Infinite Horizon Models of a Limit Order Book

Alberto Bressan and Hongxu Wei

Department of Mathematics, Penn State University.
University Park, PA 16802, USA.

e-mails: axb62@psu.edu, xiaoyitang.wei@gmail.com

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Abstract

The paper considers various models of a Limit Order Book, whose shape is determined as a Nash equilibrium among a large number of agents, in infinite time horizon, with exponentially discounted payoff.

After proving some general existence and uniqueness results, we analyze how the expected payoff of traders posting limit orders varies, depending on the size of the LOB, the volatility of the stock, and the possible presence of better informed external agents.

In particular, we prove formulas showing how the presence of better informed agents determines an increase in the bid-ask spread. Moreover, the increase in volatility determines a liquidity reduction, i.e. a decrease in the total amount of stocks posted for purchase or for sale.

Keywords: Limit Order Book, Nash equilibrium, optimal pricing strategy, bidding game, infinite horizon.

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1 Introduction

A bidding game related to a continuum model of the Limit Order Book (LOB) was recently considered in [5, 6, 8, 9], proving the existence and uniqueness of a Nash equilibrium and determining the optimal strategies for the various agents. In the basic model, it is assumed that an external buyer asks for a random amount $X > 0$ of a given asset. This external agent will buy the amount $X$ at the lowest available price, as long as this price does not exceed some (random) upper bound $\mathcal{P}$. One or more sellers offer various quantities of this same asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.
Having observed the prices asked by his competitors, each seller must determine an optimal strategy, maximizing his expected payoff. Because of the presence of the other sellers and of the upper bound \( P \), asking a higher price for an asset reduces the probability of selling it.

The models considered in [5, 6, 8, 9] all have the form of a “one shot game”. All players’ payoffs are completely determined as soon as one single incoming order is received. The later paper [7] considered a two-sided LOB, where both sell and buy orders are posted. Moreover, it included a time-dependent model, with a finite number of incoming buy or sell orders. However, the fundamental value of the stock was always assumed to remain constant in time.

The present paper is concerned with problems in infinite time horizon, also allowing for random fluctuations in the fundamental value \( \beta = \beta(t) \) of the traded asset. As customary, we consider an exponential discount, so that the expected payoff will remain finite.

In the first part of the paper we consider the case where the fundamental value of the stock is constant in time. Details of the model are discussed in Section 2. The existence and uniqueness of a LOB profile, determined as a Nash equilibrium among a large number of players, are then proved in Sections 3 and 4, in increasing generality.

In the second part of the paper we analyze how the shape of the LOB, and the expected payoff for the traders, are affected by various modeling features. More in detail:

(i) In Section 5 we show how the size of the LOB (i.e. the total amount of asset posted for buy or for sale) negatively affects the expected payoff of the traders. Indeed, as the size of LOB increases, there is more competition among traders. This lowers each player’s individual profit.

(ii) In Section 6 we study the case where, at random times, the fundamental value \( \beta(t) \) of the stock is subject to random jumps. In the case where this random value is a martingale, the existence and uniqueness of a LOB profile can still be proved as in the deterministic case, with minor modifications.

(iii) The remaining two sections deal with a model where all agents posting limit orders have the same information, but better informed external buyers or sellers may occasionally anticipate random jumps in the fundamental value of the stock. When this happens, one of the two sides of the LOB can be wiped out. As the volatility increases, we wish to understand how the strategy of the agents posting limit orders should be modified, and by how much their expected payoff will be impacted.

In this setting, a Nash equilibrium profile for the LOB is constructed in Section 7. Finally, in Section 8 we prove that the presence of informed external agents determines an increase in the bid-ask spread. Moreover, in connection with the earlier analysis in Section 5, we prove the implication

\[ \text{higher volatility} \implies \text{reduced liquidity}. \]

In other words, if better informed external agents cut into the traders’ profit, some of the traders posting limit orders will leave the LOB. As a consequence, the size of the LOB will shrink, until the expected traders’ profit will be kept up to the same acceptable level.

Throughout our analysis we consider a “mean field limit”, modeling a large number of traders. Each trader posts buy or sell orders for an amount of stock which is small, compared with
the overall size of the LOB. This assumption greatly simplifies all the analysis, and sometimes allows to obtain explicit formulas for the shape of the two sizes of the LOB, as well as the traders’ expected profit. For a rigorous derivation of this mean field limit see Theorem 9.1 in [5].

