The Flagg Resolution Revisited

by James Flagg and Louis H Kauffman

This essay would make a good companion to reading Hegel's discussion of the inadequacy of A=A and his subsequent definition of synthesis as both cancelling and preserving, which at the end of this essay is put in terms of the durability of identification, A = ~A.

I. Introduction

An early virtual logic column (Kauffman 1999) in Cybernetics and Human Knowing was devoted to the Flagg Resolution. This paper revisits the subject. The Flagg Resolution (FR) is a way to handle a paradoxical entity when it arises in mathematical language. Put in a nutshell, the FR avoids contradiction by modifying substitution to save the truth.

To see how this is done, suppose you find that $J = \sim J$ where $\sim$ denotes negation. We use $A \vee B$ to denote “A or B”. Consider the statement $S = J \vee \sim J$.

Since a statement is either True or it is False, we determine in standard logic that $J \vee \sim J$ is True. We have $J = \sim J$ and apparently we could substitute J for ~J and obtain

$$\text{True} = J \vee \sim J = J \vee J = J$$

and so J should be True. But we know that $J = \sim J$ and so if J is True then J is False. This implies that True = False, a paradoxical state of affairs!

Here is an imaginary conversation with James Flagg on the matter. Flagg is real. This conversation is imaginary.

LK. It seems that I cannot have an entity such that $J = \sim J$ without collapsing logic and creating serious inconsistency in the way we speak.

Flagg. Attend to it more carefully. Right now J is whatever J is. When you write $J \vee \sim J$ then at this moment if J is True then ~J is False, and if J is False, then ~J is True. So there is no contradiction. $J \vee \sim J$ is indeed always True. It cannot suddenly become False.

LK. So you are telling me that I cannot make an arbitrary substitution of J for ~J?

Flagg. Certainly not! I'll tell you what you can do. If you have an expression like $J \vee \sim J$, then you can make the substitution for both instances of J in the formula. This preserves the relationship between them.

LK. So I can write $J \vee \sim J = \sim J \vee \sim\sim J$. I see that this is innocuous. After all $\sim J \vee \sim\sim J = \sim J \vee J$, since $\sim\sim J = J$ and I am sure you will allow this substitution.

Flagg. That is correct. We have a paradoxical element $J = \sim J$ and the key substitution of $\sim J$ for J throughout, represents its fixity, or fixedness, if you will, under negation, truth, and indication (which here are a portmanteau ensemble). *This substitution must be performed for all appearances of J throughout a formula, salva veritate, as Leibniz (Leibniz 1677) would have it, or not at all. This dictum is the Flagg Resolution. I will say more about this in a separate section of our paper.*

The imaginary conversation between LK and Flagg will continue in the next section of the paper. We interrupt it here to indicate how the paper is organized. The sections of the paper are:

I. Introduction
II. Continuing the Conversation
III. The Formalism of Laws of Form
IV. James Flagg on the Flagg Resolution
V. Einstein/Penrose Tensor Networks and the FR.
VI. The Russell Paradox.
VII. Epilogue

The conversation in the Introduction continues in Section II. Section III provides more background on Laws of Form and its interpretation for logic. In Section III two new participants join the conversation. These are Cookie and Parabel, who are sentient text strings precariously existing very near the void. They have appeared in other writings of LK (Kauffman 2017) and add to the discussion by their very grounded insight into matters epistemological. Section IV is an insightful essay by James Flagg on the Flagg Resolution. Sections V and VI continue the discussions with Cookie and Parabel on analogs of the FR in tensor networks and knot theory, and in Section VI we discuss the Russell Paradox both from the point of view of the FR and from the point of view of Goedel, Bernays, von Neumann set theory (Mendelson 1997), where a Type distinction between Sets and Classes banishes the paradox. We compare that with the FR and its way of living with the paradox as a form of being. Section VII returns to the main theme of truth and substitution. By allowing the Flagg Resolution we can stand under logical circularlty and begin to understand the cybernetics of reason and the reason of cybernetics.

We dedicate this paper to George Spencer-Brown, Francisco Varela, David Van Cleave Lincicome, and John Ewell. And we wish to thank Providence for the many opportunities that have been provided for discussion and reflection on the issues brought forth in this paper.

II. Continuing the Conversation

LK. How do we know when an element is paradoxical? For example, I can have the famous Russell Set

$$R = \{ x \mid x \text{ is a set that is not a member of itself. } \}.$$

We discover a fixed point for negation in the form of the question: Is R a member of itself? According to the definition of R, R is a member of itself if and only if it is not a member of itself. The statement
J = "R is a member of itself." Has the property that if J is True, then J is False, and if J is False, then J is True. So we can apply the FR to J!

Flagg. And you are worried that perhaps R is not a well-formed set since it seems to flicker with respect to its own membership in itself.

LK. Yes. Perhaps we have to change other aspects of the discourse about sets in order to be sure there are no further contradictions.

