# Understanding Euler's Formula

A while back I was reading a wikipedia article on Euler's formula and came across the following quote by a 19th century Harvard Professor Benjamin Pierce, stated after proving Euler's Theorem: "is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth". And yet, that statement could not be further from the truth.

When most people are taught Euler's theorem the explanation given to them is through the Taylor Series proof or through differentiation of $\mathrm{e}^{\mathrm{j}x}$. Proofs via Taylor Series work, but provide no intuition to why they work. Instead, we're going to investigate some simple properties of numbers to give motivation to look deeper into the realm of complex numbers.

## Powers of Negative Numbers

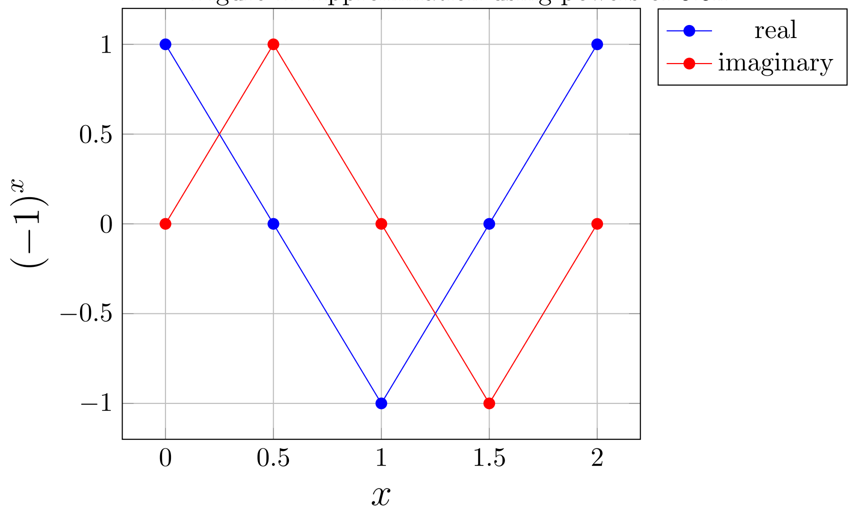
Negative Numbers multiplied by other negative numbers lead to positive numbers. Positive numbers multiplied by negative numbers lead to negative numbers. With repeated multiplication of negative numbers, numbers oscillate signs with each iteration. Let's look at $y = (-1)^x$ with integer powers

$$
\begin{aligned}
(-1)^0 &= 1 & \qquad (-1)^1 &= -1 \\
(-1)^2 &= 1 & \qquad (-1)^3 &= -1 \\
(-1)^4 &= 1 & \qquad (-1)^5 &= -1
\end{aligned}
$$

As shown, powers of negative one oscillate back and forth between positive one and negative one with integer powers. What about real non-integer powers of negative one? We immediately run into the problem of the square root of negative one : the imaginary number. This number completes the fundamental theorem of algebra and provides every polynomial equation with a well defined root. It will allow us to delve deeper into powers of negative numbers. $\mathrm{j}^2 = -1$.

$$
\begin{aligned}
(-1)^{1/2} &= \mathrm{j} & \qquad (-1)^1 &= -1 \\
(-1)^{3/2} &= \mathrm{j}^3 = -\mathrm{j} & \qquad (-1)^2 &= 1 \\
(-1)^{5/2} &= \mathrm{j}^5 = \mathrm{j} & \qquad (-1)^3 &= -1
\end{aligned}
$$

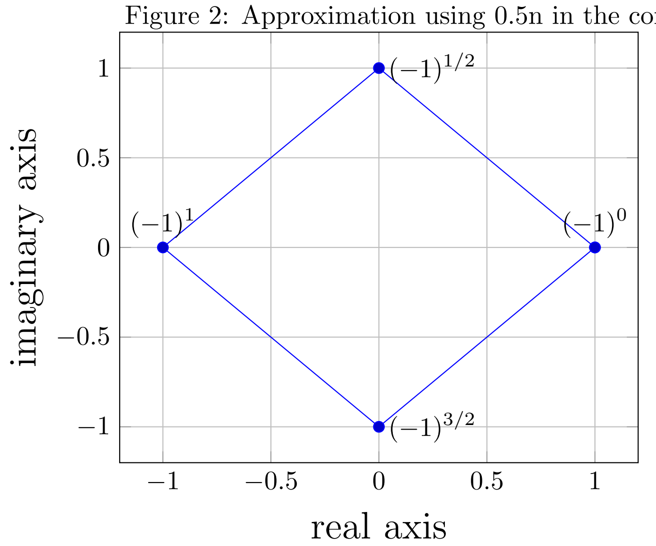
Figure 1: Approximation using powers of 0.5n




$(-1)^x$ is clearly a periodic function, with $\lim_{x\to\infty}(-1)^x$ not being defined. Its cycles are of length $T = 2$, so $(-1)^x$ can be written as $(-1)^x = (-1)^{x+2n}$, where n is some integer number.

