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Non-parametric Tests of Consumer Behaviour

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This paper shows how to test demand data for consistency with maximization, homotheticity, various forms of separability, and a rationing model without making any assumptions concerning the parametric form of underlying demand or utility functions.

The neoclassical model of consumer behaviour postulates that a consumer's choice behaviour can be described as deriving from utility maximization subject to a budget constraint. One is then naturally led to ask what this model implies about observed behaviour. This question has been addressed from two quite distinct approaches. The first approach, originating in the work of Slutsky (1915) and Antonelli (1886), derives necessary and sufficient conditions involving the derivatives of the demand functions.

The second approach, originating in the work of Samuelson (1938), (1947), (1948), derives algebraic conditions on the demand functions implied by maximizing behaviour. These conditions, known as "revealed preference" conditions, provide a complete list of the restrictions imposed by maximizing behaviour in the sense that every maximizing consumer's demand behaviour must satisfy these conditions and all behaviour that satisfies these conditions can be viewed as maximizing behaviour.

In many cases, one may wish to consider the consumer's utility function as having a specialized form; for example one might postulate that in a particular case the utility function is homogeneous of degree 1, or separable in some way. One can then ask what restriction such hypotheses place on observed behaviour. For the most part investigations in this area have restricted themselves to the first approach described above: they have derived derivative type conditions on demand functions that are implied by particular forms a utility function might take. In this paper we pursue the second approach: what algebraic, revealed preference-type conditions are imposed by restrictions on the form of the utility functions? Other papers that take this approach are those of Afriat (1967a), (1967b), (1972), (1973), (1976), (1977), (1981), Diewert (1973) and Diewert and Parkan (1978). (Afriat (1972), Diewert and Parkan (1979), Hanoch and Rothschild (1972) and Varian (1981a), apply similar techniques in a production context.)

The distinction between the two approaches is very important in empirical work. The calculus approach assumes the entire demand function is available for analysis, while the algebraic approach assumes only a finite number of observations on consumer behaviour is available. Since all existing data on consumer behaviour does consist of a finite number of observations, the latter assumption is much more realistic. Indeed the calculus approach can only proceed by assuming that demand behaviour can be adequately described by some parametric family of functional forms; one can then estimate the parameters that best describe the data by various statistical techniques and test for the restrictions imposed by the particular hypothesis one has in mind. This procedure suffers from the defect that one is always testing a joint hypothesis: whatever restrictions one wants to test *plus* the maintained hypothesis of functional form.

The revealed preference approach on the other hand is *non-parametric*: it provides a complete test of the hypothesis in question alone, with no additional assumptions concerning functional form. For this reason I would argue that the revealed preference approach is to be much preferred.

However, it must be added that the revealed preference approach does have some drawbacks. In certain cases, the tests involved may be computationally infeasible for large data sets. Also, the techniques do not typically summarize the data in a useful way. Furthermore it may be rather difficult to incorporate stochastic considerations in a satisfactory manner. (However see Varian (1981*b*).)

The general method for constructing non-parametric tests for models of consumer behaviour was first discovered by Afriat (1967) and applied to the question of utility maximization. He subsequently applied this method to tests of separability (Afriat (1967*b*)) and homotheticity (Afriat (1981)). I was unacquainted with the latter two works when I wrote the first version of this paper and wish to thank Professor Afriat for providing me with copies of these papers. Most of the tests described in this paper can be found in these much earlier papers by Afriat.

However our treatments differ in some ways. In particular I use a somewhat different proof for Theorem 2, and a different algorithm to check the condition in this theorem. Statement (3) in Theorem 3 seems to be new as does the proof of Theorem 4. Theorem 5 and 6 are also new although they are rather direct applications of the Afriat methods. Afriat's approach to these tests will be described in his forthcoming book, Afriat (1983).

Diewert and Parkan (1978) have applied some of Afriat's methods to actual demand data. However, the versions of the theorems and the computational techniques they used do not appear to be as easy to apply as those described in this paper.

1. TESTING THE MAXIMIZATION HYPOTHESIS

Suppose that we are given a finite number of observations on k -vectors of prices and quantities, $(p^i, x^i) \ i = 1, \dots, n$. When is this data consistent with the utility maximization model?

