Lecture 17 - 10/03/2014 - MATH 497C, Fall 2014

Today we discuss such an important transformation as inversion/reflection in a circle. Furthermore, we describe the disk model of the hyperbolic plane.
Comment. Most of the materials for these lecture notes were taken from the book "Geometries" A.B. Sossinsky.

Denote by $\mathcal{R}$ the plane $\mathbb{R}^2$ with an added extra point (called the *point at infinity* and denoted by ∞).
There are two ways to understand $\mathbb{R}^2$ with the point at infinity:

1. Consider the complex numbers $\mathbb{C}$ with the "point at infinity" added, $\bar{\mathbb{C}}$. It is called the Riemann sphere.

2. Consider the stereographic projection of a sphere onto a plane. It is a map projection obtained by projecting points P on the surface of sphere from the sphere's north pole N to the points P' in the plane tangent to the south pole S (Coxeter 1969, p. 93). (See Figure 1.) The only point which does not have the image on the plane is the north pole N, the image of which we consider as the point at infinity.

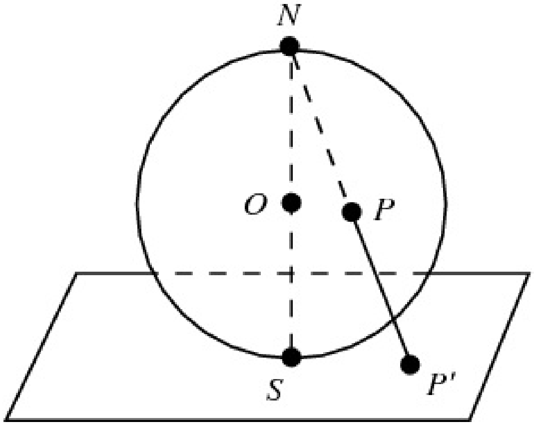

Figure 1: Stereographic projection. (http://mathworld.wolfram.com/StereographicProjection.html)

An inversion of center $O \in \mathbb{R}^2$ and radius $r > 0$ is the transformation of $\mathcal{R}$ that maps each point M to the point N on the ray OM so that

$$|OM| \cdot |ON| = r^2$$

and interchanges the point O and ∞. Sometimes inversions are called reflections with respect to the circle of inversion, i.e., the circle of radius r centered at O. As points inside (outside) the circle of inversion are mapped outside (inside) it. The points on the circle are fixed under the inversion. It follows from the definition that inversions are bijections of $\mathcal{R}$. The composition of the inversion with itself is the identity.

If the extended plane $\mathcal{R}$ is interpreted as the Riemann sphere $\bar{\mathbb{C}}$, then an example of an inversion (of center O and radius 1) is the map $z \mapsto \frac{1}{\bar{z}}$, where if $z = x + iy$, $x, y \in \mathbb{R}$, then the bar over z denotes complex conjugation, $\bar{z} = x - iy$.
There is a simple geometric way of constructing the image of a point M under an inversion of center O and radius r (see Figure 2): draw the circle of inversion, lower the perpendicular to OM from M to its intersection point T with the circle and construct the tangent to the circle at T to its intersection point N with the ray OM; then N will be the image of M under the given inversion. Indeed, the two right triangles OMT and OTN are similar (they have a common acute angle at O), and therefore

$$\frac{|OM|}{r} = \frac{|OM|}{|OT|} = \frac{|OT|}{|ON|} = \frac{r}{|ON|},$$

i.e. $|OM| \cdot |ON| = r^2$.

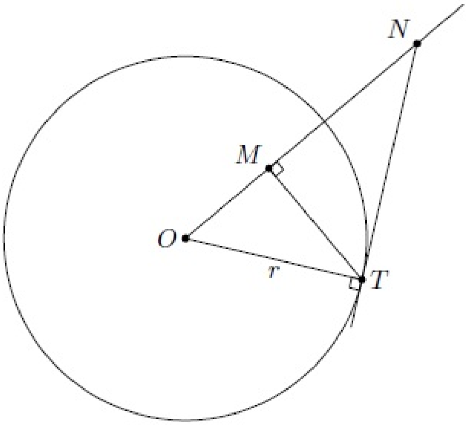

Figure 2: Inversion.