As it turns out, integer powers of negative numbers ($(-1)^2, (-1)^3$,etc) will result in purely real numbers. Real non-integer powers ($(-1)^{1.2}, (-1)^{5.8}$,etc) must be complex numbers in order for this function to be continuous. Perhaps it would be useful to look at $(-1)^x$ in the complex-plane. Based on the following graph, we should expect real non-integer powers such as $(-1)^{1/4}$ to have both real and imaginary components.

Figure 2: Approximation using 0.5n in the complex plane



## Fourth Root of Negative One

We should expect it to be some complex number with real and imaginary parts, $(-1)^{1/4} = a + \mathrm{j}b$. $(-1)^{1/4}$ is just $\mathrm{j}^{1/2}$, so $\mathrm{j}^{1/2} = a + \mathrm{j}b$

$$
\begin{aligned}
\mathrm{j}^{1/2} &= a + \mathrm{j}b \\
\mathrm{j} &= (a + \mathrm{j}b)^2 \\
\mathrm{j} &= (a + \mathrm{j}b)(a + \mathrm{j}b) \\
\mathrm{j} &= a^2 + \mathrm{j}2ab + \mathrm{j}^2 b^2 \\
\mathrm{j} &= a^2 - b^2 + \mathrm{j}2ab \\
0 &= a^2 - b^2 : \textit{Real part of both sides}: \\
a &= b \\
\mathrm{j} &= \mathrm{j}2a^2 : \textit{Imaginary part of both sides}: \\
\frac{1}{2} &= a^2 \\
a = b &= \frac{\sqrt{2}}{2} \\
(-1)^{1/4} &= \frac{\sqrt{2}}{2} + \mathrm{j}\frac{\sqrt{2}}{2}
\end{aligned}
$$

From this we can easily gain three other points to use by raising it to 3rd, 5th, and 7th powers. We can use the fact that $(-1)^{2/4} = \mathrm{j}$

$$
\begin{aligned}
(-1)^{3/4} &= (-1)^{1/4}(-1)^{1/2} \\
(-1)^{3/4} &= -\frac{\sqrt{2}}{2} + \mathrm{j}\frac{\sqrt{2}}{2} \\
(-1)^{5/4} = j(-1)^{3/4} &= -\frac{\sqrt{2}}{2} - \mathrm{j}\frac{\sqrt{2}}{2} \\
(-1)^{7/4} = j(-1)^{5/4} &= \frac{\sqrt{2}}{2} - \mathrm{j}\frac{\sqrt{2}}{2}
\end{aligned}
$$