For various other models of the LOB considered in the literature we refer to [1, 2, 3, 4, 10]. See also the surveys [11, 12] and references therein.

2 Outline of the model

We consider here a situation involving an infinite sequence of incoming orders $X_1, X_2, \ldots$. Each can be either a buy order or a sell order. We assume that the random variables $X_j, j \geq 1$, describing the amount of stock that the external agents want to buy (or sell), are mutually independent and have absolutely continuous density w.r.t. Lebesgue measure.

At any given time, we describe the limit order book in terms of an absolutely continuous function $U(\cdot)$, as shown in Fig. 1. The following notation will be used:

- $\beta > 0$ denotes the fundamental value of the stock.
- For $p < \beta$, $U = U_b(p)$ is the total amount of stock that agents bid to buy at price $\geq p$.
- For $p > \beta$, $U = U_s(p)$ is the total amount of stock that agents offer to sell at price $\leq p$.
- The maximum bid price $p_B$ and the minimum ask price $p_A$ are defined as
  \[ p_B = \sup\{p < \beta; U_b(p) > 0\}, \quad p_A = \inf\{p > \beta; U_s(p) > 0\}. \tag{2.1} \]
- $p^-, p^+$ are the lowest and highest prices at which some bid is made.
- $K_b, K_s$, the total amount of stocks that agents posting orders on the LOB offer to buy or sell, respectively,

According to the above definitions, one has

\[
U(s) = \begin{cases} 
0 & \text{if } s \in [p_B, p_A], \\
K_b & \text{if } s \leq p^-,
K_s & \text{if } s \geq p^+.
\end{cases} \tag{2.2}
\]

As in [9], we assume that the sizes of the incoming buy or sell orders $X_{buy}, X_{sell}$ are i.i.d. random variables, with tail distributions

\[
\text{Prob}\left\{X_{buy} > s\right\} = \Psi_b(s), \quad \text{Prob}\left\{X_{sell} > s\right\} = \Psi_s(s). \tag{2.3}
\]

Moreover, the maximum price that a buyer is willing to pay (or the minimum price that an external seller is willing to accept) is not known a priori. More precisely, we assume that the
external agent will agree to the transaction only as long as the price ranges within an interval \([Q_s \beta, Q_b \beta]\), where also \(Q_s, Q_b > 0\) are independent random variables (Fig. 2), say

\[
\text{Prob.}\left\{ Q_b \geq s \right\} = h_b(s), \quad \text{Prob.}\left\{ Q_s \leq s \right\} = h_s(s). \tag{2.4}
\]

To fix ideas, in the following we assume that, for some \(\delta_1, \delta_2 > 0\),

\[
h_s(1 - \delta_1) = 0 = h_b(1 + \delta_2). \tag{2.5}
\]

In other words, no external buyer will accept a price \(p > (1 + \delta_2) \beta\), and no external seller will accept a price \(p < (1 - \delta_1) \beta\).

An external order of size \(X\) is thus executed as follows (Fig. 3).

**CASE 1:** a buy order in the amount \(X_b\). In this case the external buyer will take all stocks whose price ranges in the interval \([\beta, p(X, Q)]\), where

\[
p(X, Q) = \max\left\{ p \in [\beta, Q_b \beta], \ U_s(p) \leq X_b \right\}. \tag{2.6}
\]

**CASE 2:** a sell order in the amount \(X_s\). In this case the external seller will fulfill all the bids whose price ranges in the interval \([p(X, Q), \beta]\), where

\[
p(X, Q) = \min\left\{ p \in [Q_s \beta, \beta], \ U_b(p) \leq X_s \right\}. \tag{2.7}
\]
Figure 3: A possible shape of the limit order book. For \( p > \beta \), here \( \phi(p) = U_s'(p) \) is the density of stock put on sale at price \( p \). For \( p < \beta \), \( \phi(p) = -U_b'(p) \) is the density of stock that agents offer to buy at price \( p \). If the external order has size \( X > 0 \) and is a buy order, all the stocks in the shaded region with price \( p \in [p, p(X, Q)] \) as in (2.6) will be sold. If the external order is a sell order, all the buy orders in the shaded region with price \( p \in [p(X, Q), p_B] \) as in (2.7) will be executed.