Flagg. Ah! You have not got the whole picture. I do not attempt to create mathematics that is free of all contradiction. If there are contradictions there will occur instances of J with J = ~J. Whenever this happens, the J is treated specially with regard to its fixedness under negation. That is all we do. Otherwise life and logic go on as before. There are no new special logical values, and nothing is forbidden to the discussion.

LK. I want to analyze systems and formal systems to see if they are consistent. A consistent system cannot produce a J with J = ~J.

Flagg. After the discovery of FR, whose discovery emerged from correcting a misconception of the great Francisco Varela in his 'Calculus For Self Reference" regarding Spencer-Brown's

$$F = \sim F,$$

I never went back and revisited a single paradox, as I knew its resolute power. I knew it wiped out 2800 years of paradox. FR is what the ancient Egyptian and Vedic gods do. The only difference is that we now have an actual notation for their skullduggery. The ancient gods are living functions, or mythic functions, who fulfill all the requirements of Spencer-Brown's axiom of boundaries: a thing is what it is not. Even Wittgenstein said as much, when it was said by him, that a proposition is like a ruler laid against the side of reality: such and such is so and so ... but it is also, at the same time,   not...this or that. I paraphrase the great W.
And so, the Vedic gods, or mythic functions, sit at the boundary (Being) between Asat (Non-Existence) and Sat (Existence). For ancient Egypt, Zep Teti (The First Time) took place inside the everlasting living waters of Nu as Atum slumbered. He, in an act of supreme self-reference calls his own name and moves from one state to another. In Being, Sat and Asat lay side side, Nu and Atum move together. It's Jimi Hendrix: "Good and Evil lay side by side, while electric love penetrates the sky." It's not new, but old: implicate order, wholeness, Tao.

Once anyone thinks about God, or Love, they exist. So, we now understand why the negative postulation "Don't think about elephants.” doesn't work. Any negative postulation posits its counterpart and complement. This is the source of all paradox.

LK. So in FR, “both” is neither a superposition nor a contradiction, it just is.

Flagg. Yes. It is common enough in everyday life. The most interesting entities are round squares. They should not be banished. And what do you do when you meet them in mathematics. You declare new worlds! The square root of negative unity is both +1 and -1 and eventually mathematicians learned to use it by endowing it with the symbol I, and explaining that ii = -1 and i stood at right angles to the line of real numbers. The square root of minus one crossed the boundary between number and algebra and became a rotation of ninety degrees. This is a specific action of FR. We shall see more of this.

LK. Consider the sentence S that says "If this sentence is True, then unicorns can fly." We see that if S is False then it would be of the form "False implies unicorns can fly." and since an implication of the form "False implies P" is necessarily True, it would follow that if S is False then S is True. So S cannot be false. Therefore S is true and hence unicorns can fly.

Flagg. That is an admirable proof that unicorns can fly.

LK. But I could have also used the same form of proof to show that unicorns cannot fly!

Flagg. Indeed you could. Does this worry you?

L.K. Of course it does. I can use this method to prove any statement that I like. It renders reasoning inconsistent at all levels.

Flagg. Of course you are making a mistake. "A implies B" = ~A $\vee$ B as we all know. (When A is true then B follows.) So your statement S is of the form

$$S = \sim S \vee U$$

where U is the Unicorn Statement. You are trying to avoid S = ~S but you cannot do that without being inconsistent. It was by assuming that S and ~S are distinct that you found yourself concluding the Unicorn Statement. You don't want that and so you must accept that S = ~S. Then there is no problem. Indeed if S is False then S is True, and if S is True , then S is False. Just apply FR in using S.

LK. I shall have to think about this.

III. The Formalism of Laws of Form

In thinking further about the Flagg resolution it is useful to have the formalism of “Laws of Form” by G. Spencer-Brown (Spencer-Brown 1969) . In this formalism we have a mark, $\rceil$, that represents the distinction that this mark makes between its inside and its outside. We may think of the mark as referent to a single distinction that is given and sometimes called “the first distinction.” The mark is also seen as a transformation from the state indicated on its inside to the state indicated on its outside. Thus the empty mark we have drawn above can be seen as a transformation from the unmarked state on its inside to the marked state on its outside. We can further notate this by $\overline{u}\rceil = m$ where u denote the interior unmarked state and m denotes the exterior marked state. By keeping the language as parsimonious as possible, we allow u to be replaced by nothing since u denotes the unmarked state. Thus we can rewrite $\overline{u}\rceil = m$ as $\rceil = m$, and so we see that the symbol for the marked state can indeed be taken to be the mark itself. We agree that “The value of a name called again is the value of the name.” (Spencer Brown’s Law of Calling) and so we can write mm = m and uu= u since each of these constitutes a repetition of a name. Then mm = m becomes $\rceil\rceil = \rceil$ and this equation is the version of the Law of Calling that is expressed by the mark. Each mark can be seen as the name of the other mark. Each mark can be seen as the state resulting from crossing from the unmarked state.