Definition 1. A utility function $u(x)$ rationalizes the data $(p^i, x^i) \ i = 1, \dots, n$ if $u(x^i) \geq u(x)$ for all x such that $p^i x^i \geq p^i x$, for $i \geq 1, \dots, n$.

Definition 2. An observation x^i is directly revealed preferred to a bundle x , written $x^i R^0 x$, if $p^i x^i \geq p^i x$. An observation x^i is revealed preferred to a bundle x , written $x^i R x$, if there is some sequence of bundles $(x^j, x^k, \dots, x^l)$ such that $x^i R^0 x^j, x^j R^0 x^k, \dots, x^l R^0 x$. In this case we say R is the transitive closure of the relation R^0 .

Definition 3. The data satisfies the Generalized Axiom of Revealed Preference (GARP) if $x^i R x^j$ implies $p^j x^j \leq p^j x^i$.

Afriat's Theorem. The following conditions are equivalent:

- (1) there exists a non-satiated utility function that rationalizes the data;
- (2) the data satisfies GARP;
- (3) there exists numbers $U^i, \lambda^i > 0 \ i = 1, \dots, n$ that satisfy the Afriat inequalities: $U^i \leq U^j + \lambda^j p^j (x^i - x^j)$ for $i, j = 1, \dots, n$;
- (4) there exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.

Afriat's theorem with a different but equivalent version of condition (2) was first proved by Afriat (1967*a*). Diewert (1973) provided a somewhat different proof, omitting consideration of condition (2). Varian (1982) introduced condition (2), GARP, to replace

a more unweildy condition Afriat called “cyclical consistency”. GARP is to be preferred to Afriat’s cyclical consistency since it is much easier to test in practice. This is discussed in detail in Varian (1982).

As this theorem serves as a general model for the tests described subsequently it is worthwhile to provide a sketch of the proof. A complete proof can be found in Afriat (1976), Diewert (1973) or Varian (1982). The motivation I will describe was suggested by Afriat (1967a) and Diewert and Parkan (1978).

Let us suppose that the data in question were generated by a non-satiated differentiable, concave utility function $u(x)$. Then we know from the standard properties of concave functions that for all x and y ,

$$u(x) \leq u(y) + Du(y)(x - y). \quad (1)$$

In particular for the data x^i $i = 1, \dots, n$ we have:

$$u(x^i) \leq u(x^j) + Du(x^j)(x^i - x^j) \quad i, j = 1, \dots, n. \quad (2)$$

Furthermore the hypothesis of utility maximization implies that the usual first order conditions must be satisfied by the data:

$$Du(x^i) = \lambda^i p^i \quad i = 1, \dots, n \quad (3)$$

with $\lambda^i > 0$ under the non-satiation hypothesis. Substituting (3) into (2) we have:

$$u(x^i) \leq u(x^j) + \lambda^j p^j (x^i - x^j) \quad i, j = 1, \dots, n. \quad (4)$$

This is of the same form as the condition (3) of the theorem; hence maximization of utility implies that the numbers $U^i = u(x^i)$ and λ^i exist: the U^i ’s can be interpreted as utility levels and the λ^i ’s can be interpreted as marginal utilities of income at the various levels of observed consumption.

Let us now prove the converse assertion; if we have numbers that satisfy conditions (3) of the theorem—the “Afriat inequalities”—then we can find a non-satiated, concave, monotonic utility function that rationalizes the data. The trick is to note that the concavity conditions give us directly n overestimates of the utility of an arbitrary bundle x , since

$$U(x) \leq U^i + \lambda^i p^i (x - x^i) \quad i = 1, \dots, n. \quad (5)$$

So we simply choose the minimum of these overestimates; that is, we choose as our utility function the lower envelope of the hyperplanes given in (5):

$$U(x) = \min_i \{U^i + \lambda^i p^i (x - x^i)\}. \quad (6)$$

Let us verify that this construction works. We must show that given any x with $p^i x^j \geq p^j x$ we have $U(x^j) \geq U(x)$.

First we show $U(x^i) = U^i$. We have that for some m :

$$U(x^i) = U^m + \lambda^m p^m (x^i - x^m) \leq U^i + \lambda^i p^i (x^i - x^i) = U^i. \quad (7)$$

If the inequality were strict, we would violate one of the Afriat inequalities. Hence $U(x^i) = U^i$ for $i = 1, \dots, n$.