It is natural to assume that each agent will try to maximize the combined value of cash and stock held over time. However, since this amount may grow without bounds as \( t \to +\infty \), to achieve a well defined measure of performance one needs to insert an exponential discount. Denoting by \( C(t) \) and \( S(t) \) the amount of cash and stock held by an agent at time \( t \), we shall thus seek to maximize the expected discounted payoff

\[
J \doteq E \left[ \int_0^{+\infty} e^{-\gamma t} (C(t) + \beta(t) S(t)) \, dt \right].
\]

(2.8)

In the following, we consider a sequence of incoming orders \( X_1, X_2 \ldots \), and price limitations \( Q_1, Q_2 \ldots \), assuming that the \( X_j \) as well as the \( Q_j \) are independent, identically distributed random variables. Each order will be

- a buy order with probability \( \theta \in ]0, 1[ \),
- a sell order with probability \( 1 - \theta \).

The arrival times \( 0 = t_0 < t_1 < t_2 < \cdots \) of these incoming orders are assumed to be random. More precisely, the differences \( t_j - t_{j-1} \) are independent random variables, with probability distribution

\[
\text{Prob.}\{t_j - t_{j-1} > s\} = e^{-\mu s}.
\]

(2.9)

As a first step, in this section we consider the case where the fundamental value \( \beta(t) = \beta \) of the stock remains constant in time. To obtain a mathematically tractable problem, some simplifying assumptions will be made.

(A1) The size of the LOB remains constant in time. In other words, after an external order has been executed, new agents post additional limit orders, so that the total amount \( K_b, K_s \) of buy and sell orders does not change.

(A2) The amount of buy or sell orders posted by each individual agent is small, compared with the total amounts of orders \( K_b, K_s \) on the entire LOB.

(A3) At any time \( t \geq 0 \) the LOB is in equilibrium. In other words, any ask price \( p \in [p_A, p^+] \) or any bid price \( p \in [p^-, p_B] \) yields the same expected payoff.
Remark 2.1 Assumption (A1) may seem quite restrictive. However, results very similar to the ones obtained in this paper can be proved also in the case where the size of the LOB randomly changes after each incoming order, provided that these sizes are independent of each other, see Section 3.1 in this paper.

Thanks to (A2) we can make a linear approximation, and compute the expected payoff (2.8) as a linear function of the initial amount $C_0$ of cash and the initial amount $S_0$ of stock held by an individual agent:

$$J = aC_0 + b\beta S_0.$$  

Throughout the following, our main concern will be to determine the values of $a, b$, and understand how they depend on the various constants $K_b, K_s, \gamma, \ldots$ that define the model.

To obtain a system of two equations for $a$ and $b$, we observe that, since the LOB is in equilibrium, the expected payoff to the various agents does not depend on the price at which they post their limit orders. We can thus compute $a, b$ by looking at the expected payoff of the agent that posts limit orders at the prices $p^+$ and $p^-$. Let $\tau > 0$ be the first time when an external order arrives. We consider two cases.

For an agent who initially holds a unit amount of cash, and posted a bid to buy stocks at price $p^-$, the expected payoff will be

$$J_C = \int_0^\tau e^{-\gamma s} ds + ae^{-\gamma \tau} \left[ \theta + (1-\theta)(1-\Psi_s(K_b)h_s(p^-/\beta)) \right] + b\beta e^{-\gamma \tau} \left[ (1-\theta)\Psi_s(K_b)h_s(p^-/\beta) \right] \frac{1}{p^-}.$$  

(2.11)

Indeed, with probability $\theta$ the external order will be a buy order. Hence after time $\tau$ the agent will still hold the same amount of cash, and no stock. On the other hand, with probability $(1-\theta)$ the external order will be a sell order. When this happens, our agent will have probability $\Psi_s(K_b)h_s(p^-/\beta)$ of actually buying stock at price $p^-$, while with probability $1 - \Psi_s(K_b)h_s(p^-/\beta)$ he will remain with the original cash. This accounts for the last two terms on the right hand side of (2.11).

Next for an agent who initially holds a unit amount of stock, and posted a bid to sell it at price $p^+$, the expected payoff will be

$$J_S = \int_0^\tau e^{-\gamma s} \beta ds + ae^{-\gamma \tau} \theta \Psi_b(K_s)h_b(p^+ / \beta) p^+ + b\beta e^{-\gamma \tau} \left[ \theta (1-\Psi_b(K_s)h_b(p^+ / \beta)) + (1-\theta) \right].$$  

(2.12)

Indeed, with probability $1 - \theta$ the external order will be a sell order. Hence after time $\tau$ the agent will still hold the same amount of stock, and no cash. On the other hand, with probability $\theta$ the external order will be a buy order. When this happens, our agent will have probability $\Psi_b(K_s)h_b(p^+ / \beta)$ of actually selling his stock at price $p^+$, while with probability $1 - \Psi_b(K_s)h_b(p^+ / \beta)$ he will not sell, and remain with the original amount of stock. This accounts for the last two terms on the right hand side of (2.12).