If we iterate the transformation we see that crossing from the marked state results in the unmarked state: $\overline{m}\rceil = u$. This follows from the fact that we are describing the two and only two states related to a first distinction. One can cross from the unmarked state to the marked state and crossing from the marked state yields the unmarked state. What is not marked is unmarked. Now, unpacking the equation above and using the mark for m and nothing for u, we find $\overline{\rceil}\rceil = \quad$. This is the Law of Crossing, “The value of a crossing made again is not the value of the crossing.”, written in terms of the mark. In this way, consideration of one distinction leads to a calculus of indications, a language using only the one mark and this language has the two basic rules:

$$\rceil\rceil = \rceil,$$
$$\overline{\rceil}\rceil = \quad .$$

More complex expressions in the mark reduce uniquely to either the marked state or the unmarked state. For example, $\overline{\rceil\rceil}\rceil\rceil = \overline{\rceil}\rceil\rceil = \rceil$.

The reader may wonder, what does it mean to have two expressions E and F, standing next to one another as in EF or $\overline{\overline{}\rvert}\rvert\overline{}\rvert$? In this calculus of indications each expression stands either for the marked state or the unmarked state. We have that mm = m and uu = u. What about um? Since u can be replaced by nothing (it represents the unmarked state), we have that um = m and mu = m. Thus m is dominant and it now makes sense that $\overline{\overline{}\rvert}\rvert\overline{}\rvert = \overline{}\rvert$. One can, in fact evaluate an expression by letting its deepest (empty) spaces be unmarked and then propagating state evaluations upward to the top of the expression. For example:

$\overline{\overline{\overline{\overline{\overline{\overline{\overline{u}\rvert m}\rvert u}\rvert m}\rvert u}\rvert m}\rvert u}\rvert m$ shows that $\overline{\overline{\overline{\overline{\overline{\overline{\overline{}\rvert}\rvert}\rvert}\rvert}\rvert}\rvert}\rvert = \overline{}\rvert$. The attentive reader may remark that a nest of seven marks is marked,and indeed a nest of an odd number of marks will be marked.

We can then write algebraic expressions such as $P\overline{P}\rvert$ and ask how they will evaluate. Since P is either marked or unmarked we find here two possibilities: $\overline{u}\rvert u = \overline{}\rvert$ or $\overline{m}\rvert m = \overline{\overline{}\rvert}\rvert\overline{}\rvert = \overline{}\rvert$. Thus we can assert that $P\overline{P}\rvert = \overline{}\rvert$ in the *primary algebra.*

We are now in a position to show the translation of this algebra for logic. We can write ab for “a or b” because the algebraic expression ab is marked exactly when either a is marked or b is marked or both a and b are marked. In symbolic logic one writes $a \vee b$ for “a or b”. Thus we can state that $a \vee b = ab$ in the Primary algebra. Similarly $a \wedge b$ is the standard notation for “a and b”. We can write $a \wedge b = \overline{\overline{a}\rvert\overline{b}\rvert}\rvert$. Note that the only way that $\overline{\overline{a}\rvert\overline{b}\rvert}\rvert$ can be marked is if both a and b are marked! Finally it turns out that $\overline{a}\rvert b$ represents “a implies b” when we take $\overline{}\rvert = True$ and $\overline{\overline{}\rvert}\rvert = False$. There is more to say in this domain, but we stop here with the interpretation for logic.

On the other hand consider

$$J = \overline{\overline{\overline{\overline{\overline{\overline{\ldots}\rvert}\rvert}\rvert}\rvert}\rvert}\rvert = \overline{J}\rvert.$$

Here we see an infinite nest of marks and we note that it appears literally underneath its own outermost mark. The infinite nest of marks re-enters its own indicational space and satisfies the equation $J = \overline{J}\rvert$. Since J has no deepest space, there is no way to evaluate it as marked or unmarked. J is neither marked nor is J unmarked. What shall we do with $J\overline{J}\rvert$? If we wish to say that $J\overline{J|} = \overline{|}$, then we should go back to J and ask in what sense J could be marked or unmarked. Look at this state of J:

$$J = \overline{\overline{\overline{\overline{\overline{\overline{\overline{\ldots|}\,m|}\,u|}\,m|}\,u|}\,m|}\,u|}\,m\,.$$

We see that it is possible to imagine J as marked, but it is also possible to imagine J as unmarked using the same procedure, labeling an infinity of spaces downward from the outside of the expression. Furthermore, if we take J as above, then

$$\overline{J|} = \overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\ldots|}\,m|}\,u|}\,m|}\,u|}\,m|}\,u|}\,m|} = \overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\ldots|}\,m|}\,u|}\,m|}\,u|}\,m|}\,u|}\,m|}\,u\,.$$

And so we see that if J is marked then $\overline{J|}$ is unmarked and indeed $J\overline{J|} = \overline{|}$ Just so long as we understand that *the two J's in this equation are identical*. This is the Flagg Resolution. We can incorporate the re-entering mark into the primary algebra with the understanding that it is an entity that can be either marked or unmarked and that J must be treated according to the FR.