Now we suppose that we have some x such that $p^i x^j \geq p^j x$. Then we have

$$U(x) = \min_i \{U^i + \lambda^i p^i (x - x^i)\} \leq U^j + \lambda^j p^j (x - x^j) \leq U^j = U(x^j). \quad (8)$$

Hence $U(x)$ rationalizes the data. Note that since $U(x)$ is a piecewise linear function it is straightforward to show that it is non-satiated, monotonic and concave.

Let us summarize the above discussion in the following statements:

(a) If the data can be rationalized by a differentiable, concave, monotonic non-satiated utility function then there must exist numbers $U^i > 0$ and $\lambda^i > 0$ that satisfy the

Afriat inequalities:

$$U^i \leq U^j + \lambda^j p^j (x^i - x^j) \quad i, j = 1, \dots, n.$$

(b) If there exist some numbers $U^i, \lambda^i, i = 1, \dots, n$ that satisfy the Afriat inequalities for some data $(p^i, x^i), i = 1, \dots, n$, then there exists a continuous, concave, monotonic non-satiated utility function that rationalizes the data.

Let us first note that the differentiability requirement in (a) can be weakened to a continuity requirement. Since we assume $u(x)$ is concave, it has a subgradient at every point, and this is enough for equations (2) and (3) to be satisfied. (On subgradients, see Rockafellar (1970).)

Secondly, let us compare (a) and (b) to Afriat's theorem. Here we note that Afriat's theorem makes a much stronger statement: if the data can be rationalized by any non-satiated utility function at all then it must satisfy the Afriat inequalities.

In order to establish this result it must be shown that the existence of a positive solution of the Afriat inequalities is equivalent to GARP or some similar combinatorial condition. This is exactly what is done in the proofs cited above.

This is an instance of a general pattern that will appear in what follows. The concavity-differentiability argument described above can be used to generate finite necessary and sufficient tests for a variety of problems in demand analysis. These tests take the form: does there exist a positive solution to a certain set of linear inequalities? For some of the problems we consider, this question turns out to be equivalent to some sort of combinatorial condition being satisfied. Such combinatorial conditions are often much simpler to verify than the inequality conditions.

2. TESTING FOR HOMOTHETICITY

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is homothetic if it is a positive monotonic transformation of a function that is homogeneous of degree 1; that is, if $f(x) \equiv g(h(x))$ where $h(x)$ is homogeneous of degree 1 and $g(h)$ is positive monotonic. In applications involving utility functions we know that behaviour is invariant with respect to monotonic transformations of utility so we might as well restrict ourselves to homogeneous functions in the first place: when is observed demand behaviour compatible with a homogeneous utility function?

As in Afriat's theorem we will first motivate a condition using concavity and homogeneity; then we will prove that the condition is valid in more general circumstances.

First let us note that the observed behaviour must be compatible with the utility maximization conditions of Section 1:

$$U^i \leq U^j + \lambda^j p^j (x^i - x^j) \quad (9)$$

where λ^i is interpreted as the marginal utility of income at consumption x^i . In what follows it will be convenient to normalize the prices by the level of expenditure at each observation so that $m^i = p^i x^i = 1$ for $i = 1, \dots, n$.

It is now easy to calculate an expression for the marginal utility of income at any level of observed consumption since:

$$u(x(p^i, m)) = u(x(p^i, m^i)m) = yu(x(p^i, m^i)) = mU^i. \quad (10)$$

We therefore have:

$$\lambda^i = \frac{du(x(p^i, m))}{dm} = U^i. \quad (11)$$

Hence (9) becomes

$$U^i \leq U^j + U^j p^j (x^i - x^j). \quad (12)$$

Recalling that $p^i x^i = 1$ we can simplify this to:

$$U^i \leq U^j p^j x^i. \quad (13)$$

These homogeneity inequalities were first derived by Afriat (1972) and Diewert (1973). In the next theorem we show that they are equivalent to a combinatorial condition that is substantially easier to test.

