OFF DIAGONAL.

11 IS OF ODD INDEX; SO ON DIAGONAL.

X	X					
	X	X				
		X	X			

NOW READING ROWS WE RETRIEVE $7 + 4 + 4 + 3 + 1$.

MORK'S SOLUTION WAS QUITE INGENIOUS. HOWEVER, ITS VERY CLEVERNESS MAKES IT HARD TO SEE THIS PROBLEM AS ONE OF A GENERAL CLASS OF RESULTS.

THE FOLLOWING IS THE GENERATING FUNCTION FOR
PARTITIONS WITH n DISTINCT PARTS WITH x_i
ACCOUNTING FOR THE i^{th} PART

$$\begin{aligned}
 & \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 - j_2 \geq 1, j_2 - j_3 \geq 1, \dots, j_{n-1} - j_n \geq 1}} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \\
 = \Omega_{\geq} & \sum_{j_1, \dots, j_n \geq 0} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \lambda_1^{j_1 - j_2 - 1} \lambda_2^{j_2 - j_3 - 1} \cdots \lambda_{n-1}^{j_{n-1} - j_n} \lambda_n^{j_n} \\
 & = \frac{x_1(x_1 x_2)(x_1 x_2 x_3) \cdots (x_1 x_2 x_3 \cdots x_r)}{(1 - x_1)(1 - x_1 x_2) \cdots (1 - x_1 x_2 \cdots x_r)}
 \end{aligned}$$

NOW SET $n = 2k$

$$x_1 = q, x_2 = 1, x_3 = q, x_4 = 1, x_5 = q, \dots, x_{2k-1} = q, x_{2k} = 1.$$

THIS WILL GENERATE THOSE PARTITIONS INTO DISTINCT PARTS WHERE ONLY THE 1st, 3rd, 5th, ..., $(2k - 1)$ st PARTS ARE ADDED:

$$\frac{q^{k^2}}{(1 - q)^2(1 - q^2)^2 \cdots (1 - q^k)^2}$$

SUMMING OVER ALL $k \geq 0$, WE FIND

$$\begin{aligned}\sum_{n \geq 0} f(n)q^n &= \sum_{k \geq 0} \frac{q^{k^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} \\ &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \text{ (JACOBI)} \\ &= \sum_{n \geq 0} p(n)q^n\end{aligned}$$



RETURNING TO PARTITION DIAMONDS FOR THE MOMENT,
WE FOUND (IN SCHMIDT STYLE) THAT IF WE ONLY ADDED
SUMMANDS

$$a_1 + a_4 + a_7 + a_{10}$$

THE GENERATING FUNCTION WAS

$$\prod_{n=1}^{\infty} \frac{1 + q^n}{(1 - q^n)^3},$$

EFFECTIVELY A MODULAR FORM.

MY FINAL TOPIC ARISES FROM MACMAHON'S PAPER

DIVISORS OF NUMBERS AND THEIR CONTINUATION IN THE
THEORY OF PARTITIONS

MACMAHON'S STARTING POINT IS THE OBSERVATION
THAT THE NUMBER OF DIVISORS OF n IS THE NUMBER OF
PARTITIONS OF n INTO PARTS OF ONLY ONE SIZE.
THE DIVISORS OF 6 ARE

1, 2, 3, 6.

THE PARTITIONS OF 6 WITH ONLY ONE PART SIZE ARE

$1 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2, 3 + 3, 6.$

THE RELEVANT GENERATING FUNCTION IS

$$\sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

IN ADDITION, THE SUM OF THE DIVISORS OF n DENOTED $\sigma_1(n)$ IS OF INTEREST, AND THE GENERATING FUNCTION IS

$$A_1(q) := \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2}.$$

$A_1(q)$ IS ESPECIALLY INTERESTING BECAUSE

$$A_1(q) = \frac{1 - E_2(q)}{24}$$

WHERE $E_2(q)$ IS THE FUNDAMENTAL QUASI-MODULAR
FORM.