Back to the dialogue, but now we include two new participants. Cookie and Parabel are sentient text strings who comment on all these matters from a somewhat different point of view. They are significantly closer to the void than either LK or Flagg.

Cookie. I am worrying about your infinite nest of marks. You say that if

$$J = \overline{\overline{\overline{\overline{\overline{\ldots|}|}|}|}|}\,,$$

then $J = \overline{J|}$. But I have written

$$J\overline{J|} = \overline{\overline{\overline{\overline{\overline{\ldots|}|}|}|}|}\;\overline{\overline{\overline{\overline{\overline{\overline{\ldots|}|}|}|}|}|}\,,$$

and I see clearly that by juxtaposing J and $\overline{J|}$,
it is clear that $\overline{J|}$ is not equal to J. It is "taller" by one mark!

Parabel. Do you see the three dots deep inside J and $\overline{J|}$? This means that each of these expressions is infinite.

Cookie. Well, to me J is seven nested marks with three dots down in the bottom, while $\overline{J|}$ is eight nested marks with three dots down in the bottom. So you see, $\overline{J|}$ stands one mark taller than J.

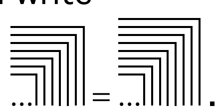
Parabel. But adding one more mark to an infinite nest of marks does not change it. =“idempotency”

Cookie. Why not?

Parabel. You can match them up. Its like this. Suppose I define a correspondence of the natural numbers to themselves by the function F(n) = n +1. Then F(1) = 2, F(2) = 3, ... and I can make a 1-1 correspondence between the set {1,2,3,...} and the set {2,3,4,...}.
Cookie. The way I see it is like this

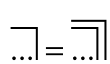
{1,2,3,4,5,6,7,...}
{2,3,4,5,6,7,8,...}

You correspond 1 to 2, 2 to 3, 3 to 4, 4 to 5, 5 to 6, 6 to 7, 7 to 8 and ... to ... . Easy! Each set has eight elements (letting ... be one element). Nothing strange here and no infinities.
I appreciate that you can fantasize about infinities as much as a string can fantasize, but these nests of marks you have written are finite.
It is just not true that you can write

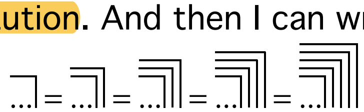


Anyone can see they are not equal.
One has seven marks and three dots. The other has eight marks and three dots. Seven is not equal to eight. I may be just a string, but I know that!

Parabel. You are right. One can fantasize about a nest of marks that goes down forever, but the logical syntax of such notations is based on an equivalence relation. I can write



and this is not a literal equality. It is the generator of an equivalence relation, a rule for substitution. And then I can write



by using this rule of substitution again and again.

Cookie. Oh. You are doing recursion by substitution. Every string knows about this. But wait a moment. Suppose I write the string

$$1 + x + x^2 + x^3 + \ldots$$

Then I could say that

$$1 + x + x^2 + x^3 + \ldots = 1 + x + x^2 + x^3 + x^4 + \ldots$$

In fact the recursive rule is now that

$$x^n + \ldots = x^n + x^{n+1} + \ldots.$$

It is a different rule!

You can't just substitute $\overline{\ldots}\rceil = \overline{\overline{\ldots}\rceil}\rceil$ when doing this algebra.

Parabel. You are right. We handle the ellipsis ... just as we handle a paradoxical element J in the Flagg resolution. The substitutions that are allowed for the ellipsis ... are context dependent, and we learn to behave correctly with respect to the ellipsis ... as we learn to use mathematical language.

Cookie. So the Flagg Resolution has been around all along!

Parabel. Yes it has, but James Flagg was surely the first person to become conscious of the FR. That was a giant step for mathematical consciousness.

Cookie. I have heard these peculiar words: The form re-enters its own indicational space. I think earlier you were trying to convince me that If $J = \overline{\overline{\overline{\ldots}\rceil}\rceil}\rceil$, then $J = \overline{J}\rceil$.

Parabel. Yes! And now we can see this on our own grounds. You have agreed that $\overline{\overline{\overline{\ldots}\rceil}\rceil}\rceil = \overline{\overline{\overline{\overline{\ldots}\rceil}\rceil}\rceil}\rceil$,and so, with $J = \overline{\overline{\overline{\ldots}\rceil}\rceil}\rceil$, we have to simply substitute J into that formula and obtain $J = \overline{J}\rceil$.

Cookie. It is not literally so unless we use the generating property of the ellipsis. I have heard that some of our readers have a fantasy that the nest of marks with an ellipsis at the bottom "means" an endless descent of marks. From the point of view of a string this is complete nonsense. Nothing written goes on forever. I am glad that you explained the generative property of the ellipsis so that all this is clear to a simple string such as myself.