Theorem 2. *The following conditions are equivalent:*

- (1) *there exists a non-satiated homothetic utility function that rationalizes the data;*
- (2) *the data satisfies the Homothetic Axiom of Revealed Preference (HARP): for all distinct choices of indices $(i, j, \dots, m)$ we have $(p^i x^j)(p^j x^k) \cdots (p^m x^i) \geq 1$;*
- (3) *there exist numbers $U^i > 0, i = 1, \dots, n$ such that $U^i \leq U^j p^j x^i, i, j = 1, \dots, n$;*
- (4) *there exists a concave, monotonic, continuous, non-satiated, homothetic, utility function that rationalizes the data.*

Proof. First we show (1) implies (2). For $(i, j, \dots, m)$ let:

$$\begin{aligned} s^j &= p^i x^j \\ s^k &= (p^i x^j)(p^j x^k) = s^j (p^j x^k) \\ &\vdots \\ s^m &= (p^i x^j)(p^j x^k) \cdots (p^l x^m) = s^l (p^l x^m) \end{aligned}$$

Note that $p^i x^i = p^i (x^j/s^j) = 1$. Since $u(x)$ rationalizes the data, this implies $u(x^i) \geq u(x^j/s^j)$.

Similarly $(s^j p^j)(x^i/s^i) = (s^j p^j)(x^k/s^k) = 1$. Since $u(x)$ is homothetic and x^i is optimal at prices p^i , (x^j/s^j) must be optimal at prices $(s^j p^j)$. Hence $u(x^j/s^j) \geq u(x^k/s^k)$.

Continuing in this manner we find $u(x^l/s^l) \geq u(x^m/s^m)$. Combining these inequalities we have $u(x^i) \geq u(x^m/s^m)$.

Now suppose that $p^m (s^i x^i) < p^m x^m$. Since $u(x)$ is non-satiated this implies $u(s^i x^i) < u(x^m)$. By homotheticity this is equivalent to $u(x^i) < u(x^m/s^i)$ which contradicts the above inequality.

Hence $s^m p^m x^i \geq p^m x^m = 1$ which is precisely the Homothetic Axiom of Revealed Preference.

Next we show (2) implies (3). We define

$$U^i = \min_{(j,k,\dots,m,i)} \{(p^j x^k)(p^k x^l) \cdots (p^m x^i)\} \quad (14)$$

That is U^i is a minimum of the given expression over all paths starting anywhere and terminating at i . First let us note that this is well defined—i.e. a minimum actually exists. By HARP we only need consider strings $(j, k, \dots, m, i)$ without cycles—since if a cycle $(i, \dots, i)$ is removed U^i can never increase. But there are only a finite number of strings without cycles, and hence a minimum exists.

Next we note that the U 's defined in this way do satisfy the inequalities given in (3). Let:

$$U^i = (p^j x^k)(p^k x^l) \cdots (p^m x^i) \quad (15)$$

$$U^j = (p^n x^o)(p^o x^q) \cdots (p^r x^j). \quad (16)$$

Then

$$U^i = (p^j x^k)(p^k x^l) \cdots (p^m x^i) \leq (p^n x^o)(p^o x^q) \cdots (p^r x^j)(p^j x^i) \quad (17)$$

$$= U^j p^j x^i \quad (18)$$

since the value on the left of the inequality is a minimum over all paths to i . Hence (2) implies (3).

Next we show that (3) implies (4). We simply define

$$U(x) = \min_i \{U^i p^i x\}. \quad (19)$$

It is straightforward to verify that $U(x)$ has the desired properties. Finally, it is trivial that (4) implies (1). $\parallel$

There are several bibliographic remarks concerning this proof. First, I wish to thank an anonymous referee for providing an improvement on my original demonstration that (1) implies (2). Second, I wish to thank the editor of this journal for informing me that the argument that (1) implies (2) is nearly identical to a construction used by Shafer (1977), p. 1176. Third, the fact that (3) is a necessary and sufficient condition for homogeneity was pointed out by Afriat (1972) and Diewert (1973). Subsequently Afriat (1977) showed that

$$(p^1 x^2)(p^2 x^1) \geq 1 \quad (20)$$

is necessary and sufficient for homogeneity when $n = 2$.