REFRESHER COURSE ON QUASI-MODULAR FORMS:

$$E_{2k}(q) := 1 + \frac{2}{\zeta(1-2k)} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

WHERE

$$\sigma_m(n) = \sum_{d|n} d^m.$$

THE RING OF QUASI-MODULAR FORMS IS GENERATED BY
 $E_2(q)$, $E_4(q)$, AND $E_6(q)$, OR EQUIVALENTLY BY
 $E_2(q)$ AND $q \frac{d}{dq}$.

MACMAHON NOW CONSIDERS WHAT HAPPENS WHEN WE CONSIDER PARTITIONS OF n WITH EXACTLY k DIFFERENT PART SIZES. THE GENERATING FUNCTION IS CLEARLY

$$\sum_{1 \leq n_1 < n_2 < \dots < n_k} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1 - q^{n_1})(1 - q^{n_2}) \dots (1 - q^{n_k})}.$$

THE GENERALIZATION FOR SUM OF DIVISORS IS

$$\sum_{1 \leq n_1 < n_2 < \dots < n_k} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1 - q^{n_1})^2 (1 - q^{n_2})^2 \dots (1 - q^{n_k})^2}.$$

MACMAHON ASSERTS LATE IN HIS PAPER

The function $A_k = \sum a_{n,k} q^n$ has apparently the property that the coefficient $a_{n,k}$ is expressible as a linear function of the sum of the uneven powers of the divisors of n . I have not succeeded in reaching the general theory, but by the help of certain results of Glaisher, obtained by means of elliptic functions, I am able to give the expressions in the cases $n = 2$, $n = 3$, and $n = 4$.

IN 2013, SIMON ROSE AND I PROVED MACMAHON'S
CONJECTURE. $T_n(x)$ is the n^{th} Chebychev polynomial:

$$T_n(\cos \theta) = \cos(n\theta).$$

NOW DEFINE

$$F(x, q) := \sum_{n=0}^{\infty} T_{2n+1} \left(\frac{x}{2} \right) q^{n^2+n}.$$

THEN

Theorem

$$F(x, q) = (q^2; q^2)_{\infty}^3 \sum_{k=0}^{\infty} A_k(q^2) x^{2k+1}.$$

Corollary (1)

$$A_k(q) = \frac{(-1)^k}{(2k+1)!(q; q)_\infty^3} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{(n+k)!}{(n-k)!} q^{\binom{n+1}{2}}.$$

Corollary (2)

$$A_k(q) = \frac{1}{(2k+1)2k} \left((6A_1(q) + k(k-1))A_{k-1}(q) - q^2 \frac{d}{dq} A_{k-1}(q) \right).$$

Corollary (3)

THE FUNCTIONS $A_k(q)$ ARE IN THE RING OF QUASI-MODULAR FORMS.

PROOF OF THE THEOREM

LET $x = 2 \cos \theta$.

THEN

$$\begin{aligned} F(x, q) &= 2 \sum_{n=0}^{\infty} T_{2n+1}(\cos \theta) q^{n^2+n} \\ &= 2 \sum_{n=0}^{\infty} \cos((2n+1)\theta) q^{n^2+n} \\ &= \sum_{n=0}^{\infty} (e^{i(2n+1)\theta} + e^{-i(2n+1)\theta}) q^{n^2+n} \\ &= e^{i\theta} \sum_{n=-\infty}^{\infty} e^{i2n\theta} q^{n^2+n} \\ &= e^{i\theta} (-e^{-i2n\theta}; q^2)_{\infty} (-q^2 e^{i2n\theta}; q^2)_{\infty} (q^2; q^2)_{\infty} \\ &\vdots \end{aligned}$$

PROOF OF THE THEOREM (CONTINUED).

$$\begin{aligned} &= (e^{i\theta} + e^{-i\theta})(q^2; q^2)_{\infty} \prod_{n=1}^{\infty} (1 + 2 \cos(2\theta) q^{2n} + q^{4n}) \\ &= x(q^2; q^2)_{\infty} \prod_{n=1}^{\infty} (1 + (x^2 - 2) q^{2n} + q^{4n}) \\ &= x(q^2; q^2)_{\infty} \prod_{n=1}^{\infty} ((1 - q^{2n})^2 + x^2 q^{2n}) \\ &= x(q^2; q^2)_{\infty}^3 \prod_{n=1}^{\infty} \left(1 + \frac{x^2 q^{2n}}{(1 - q^{2n})^2} \right) \\ &= x(q^2; q^2)_{\infty}^3 \sum_{n=0}^{\infty} A_k(q^2) x^{2k+1} \end{aligned}$$