Parabel. I have noticed something curious. This use of the ellipsis is actually the same as the use of the reentry turn in the reentering mark. Look at this glyph in Figure 0.

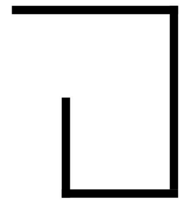


Figure 0. The Re-entering Mark.

The in-turning line indicates where the form is to be placed inside itself. To make matters clear, let me make a notation that we can use for our discussion. I will write as in Figure 0' for the Re-entering Mark using a "Hat" $\wedge$ to indicate the point of re-entry.

Figure 0'. Our notation for the Re-entering Mark.

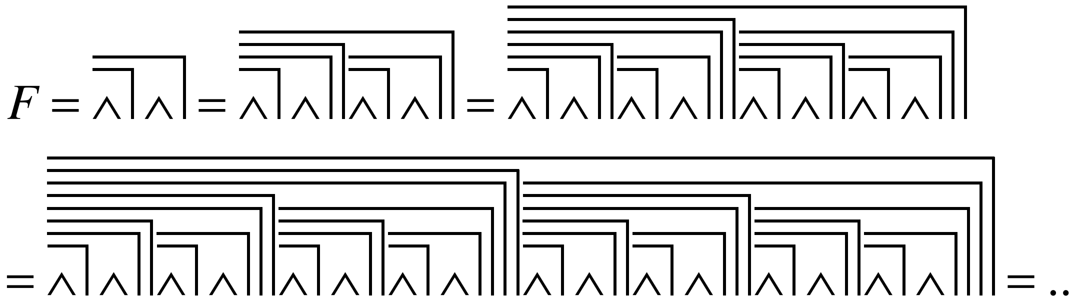
This means that we can write that if $J = \overline{\wedge\,}|$, then $J = \overline{J\,}|$, and indeed

$$\overline{\wedge\,}| = \overline{\overline{\wedge\,}|\,}|.$$

But now you see that the "hat" ^ behaves just like the ellipsis except that it is specialized for re-entry.

Cookie. That helps even more. Those readers who think that they are having infinity with the ellipsis in a really different way than they are having infinity with the re-entry symbol can now understand that there is no difference at the base of things in our string world.

Parabel. We can make other re-entry forms this way. For example, I like $F = \overline{\overline{\wedge\,}|\wedge\,}|$ which implies that $F = \overline{\overline{F\,}|F\,}|$. This is the Fibonacci Form because as you keep substituting it into itself it produces an architecture with Fibonacci numbers of divisions at successive depths of the form.
We can write for example:



and keep going as far as we like.

Cookie. That looks like it was produced by the Mad Hatter! If you see a re-entry form, you know from its Hats that it has to be handled with the Kid Gloves of the Flagg Resolution.

IV. James Flagg on the Flagg Resolution

Since the first publication of Laws of Form, it is well known that indication cuts deeper than truth. Truth is a token of indication. In the form of indication we can treat self-reference as a function of self-indication.

In so doing, Spencer-Brown was able to discard Russell and Whitehead's Theory of Types, and uncover - from within the original paradoxes themselves - expressions of higher degree, expressions concealed by the semantic, syntactical and alethic clothing of the paradoxes themselves, shorn of theory.

Varela (Varela 1975) , in reviewing the application of self-indicative expressions in autonomous “autopoietic” domains, conceived self-indicative expressions as a "departure from the calculus of indications proper, into re-entering forms...(as)...not without its difficulties which render the treatment of higher degree equations, as it now stands, in need of revision. Spencer-Brown's claims that "it is evident that (the two algebraic initials) hold for all equations, whatever their degree." “

Thereafter, Varela considers " the simple second degree equation"

$$F = \overline{F}| \qquad (1)$$

with respect to the first algebraic initial, J1

$\overline{\overline{P|}P|} =$ (J1)

Which he rewrites as

$\overline{P|}P = \overline{\ \ |}$,

which is proper, since in this form "all the relevant properties of the point p in Figure 1 (LOF, Chapter 11) appear in two successive spaces of expression. In this case the superimposition of the two square waves in the outer space, one of them inverted by the cross, would add up to a continuous representation of the marked state there" as in Varela's J1.

Varela goes on to say, "and hence, if Spencer-Brown is right, we have

$\overline{\ \ |} = \overline{F|}F$ (J1)
$= FF$ (by 1 and substitution)
$= F$ (C1, C3).
"But" he adds, "this is clearly untrue, since replacing in (1) we obtain

$\overline{\overline{\ \ |}|} = \overline{\ \ |}$

contrary to J1 itself. So, by allowing re-entry we lose connection with both the arithmetic and the algebra."

However, Varela could have remained in connection with both the arithmetic and the algebra if he had conceived

$F = \overline{F|}$ (1) as a formal application of re-entry to itself. Which is to say, if an act of self-reference gets us into paradox, then a second act of self-reference reapplied to the first will get us out.