Afriat (1981) generalized this result to an arbitrary number of observations. Afriat's argument in this work is based on an earlier mathematical paper (Afriat (1963)) concerning solutions to systems of inequalities. The resulting statement of necessary and sufficient conditions for homogeneity is virtually identical to Theorem 2, but the proof is somewhat different.

Let us now consider how we might test HARP in practice. For reasons of numerical stability it is more convenient to express HARP in the form:

$$\log(p^i x^i) + \log(p^j x^k) + \cdots + \log(p^m x^i) \geq 0. \quad (21)$$

Let us consider a graph with n vertices and associated n^2 matrix of "costs" $C = (c^{ij}) = (\log p^i x^j)$. Here we interpret $\log p^i x^j$ to be the cost of moving from vertex i to vertex j in this graph. Then condition (21) just asks whether the cost of moving from node i to itself can be made cheaper than zero; in other words, are there any negative cost cycles in the data?

There are well known algorithms in the operations research literature that can be used to answer this question. Typically the computational costs involved are on the order of n^3 computer additions which is quite feasible for most problems. I describe one of these algorithms in the Appendix. See also Afriat's (1981) discussion of similar computational issues.

3. TESTING FOR WEAK SEPARABILITY

Let us partition our data into two sets of goods and prices $(p^i, x^i), (q^i, y^i) \ i = 1, \dots, n$. We let x and y be arbitrary bundles of the x -goods and the y -goods respectively. Then we say a utility function u is (weakly) separable in y -goods, if we can find a "subutility function" $v(y)$ and a "macro function" $\bar{u}(x, v)$ with $\bar{u}(x, v)$ strictly increasing in v such that:

$$u(x, y) \equiv \bar{u}(x, v(y)).$$

(The terminology and definition is adapted from Blackorby, Primont, and Russell (1978) which can be consulted as a definitive reference concerning separability.)

Let us seek a criterion for separability. The first thing we note is that if the data is generated by a separable utility function then the data $((p^i, q^i), (x^i, y^i))$ and the data $(q^i, y^i) \ i = 1, \dots, n$ must both satisfy GARP. (We assume $u(x, y)$, $\bar{u}(x, v)$ and $v(y)$ are all non-satiated.)

Clearly the entire data set must satisfy GARP since it comes from maximization of $u(x, y)$. The subdata set must satisfy GARP since each y^i must solve the problem:

$$\begin{aligned} & \max v(y) \\ & \text{s.t. } q^i y \leq q^i y^i \end{aligned}$$

(Suppose y^* satisfied the budget constraint and yielded higher subutility. Then

$$\bar{u}(x^i, v(y^*)) > \bar{u}(x^i, v(y^i))$$

and

$$p^i x^i + q^i y^* \leq p^i x^i + q^i y^i,$$

contradicting maximization.)

However, these are only necessary conditions. Let us seek sufficient conditions through the concavity inequalities for u , $\bar{u}$, and v :

$$u(x^i, y^i) \leq u(x^j, y^j) + \lambda^j p^j (x^i - x^j) + \lambda^j q^j (y^i - y^j) \quad (22)$$

$$\bar{u}(x^i, v^i) \leq \bar{u}(x^j, v^j) + \lambda^j p^j (x^i - x^j) + \rho^j (v^i - v^j) \quad (23)$$

$$v(x^i) \leq v(x^j) + \mu^j q^j (y^i - y^j) \quad (24)$$

where ρ^j is interpreted as $\partial \bar{u} / \partial v(x^j, v^j)$ and μ^j is interpreted as the marginal utility of income at y^j . Arguing loosely from the chain rule for any y -good l :

$$\frac{\partial \bar{u}}{\partial y_l} = \frac{\partial \bar{u}}{\partial v} \frac{\partial v}{\partial y_l}. \quad (25)$$

Thus we have:

$$\frac{\partial \bar{u}}{\partial y_l} = \lambda^j p_l = \rho^j u_l p_l = \frac{\partial \bar{u}}{\partial v} \frac{\partial v}{\partial y_l}. \quad (26)$$

Hence $\rho^j = \lambda^j / \mu^j$ and we can rewrite (33) as:

$$U^i \leq U^j + \lambda^j p^j (x^i - x^j) + \frac{\lambda^j}{\mu^j} (V^i - V^j). \quad (27)$$

This turns out to be the condition we want. Let us now turn to a formal proof.