Figure 3: Approximation using 0.25n in the complex plane

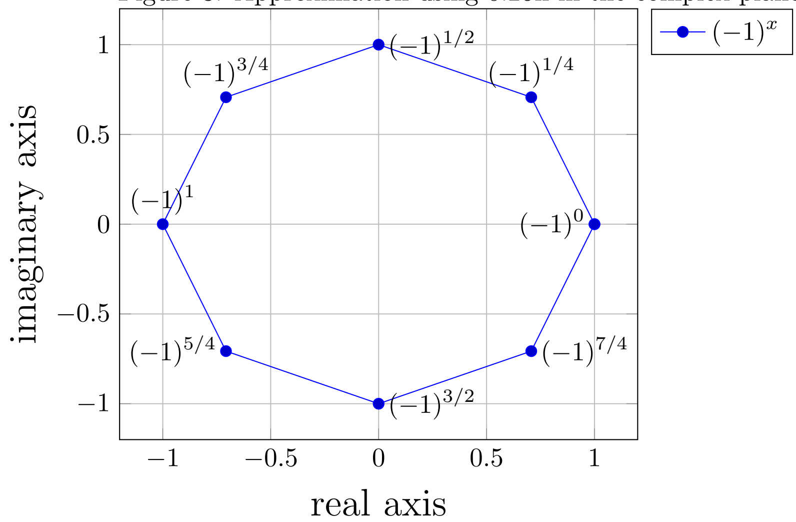
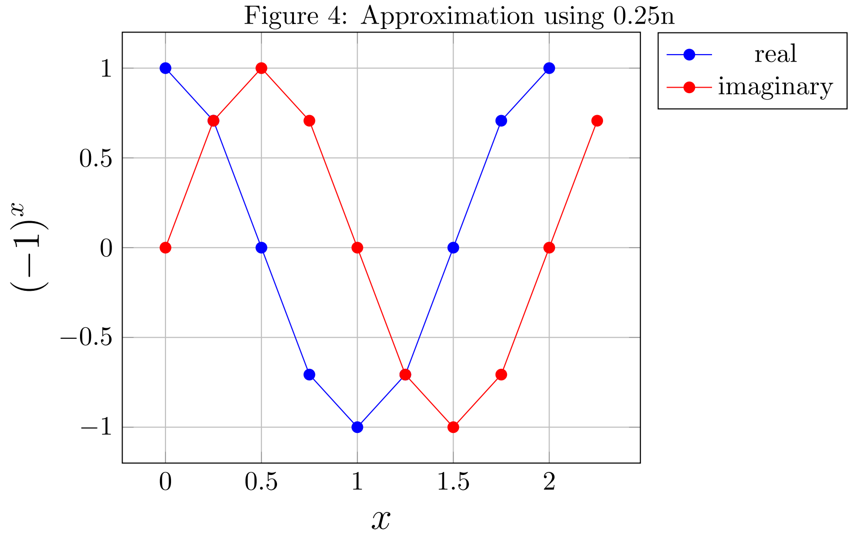



Figure 4: Approximation using 0.25n



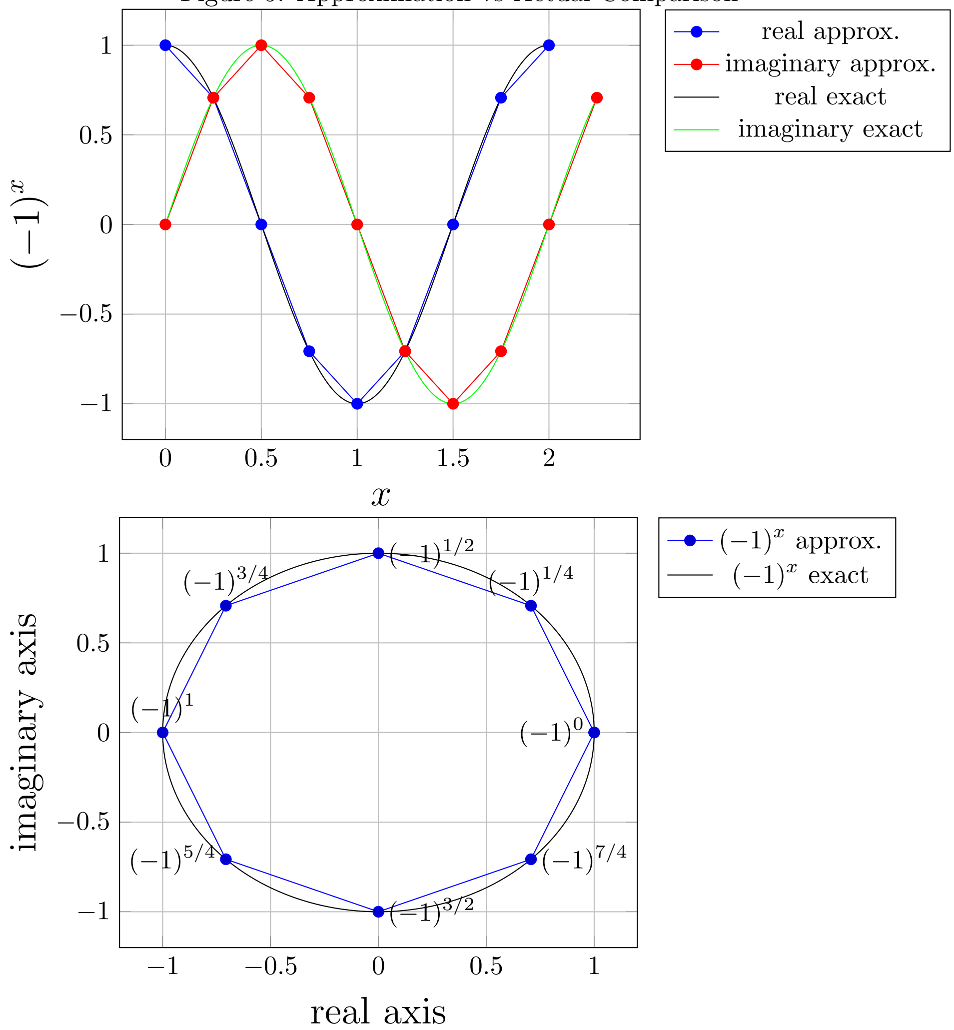
## So it's a circle in the complex plane?