Take $F = \overline{F|}$ as a formal definition of a second act of self-reference in the form, of substitution of the (formal) defiendum for the (formal) definiens. It doesn't matter which is taken as which, so long as *this substitution is performed for all appearances of J throughout a formula or not at all.* This is the formal statement of the Flagg Resolution (FR).

Thus, in $F = \overline{F|}$, F has been re-defined as $\overline{F|}$, so for every occurrence of F we must substitute $\overline{F|}$ throughout, salva veritate.

Varela's J1 is $\overline{\ \ |} = \overline{F|}F$, and is transformed by FR substitution to $\overline{\ \ |} = \overline{\overline{F|}|}\,\overline{F|} = F\overline{F|}$

thus preventing the collapse (by substitution) of J1 into $\rceil = FF$, and F.

A second pass of the resolution returns

$\rceil = \overline{F}\rceil F$ to $\rceil = F\overline{F}\rceil = \overline{F}\rceil F$

which accords with Spencer-Brown's statement regarding two successive spaces of expression of all the relevant properties of the point F.

Furthermore, from

$\rceil = \overline{F}\rceil F$

we may set J1 to either the marked state, as Varela did, or the unmarked state as Spencer-Brown does in as the first initial of the primary algebra. All we need do is apply the dictum of the resolution once again, what we do locally, we do globally, and from

$\overline{P}\rceil P = \rceil$

we obtain $\overline{\overline{P}\rceil P}\rceil = \overline{\rceil}\rceil = \quad$.

We are now back in the primary algebra, and need only recall J2 to once again fully avail ourselves of its powers.

The reader should note that the identity $F = \overline{F}\rceil$ can be seen as asserting that any given form is identical with what it is not. This principle of identity is entirely coherent with the primary algebra via the Flagg Resolution.

The key philosophical precursor to the Flagg Resolution is the statement of Leibniz (Leibniz 1677) on intersubstitutivity: “salva veritate, two expressions may be interchanged, without loss of generality, or altering the value, or truth of the constituent statements in which the expressions occur, if they satisfy the condition that they *are the same if one can be substituted for the other without altering the truth of any (constituent) statement*.”

V. Einstein/Penrose Tensor Networks and the FR.

Cookie. Are there other examples of implicit FR in ordinary mathematics?

Parabel. Here is an example. Do you remember the Einstein Summation Convention?

Cookie. That is the one thing that Einstein did that I do understand. Einstein said that we could write $G = T_{ab} S^a R^b$ and it would be understood that this quantity is obtained by summing over the instances of the repeated indices. Thus if a and b can range over the values 1 and 2, then

$$G = T_{11}S^1R^1 + = T_{12}S^1R^2 + T_{21}S^2R^1 + T_{22}S^2R^2.$$

In order for the value of G to make sense we do have some leeway in handling the indices. For example, we can write

$$T_{ab} S^a R^b = T_{ij} S^i R^j$$

but we must follow the rules:

1. Since a and b are distinct, a new variable substituted for a must be distinct from b. And a new variable substituted for b must be distinct from a.
2. If a new variable is to replace x, then it must replace all instances of x in the given formula.

It should be clear that these rules are needed in order to insure that the summation the condensed formula represents does not change when we change the indices. The rules are another version of the FR. Note particularly that the index a may appear in many places in a formula and it has the same property as our J in the first example of FR: If you change it anywhere then you must change it everywhere in the same way.

Cookie. That is a wonderful example! I have a hobby that is regarded as a bit racy by some strings. I like to rewrite Einstein expressions in diagrammatic language. For example look at Figure 1 just below my present string. In that figure you see that T_{ab} is a circular blob with two legs labeled a and b. And S^a and S^b are each circular blobs with antennae labeled a and b. The antennae of S^a and S^b are connected directly to the legs of T_{ab}. Thus in the diagrammatic there is one connecting line labeled a and another connecting line labeled b. It is clear that if we were to change the label a to a' it would have to be done for the whole line, and so in the algebraic expression, both a's would become a'. Here we see that the FR is working because there really is only one a and one b. In fact,in the Figure we have shown how the structure of the T and the two S blobs is actually standing independent of any particular labeling. The labeling is a way to communicate the connections in the structure of the diagram. This has many implications for numerous subjects where diagrams and interconnections are important to articulate. My favorite paper on this subject is by Roger Penrose (Penrose 1971).