Theorem 3. *The following conditions are equivalent:*

- (1) *there exists a weakly separable concave, monotonic, continuous non-satiated utility function that rationalizes the data;*
- (2) *there exist numbers $U^i, V^i, \lambda^i > 0, \mu^i > 0$ $i = 1, \dots, n$ that satisfy:*

$$\begin{aligned} U^i & \leq U^j + \lambda^j p^j (x^i - x^j) + (\lambda^j / \mu^j) (V^i - V^j) \\ V^i & \leq V^j + \mu^j q^j (y^i - y^j) \end{aligned}$$

for $i, j = 1, \dots, n$;

- (3) *the data (q^i, y^i) and $(p^i, 1/\mu^i; x^i, V^i)$ satisfy GARP for some choice of (V^i, μ^i) that satisfies the Afriat inequalities.*

Proof. We have argued above that (1) implies (2) in the differentiable concave case. The general concave case is a similar argument involving subgradients. We omit the details.

To show (2) implies (1), we define:

$$V(y) = \min_j \{V^j + \mu^j q^j(y - y^j)\}$$

$$U(x, V) = \min_i \left\{ U^i + \lambda^i p^i(x - x^i) + \frac{\lambda^i}{\mu^i} (V - V^i) \right\}.$$

Then as before it is straightforward to show that $U(x^j, V(y^j)) = U^j$ for all $j = 1, \dots, n$.

Now suppose that

$$p^l x^l + q^l y^l \geq p^l x + q^l y.$$

We have:

$$\begin{aligned} U(x, V(y)) &= \min_i \left\{ u^i + \lambda^i p^i(x - x^i) + \frac{\lambda^i}{\mu^i} (\min_j \{V^j + \mu^j q^j(y - y^j)\} - V^i) \right\} \\ &\leq U^l + \lambda^l p^l(x - x^l) + \frac{\lambda^l}{\mu^l} (V^l + \mu^l q^l(y - y^l) - V^l) \\ &= U^l + \lambda^l p^l(x - x^l) + \lambda^l q^l(y - y^l) \\ &\leq U^l = U(x^l, V(y^l)). \end{aligned}$$

The fact that (2) and (3) are equivalent follows from Theorem 1 by interpreting $1/\mu^i$ as the price and V^i the quantity of good $k+1$. $\parallel$

An interesting application of Theorem 3 is the following simple proof of the Hicksian composite commodity theorem (Hicks (1956), Diewert (1978)).

Theorem 4. *Let the data $(p^i, q^i; x^i, y^i)$ $i = 1, \dots, n$ have the property that $q^i = t^i q^0$ for scalars $t^i > 0$ $i = 1, \dots, n$ and a fixed vector q^0 . Then if the data can be rationalized at all, it can be rationalized by a utility function of the form $U(x, V(y))$ where $V(y) = q^0 y$.*

Proof. According to Theorem 3 we only need show that (x^i, V^i) satisfies GARP for some choice of (V^i, μ^i) that satisfies the Afriat inequalities. Let us choose $V^i = q^0 y^i$ and $\mu^i = 1/t^i$.

(1) These choices satisfy the Afriat inequalities. We have that:

$$q^0 y^i = q^0 y^j + \frac{1}{t^j} t^j q^0 (y^i - y^j)$$

is identically true; substituting the definitions and using the fact that $q^j = t^j q^0$ we have:

$$V^i = V^j + \mu^j q^j (y^i - y^j).$$

(2) These choices satisfy GARP if and only if (p^i, q^i, x^i, y^i) satisfies GARP. For the above argument shows that

$$\frac{V^j - V^i}{\mu^j} = q^j (y^j - y^i).$$

Hence

$$(p^j, q^j)(x^j, y^j) \geq (p^j, q^j)(x^i, y^i)$$

if and only if

$$(p^j, 1/\mu^j)(x^j, V^j) \geq (p^j, 1/\mu^j)(x^i, V^i).$$

Thus $(x^j, y^j)R^0(x^i, y^i)$ if and only if $(x^j, V^j)R^0(x^i, V^i)$. $\parallel$

4. TESTING FOR HOMOTHETIC SEPARABILITY

A utility function is homothetically separable if it takes the form $U(x, V(y))$ where $V(y)$ is homothetic. This case is of considerable practical interest since it allows one to analyze the data via a two stage budgeting process. (Blackorby, Primont and Russell (1979), p. 206). Diewert and Parkan (1978) have suggested an heuristic procedure for testing for homothetic separability. The following theorem provides a formal rationalization for their procedure.