We knew from the start that $(-1)^x$ oscillates between -1 and 1 in some form, but now we have some hard evidence that it oscillates just like sinusoids do. We can create a formula based on this evidence.

The real part $(-1)^x$ is a cosine of period two in x, and the imaginary part is a sinusoid of period two in x.

$$\begin{aligned}
Real\,[(-1)^x] &= \cos(\pi x)\\
Imag\,[(-1)^x] &= \sin(\pi x)\\
(-1)^x &= Real\,[(-1)^x] + \mathrm{j}\,Imag\,[(-1)^x]\\
(-1)^x &= \cos(\pi x) + \mathrm{j}\sin(\pi x)
\end{aligned}$$

Figure 5: Approximation vs Actual Comparison



## Proving that it works

The hypothesis stated is thus: $(-1)^x$ is equal to $\cos(\pi x) + \mathrm{j}\sin(\pi x)$. In order for this statement to be true, they must have the same properties and values. An easy way of determining properties of functions when precise values aren't known is by looking at how the function changes: its derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}(-1)^x = \ln(-1)(-1)^x$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(-1)^x = \ln(-1)^2(-1)^x$$

$\ln(-1)$ must be purely imaginary otherwise $\left(\mathrm{e}^{\ln(-1)}\right)^x = e^{x\,Re\,\ln(-1)} e^{\mathrm{j}x\,Im\,\ln(-1)} = (-1)^x$ would increase in magnitude as x increased, which doesn't happen. We can say that $\ln(-1) = \mathrm{j}k$, so $\ln(-1)^2 = -k^2$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(-1)^x = \ln(-1)^2(-1)^x$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(-1)^x = -k^2(-1)^x$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -k^2 y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(\pi x) + \mathrm{j}\sin(\pi x) = -\pi\sin(\pi x) + j\pi\cos(\pi x)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(\pi x) + \mathrm{j}\sin(\pi x) = -\pi^2\cos(\pi x) - \mathrm{j}\pi^2\sin(\pi x)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(\pi x) + \mathrm{j}\sin(\pi x) = -\pi^2\left(\cos(\pi x) + \mathrm{j}\sin(\pi x)\right)$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\pi^2 y$$

$$(-1)^x = \cos(\pi x) + \mathrm{j}\sin(\pi x)$$
$$\ln(-1) = \mathrm{j}\pi$$
$$(-1)^x = \mathrm{e}^{x\ln(-1)} = \mathrm{e}^{\mathrm{j}\pi x}$$

## Conclusion

Powers of negative numbers are inherently cyclical. $(-1)^x$ varies from 1, to -1, to 1, etc. Real non-integer powers of negative numbers are complex numbers. If you were to evaluate the real portion of $(-1)^x$, it isn't simply a straight line from 1 to zero to negative one like the graph in figure 1. Instead, it curves exactly as a sinusoid. A real number to an imaginary power is the same thing as a negative number to any real power. The former isn't very intuitive, but it is easier to imagine repeated powers of negative number altering the sign. Each integer power of -1 flips the sign 180 degrees. $(-1)^1 = -1$, $(-1)^2 = 1$, $(-1)^3 = -1$, etc. But powers in between those don't flip it all the way, making it a "kind of positive, kind of negative" number, which is a complex number. Negative number being numbers that have the properties n*n = p, n*p = n. And positive number having the properties p*p=p, p*n = n. Complex numbers are an in between, a combination of positive and negative real numbers, and a "halfway" imaginary number.

## Different Representations

$$(-a)^x = a^x(-1)^x$$
$$(-a)^x = a^x\left(\cos\pi x + \mathrm{j}\,\sin\pi x\right)$$
$$(-a)^x = \mathrm{e}^{x\ln a}(-1)^x$$
$$(-a)^x = \mathrm{e}^{x\ln a}\mathrm{e}^{\mathrm{j}\pi x}$$
$$(-a)^x = \mathrm{e}^{x\ln a}\left(\cos\pi x + \mathrm{j}\,\sin\pi x\right)$$