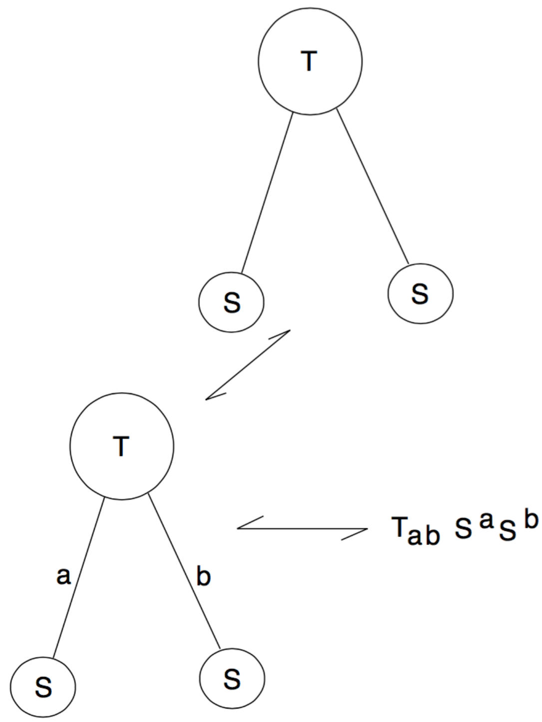
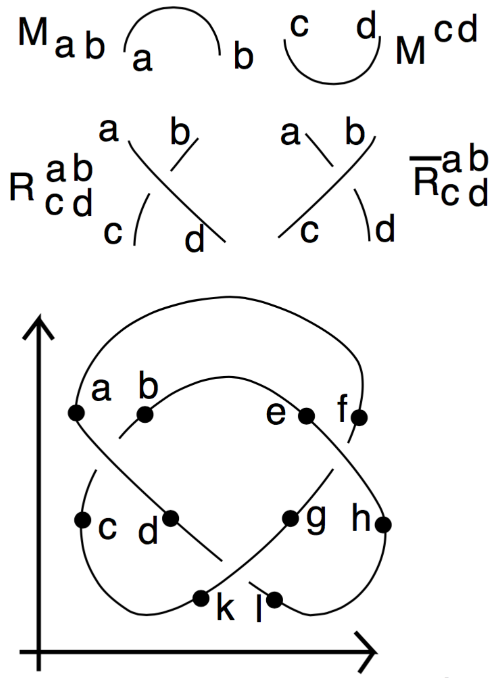


Figure 1 – Diagrammatic Einstein Summation Convention

Parabel. That is the longest speech I have ever heard you give Cookie. I can see why some strings are unhappy with your views. You have left the line for the plane. I myself am a bit leery of such excursions, but I think you are completely sound in what you say!

Cookie. I am so enthusiastic about this! Look Parabel at Figure 2. There you see the small diagrams Cup = M^{cd}, Cap= M_{ab} and crossings $R^{ab}{}_{cd}$, each with their own indices, and how they can be composed into the big tensor Z_K that represents a knot (Kauffman 2012)! Just think about this. A string Z_K can precisely represent a knot in three-dimensional space. We strings are not really restricted to one dimension any more. All we need is the correct use of Flagg Resolution for the indices of our Einstein tensors that represent these higher dimensional structures. Notice that all we need, to follow the Flagg Resolution Einstein convention, is the proper use of upper and lower indices, a tradition of long standing in the string-world.



$$Z_K = M_{af} M_{be} M^{ck} M^{lh} R^{ab}_{cd} R^{ef}_{gh} \overline{R}^{dg}_{kl}$$

Figure 2. Knotted String

Parabel. Cookie, I believe that you have invented knotted strings!

Cookie. Not bad for a string to invent knotted strings.

VI. The Russell Paradox

Cookie. Can we look at the Russell Paradox?

Parabel. Certainly. I prefer to use string notation. Do you mind?

Cookie. Not at all. You use Sx to mean “x is a member of S”.
Is that what you like?

Parabel. Exactly! The Russell set is the set of all sets x that are not members of themselves. I write it as

$$Rx = \sim xx,$$

and in LOF notation I write the Russell set R as

$$Rx = \overline{xx}\rceil$$

where it is understood that xx means “x is a member of x”.

Cookie. And then we have the paradox of “Could R be a member of R?”.

Parabel. In our notation we have $RR = \overline{RR}\rceil$ and so R is a member of R exactly when R is not a member of R. This is a clear case for the Flagg Resolution. We just treat RR with kid gloves and never change it except globally. With the new rule of substitution we do not have a problem here. The Flagg resolution says: Yes we can have R a member of R or R not a member of R, but when you shift from one stance to the other it is a global shift. The two states do not occur together. There is no contradiction.

Cookie. This is very simple. How did the old time set theorists handle the problem?

Parabel. Well you know that Russell and Whitehead, in their monumental treatise Principia Mathematica, handled the paradox by using a theory of types. I will not go into this, but the set theorists who came after used a highly simplified type theory where there were two Types of Collections. This is the Goedel – Bernays – von Neumann Set Theory (GBN) (Mendelson 1997). In their theory a collection is either a Set or it is a Class (sometimes called Proper Class). Sets have sets for members and every set is a member of another set. For example if S is a set then S is a member of the singleton {S}. Classes have members that are sets, but *no class is a member of a set.* Nor is a class a member of a class, since all members of a class are sets.

Cookie. I think I see how it works. We take the collection R of all sets that are not members of themselves. Then we ask, is R a member of R? If it were we would get the contradiction. So we conclude that R cannot be a set. Therefore R is a class. R is the class of all sets that are not members of themselves. The contradiction is gone!