Theorem 5. *The following conditions are equivalent:*

- (1) *there exists a homothetically separable, concave, monotonic, non-satiated utility function that rationalizes the data;*
- (2) *the data $(p^i, 1/V^i; x^i, V^i)$ satisfy GARP for some choice of V^i that satisfies the homotheticity inequalities;*
- (3) *there exist numbers $U^i, V^i > 0, \lambda^i > 0 \ i = 1, \dots, n$ such that;*

$$U^i \leq U^j + \lambda^j p^j (x^i - x^j) + (\lambda^j / V^j)(V^i - V^j)$$

$$V^i \leq V^j q^j y^i.$$

Proof. The proof is a straightforward application of Theorems 2 and 3. ||

5. TESTING FOR ADDITIVE SEPARABILITY

Let the data be given as in Section 3. We say a utility function $\bar{u}(x, y)$ is additively separable if there is some monotonic transformation of $\bar{u}(x, y)$ such that

$$f(\bar{u}(x, y)) = u(x) + v(y)$$

for some utility functions $u(x)$ and $v(y)$.

Let us derive a condition that can be used to test for consistency with additive utility functions. Since additive separability implies weak separability we know immediately that one condition is that there exist numbers (u^i, λ^i) and $(v^i, \mu^i) \ i = 1, \dots, n$ such that

$$u^i \leq u^j + \lambda^j p^j (x^i - x^j)$$

$$v^i \leq v^j + \mu^j q^j (y^i - y^j)$$

for $i, j = 1, \dots, n$. On the other hand, the first order conditions for overall utility maximization imply that

$$\frac{\partial u(x^i)}{\partial x_j} - \lambda^i p_j^i = 0$$

$$\frac{\partial v(y^i)}{\partial y_j} - \lambda^i q_j^i = 0.$$

Hence we can take $\lambda^i = \mu^i$ for $i = 1, \dots, n$. This in fact is a necessary and sufficient condition.

Theorem 6. *The following two conditions are equivalent:*

- (1) *there exist two concave, monotonic, continuous utility functions whose sum rationalizes the data;*
- (2) *there exist numbers $U^i, V^i, \lambda^i > 0$ such that;*

$$U^i \leq U^j + \lambda^j p^j (x^i - x^j)$$

$$V^i \leq V^j + \lambda^j q^j (y^i - y^j).$$

Proof. Analogous to the proof of Theorem 4. ||

I have been unable to find a convenient combinatorial condition that is necessary and sufficient for additive separability.

6. TESTING FOR RATIONING

Let us now consider a simple rationing model; imagine that we have n observations on choice behaviour that may have been generated by a model of the form:

$$\begin{aligned} & \max u(x) \\ \text{s.t. } & p^i x \leq m^i \quad i = 1, \dots, n \\ & a^i x \leq b^i. \end{aligned}$$

Such a model can encompass a variety of rationing behaviour. For example a^i could be a vector composed of a one and several zeros: this corresponds to a simple upper bound on the consumer's consumption of a single commodity. Or the a^i 's could represent the prices in ration coupons of the various goods. We suppose that $a^i \geq 0$ and $b^i \geq 0$ for $i = 1, \dots, n$.

We will say a utility function rationalizes observed behaviour (p^i, x^i, a^i, b^i) $i = 1, \dots, n$ if $u(x^i) \geq u(x)$ for all x such that $m^i \geq p^i x$ and $b^i \geq a^i x$. Question: what are necessary and sufficient restrictions on observed behaviour under rationing that allow behaviour to be rationalized by the utility maximization model?