IN A RECENT PAPER, AMDEBERHAN, TAURASO, AND I
EXPLORE THE ARITHMETIC RICHES REVEALED BY THE
QUASI-MODULARITY OF THESE FUNCTIONS. TO MIX
MACMAHON'S NOTATION WITH OURS, WE DEFINE

$$A_k(q) := \sum_{n \geq 0} MO(k, n) q^n.$$

THERE ARE NUMEROUS CONGRUENCES:

$$MO(2, n) \equiv 0 \pmod{3} \quad \text{WHEN } 3 \nmid n.$$

$$MO(2, 5n + 1) \equiv 0 \pmod{5}$$

$$MO(3, 7n + 3) \equiv 0 \pmod{7}$$

$$\vdots$$

PROOF THAT $MO(2, n) \equiv 0 \pmod{3}$
WHEN $n \not\equiv 0 \pmod{3}$

BY COROLLARY 1,

$$\sum_{n \geq 0} MO(2, n)q^n = A_2(n)$$

$$\begin{aligned} &= \frac{1}{(q; q)_{\infty}^3} \sum_{k \geq 2} (-1)^k \frac{2k+1}{5} \binom{k+2}{k-2} q^{\binom{k+1}{2}} \\ &= \frac{1}{(q^3; q^3)_{\infty}} \sum_{k \geq 2} (-1)^{k-1} (2k+1) \binom{k+2}{k-2} q^{\binom{k+1}{2}} \end{aligned}$$

NOW $\binom{k+1}{2} \equiv 0 \pmod{3}$ IF $k \equiv 0$ OR $2 \pmod{3}$
 $\binom{k+1}{2} \equiv 1 \pmod{3}$ IF $k \equiv 1 \pmod{3}$

PROOF THAT $MO(2, n) \equiv 0 \pmod{3}$
WHEN $n \not\equiv 0 \pmod{3}$ (CONTINUED.).

SINCE $\binom{k+1}{2} \not\equiv 2 \pmod{3}$, $MO(2, 3n+2) \equiv 0 \pmod{3}$.

WHEN $\binom{k+1}{2} \equiv 1 \pmod{3}$, $k \equiv 1 \pmod{3}$;

THUS $2k+1 \equiv 0 \pmod{3}$, SO $MO(2, 3n+1) \equiv 0 \pmod{3}$.



AT THE END OF OUR PAPER, WE CONJECTURED

$$MO(10, 11n + 7) \equiv 0 \pmod{1}.$$

THIS HAS SUBSEQUENTLY BEEN PROVEN BY
AMDEBERHAN, ONO, AND SINGH.

OF COMPARABLE INTEREST IS THE GENERALIZATION

$$\sum_{1 \leq n_1 < n_2 < \dots < n_k} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1 + \alpha q^{n_1} + q^{2n_1})(1 + \alpha q^{n_2} + q^{2n_2}) \dots (1 + \alpha q^{n_k} + q^{2n_k})},$$

ESPECIALLY WHEN $\alpha = 0, \pm 1, \pm 2$.

AS A MOST APPEALING SAMPLE, NOTE

$$\begin{aligned} \sum_{1 \leq n_1 < n_2} \frac{q^{n_1 + n_2}}{(1 + q^{n_1} + q^{2n_1})(1 + q^{n_2} + q^{2n_2})} \\ &= \sum_{1 \leq n_1 < n_2} \frac{q^{n_1 + n_2}(1 - q^{n_1})(1 - q^{n_2})}{(1 - q^{3n_1})(1 - q^{3n_2})} \\ &= \sum_{n \geq 1} \frac{q^{3n}}{(1 - q^{3n})^2} = \sum_{n=1}^{\infty} \sigma(n) q^{3n}. \end{aligned}$$

A FEW REFERENCES

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THANK
YOU!
HAPPY BIRTHDAY
BRUCE