Parabel. I think this GBN set theory that I have been describing to you is very like the Flagg Resolution. One determines that R is not a set and hence a class. Once R is a class, we cannot form {R} or other ways to treat R as anything but singular. It is not the same as changing all instances of R at once. We could try an FR where we can have RR or ~RR but if we write T = RR v ~RR we cannot substitute RR for ~RR and conclude that T = RR V RR = RR and then conclude that T = ~T. This will be forbidden directly by the FR. The GBN resolution by sets and classes is a separate move that removes the paradoxical element completely. It is like putting an imaginary value in a circuit transition to keep it from being oscillatory.

Cookie. Can you explain that?

Parabel. Why yes I can. Consider the circuit in Figure 3.

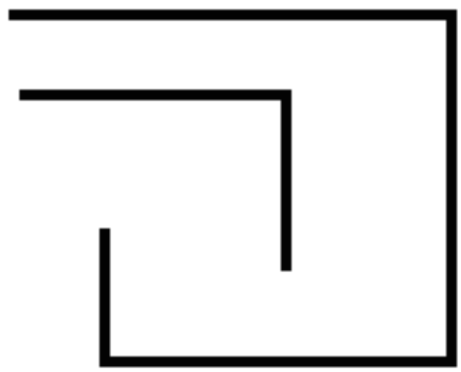


Figure 3. A Two-State Circuit

If the values in the two inner spaces of this form are both unmarked and we let the circuit perform, then if the inner mark fires first we get an unmarked state on the outside and a stable form, while if the outer mark fires first we get a marked state on the outside and a stable form. This re-entrant form has an ambiguous transition to its next stable state. In the case of the circuit it is possible to add an extra feedback that influences this transition and makes it determinate. This idea is analogous to replacing $Rx = \overline{xx}\rceil$ with

$$Rx = \overline{xx}\rceil \wedge Sx$$

where $\wedge$ denotes "and". *R now states that its members are not members of themselves and they are sets.*

Cookie. I could rewrite it as

$$Rx = \overline{xx}\rceil \wedge Sx = \overline{\overline{\overline{xx}\rceil}\rceil \overline{\overline{Sx}\rceil}\rceil}\rceil = \overline{xx\overline{Sx}\rceil}\rceil .$$

Thus

$$Rx = \overline{xx\overline{Sx}\rceil}\rceil .$$

Then

$$RR = \overline{RR\overline{SR}\rceil}\rceil$$

and if SR is unmarked (R is not a set, whence a class) then $RR = \overline{RR\,\overline{\;\;}\rceil}\rceil = \overline{\overline{\;\;}\rceil}\rceil$ and so it is simply the case that R is not a member of itself when R is not a set. If R is a set we obtain a classical contradiction. Since we assume in this classical set theory that it is free of contradiction, it follows that R is a class and not a set. R is a class and cannot be a member of any set or class. *The paradox is resolved by the extra observer in the circuit.* We simply do not get to consider the paradoxical expression $RR = \overline{RR}\rceil$. This is how the Goedel, Bernays, von Neumann Set Theory resolves the Russell Paradox. It uses the types of Set and Class to banish the paradoxical statement from every appearing in the language of the theory.

Parabel. We could have done this to resolve any paradoxes. If we have $P = \overline{P}\rceil$ we replace it by $P = \overline{P\overline{SP}\rceil}\rceil$ and then SP =True gives a contradiction, so SP must be false. That throws P in the $\overline{S}\rceil$ category and essentially into jail! Do not pass Go, go directly to Jail. This is the GBN solution placed in the simple context that a string can understand.

Cookie. The GBN seems very artificial from the point of view of a simple string. The notion of simultaneous replacement in FR is an exciting new dimension for a string.

Parabel. One small step for a string. But a giant step for stringkind.
I hope you are not stringing me along with this flattery.

Cookie. On that note we have to vanish into the void.
But maybe we can reappear in another publication.

Parabel. Where else do you find strings of symbols?
It is publish or perish for us. Either we are in print or we are not. Cookie?

Cookie.

Parabel. Gone again into the void.

VII. Epilogue

This essay on the Flagg Resolution has been an essay on the role of substitution in mathematics. When we write A=B where A and B are in fact different in some way, then one must take care to see just what it is that makes them equal and what it is that makes them different. When we have an apparent logical paradox in the form $J = \overline{J|}$, the most careful way to handle the substitution is to regard all appearances of J in a given expression as identical and let them change together or not at all. When we break a whole into parts and give remote labels on the parts to enable the reconstruction of the whole, these multiple labels for the same place must be regarded as identical. They can be changed together or not at all. In this way the whole and its parts can be identical in the form. Unity and diversity can coexist and circularity can take its proper place in logic, reason and structural understanding.

idem > identity > identification = "durability"

durability is different from the things that are declared durable by means of identity

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