Proceeding as before we write the first order Kuhn–Tucker conditions for the above problem:

$$\begin{aligned} Du(x^i) - \lambda^i p^i - \mu^i a^i &= 0 & i = 1, \dots, n \\ \mu^i &\geq 0 & \lambda^i \geq 0 \\ \mu^i &= 0 & \text{if } a^i x^i < b^i \\ \lambda^i &= 0 & \text{if } p^i x^i < m^i. \end{aligned}$$

Now substitute these into the concavity conditions and we have:

Theorem 7. *The following conditions are equivalent:*

(1) *The data can be rationalized by a continuous, concave, monotonic, non-satiated utility function;*

(2) *There exist numbers $u^i, \lambda^i \geq 0, \mu^i \geq 0$ with $\mu^i = 0$ if $a^i x^i < b^i$ and $\lambda^i = 0$ if $p^i x^i < m^i$ such that:*

$$U^i \leq U^j + (\lambda^j p^j + \mu^j a^j)(x^i - x^j).$$

Proof. That (1) implies (2) follows from a modification of the derivative argument given above to an argument involving subgradients. We omit the details.

That (2) implies (1) follows from an argument very similar to those given before. We define:

$$u(x) = \min_i \{u^i + (\lambda^i p^i + \mu^i a^i)(x - x^i)\}$$

and then note that $u(x^i) = u^i$.

Now suppose that $p^j x^j \geq p^j x$ and $b^j \geq a^j x$. There are two cases:

- (i) If $\mu^j = 0$ then we use exactly the same arguments as before to show $u(x) \leq u(x^j)$.
- (ii) If $\mu^j > 0$ then we have $a^j x^j = b^j$ and hence $\mu^j a^j (x - x^j) = \mu^j (a^j x - b^j) \leq 0$. Thus:

$$\begin{aligned} u(x) = \min_i \{u^i + (\lambda^i p^i + \mu^i a^i)(x - x^i)\} &\leq u^j + (\lambda^j p^j + \mu^j a^j)(x - x^j) \\ &\leq u^j. \quad \parallel \end{aligned}$$

7. SUMMARY

We have extended Afriat's (1967*a*) method of testing a finite amount of data for consistency with the utility maximization hypothesis in several directions by providing finite tests for homotheticity, separability, homothetic separability, additive separability and rationing models of behaviour. These tests typically take the form of asking whether there exists a solution to a certain set of inequalities. Several of these tests were derived in earlier unpublished papers by Afriat (1967*b*), (1981).

APPENDIX

A shortest path algorithm

Consider a directed graph where the cost of moving from vertex i to vertex j is given by c_{ij} , with $c_{ij} = \infty$ if vertex i and vertex j are not connected. We wish to calculate the cost of the cheapest path from any vertex i to any other vertex j . There are several existing algorithms to do this; following Aho and Ullman (1972), we will describe an algorithm attributed to Warshall (1962). But see also Ford and Fulkerson (1962), p. 130, or Christofides (1975), pp. 163–167.

Warshall's Algorithm

Input: c_{ij} = cost of moving from vertex i to vertex j

Output: $\bar{c}_{ij}$ = minimum cost of moving from vertex i to vertex j .

- (1) Set $k = 1$.
- (2) for all i and j , if $c_{ij} \geq c_{ik} + c_{kj}$, set $c_{ij} = c_{ik} + c_{kj}$.
- (3) If $k < n$, let $k = k + 1$ and go to 2. If $k = n$, set $\bar{c}_{ij} = c_{ij}$ for all i and j .

It is not immediately obvious that Warshall's algorithm actually succeeds in calculating the cheapest paths in the graph. The interested reader can consult Aho and Ullman (1972) for a proof. (This proof is also reproduced in Varian (1982).)

Note that Aho and Ullman use an assumption that $c_{ij} \geq 0$; however, a careful inspection of the proof indicates that one only need assume that the cost of moving around any directed cycle is non-negative. We state this fact here for reference:

Fact A1. *If $c_{ij} + c_{jk} + \dots + c_{li} \geq 0$ for all cycles $(i, j, \dots, l, i)$ then Warshall's algorithm correctly computes the minimum cost paths through the graph.*

Fact A1 implies that Warshall's algorithm can be applied directly to the problem described in Section 2: if there is no cycle with negative costs, Warshall's algorithm will compute the minimum cost paths. If there is a cycle with negative costs then Warshall's algorithm will detect it.

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