Inversions possess the following important properties.

1. Inversions map any circle or straight line into a circle or straight line. In particular, lines passing through the center of inversion are mapped to themselves (but are "turned inside out" in the sense that O goes to ∞ and vice versa, while the part of the line inside the circle of inversion goes to the outside part and vice versa); circles passing through the center of inversion are taken to straight lines, while straight lines not passing through the center of inversion are taken to circles passing through that center. (See Figure 3).

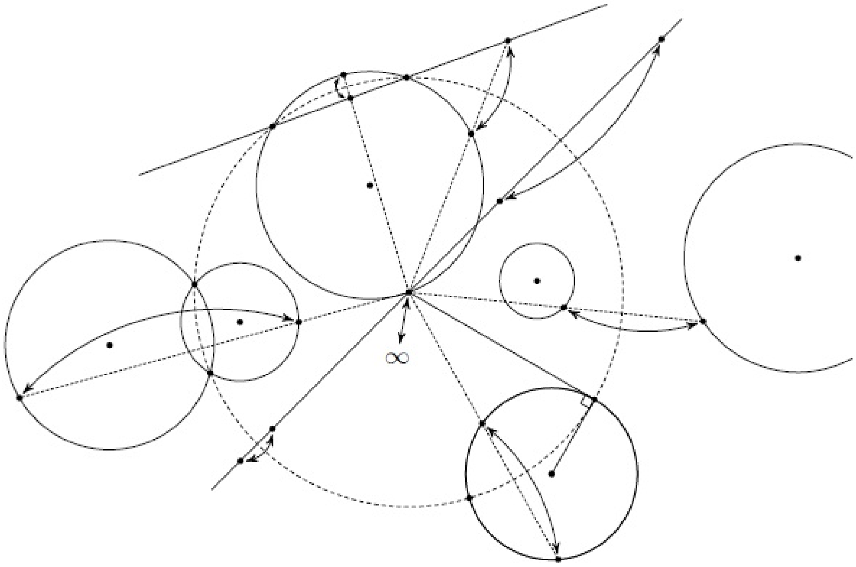

Figure 3: Images of circles and lines under inversion.

2. Inversions preserve (the measure of) angles; here by the measure of an angle formed by two intersecting curves we mean the ordinary (Euclidean) measure of the angle formed by their tangents at the intersection point. Although, inversions reverses orientations of angles. The circle inversion map is so called anticonformal map.
3. Inversions map any circle or straight line orthogonal to the circle of inversion into itself.

Hyperbolic geometry.

The disk model of the *hyperbolic plane* is the geometry $(\mathbb{H}^2, G)$ whose points are the points of the open disk

$$\mathbb{H}^2 := \{(x,y) in \mathbb{R}^2 | x^2 + y^2 < 1\},$$

and whose transformation group G is the group generated by the reflections in all circles orthogonal to the boundary circle

$$S := \{(x,y) in \mathbb{R}^2 | x^2 + y^2 < 1\}$$

of $\mathbb{H}^2$, which is called *the absolute*, and by reflections in all diameters of the circle S.
The properties formulated above show that a reflection of the type considered takes points of $\mathbb{H}^2$ to points of $\mathbb{H}^2$ and, being its own inverse, we have the implication

$$\phi \in G \Rightarrow \phi^{-1} \in G.$$

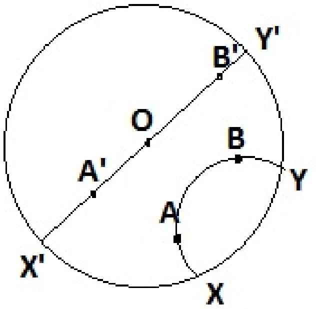

Figure 4: Distance between two points in the hyperbolic plane.

The distance between two points $A, B \in \mathbb{H}^2$ is defined the following way: consider a circle/diameter orthogonal to the absolute passing through points A, B (it exists and unique); denote by X, Y the points of intersection with the absolute (see Figure 4); let $|AX|$ mean the arclength. Then, the distance between the points A, B is equal

$$d(A, B) = |\ln|\frac{|AX|}{|BX|} : \frac{|AY|}{|BY|}||.$$