Combining the two cases (2.11)-(2.12), and using the assumption (2.9) on the distribution of
the first arrival time $\tau$, we obtain

$$aC_0 + b\beta S_0 = \int_0^{+\infty} \mu e^{-\mu t} \left\{ \int_0^\tau e^{-\gamma s}(C_0 + \beta S_0) \, ds + e^{-\gamma \tau} \left[ \theta a + (1 - \theta)(1 - \Psi_b(K_b)h_s(p^-/\beta))a + (1 - \theta)\Psi_b(K_b)h_s(p^-/\beta) \frac{b\beta}{p^-} \right] C_0 
+ e^{-\gamma \tau} \left[ \theta \Psi_s(K_b)h_b(p^+/\beta) p^+ + \theta(1 - \Psi_s(K_b)h_b(p^+/\beta)) b\beta + (1 - \theta)b \right] S_0 \right\} \, d\tau. \quad (2.13)$$

Since these quantities will appear repeatedly in our computations, we introduce the constants

$$\Psi^\sharp \doteq \Psi_b(K_b), \quad \Psi^\flat \doteq \Psi_s(K_b), \quad (2.14)$$
denoting the probability that an incoming buy or sell order is larger than the entire size of the corresponding part of the LOB, respectively.

Since (2.13) holds for every $C_0, S_0$ this yields a system of two equations for $a, b$, depending on $\beta, p^+, p^-$. Adopting vector notation, we thus obtain

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\gamma + \mu} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\mu}{\gamma + \mu} \begin{bmatrix} \theta + (1 - \theta)(1 - \Psi^\sharp h_s(p^-/\beta)) & (1 - \theta) \Psi^\flat \cdot h_s(p^-/\beta) \cdot (\beta/p^-) \\ \theta \cdot \Psi^\flat \cdot h_b(p^+/\beta) \cdot (p^+/\beta) & (1 - \theta) + \theta(1 - \Psi^\sharp h_b(p^+/\beta)) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \quad (2.15)$$

This can equivalently be written as

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (2.16)$$

where $\beta$ is the fundamental value of the stock, and

$$A \doteq \begin{bmatrix} \gamma + \mu(1 - \theta)\Psi^\flat h_s(p^-/\beta) & -\mu(1 - \theta)\Psi^\flat h_s(p^-/\beta) \cdot (\beta/p^-) \\ -\mu\theta \Psi^\flat h_b(p^+/\beta) \cdot (p^+/\beta) & \gamma + \mu\theta\Psi^\sharp h_b(p^+/\beta) \end{bmatrix}. \quad (2.17)$$

**Remark 2.2** In general, the intervals $[p^-, p_B]$ and $[p_A, p^+]$ are not known a priori, and must be determined as part of the solution. In view of the assumption (2.5), we have the inclusions

$$[p^-, p_B] \subseteq [(1 - \delta_1)\beta, \beta], \quad [p_A, p^+] \subseteq [\beta, (1 + \delta_2)\beta]. \quad (2.18)$$

As soon as the constants $a, b, p^+, p^-$ are found, in order to determine $U(p)$ for $p \in [p_A, p^+]$, we observe that the assumption that the LOB is in equilibrium implies

$$a p \cdot \text{[probability of selling at price } p \text{]} + b\beta \cdot \text{[probability of not selling at price } p \text{]} = \text{constant}. \quad (2.19)$$

Similarly, for $p \in [p^-, p_B]$, we have

$$\frac{b\beta}{p} \cdot \text{[probability of buying at price } p \text{]} + a \cdot \text{[probability of not buying at price } p \text{]} = \text{constant}. \quad (2.20)$$
With the above notations, we have

\[
\text{[probability of selling at price } p] = \text{Prob.}\left\{ X_b \geq U_s(p) \text{ and } Q_b \beta \geq p \right\} = \Psi_b(U_s(p)) \cdot h_b(p/\beta).
\]

\[
\text{[probability of buying at price } p] = \text{Prob.}\left\{ X_s \geq U_b(p) \text{ and } Q_s \beta \leq p \right\} = \Psi_s(U_b(p)) \cdot h_s(p/\beta).
\]

Inserting these expressions in (2.19)-(2.20), we obtain

\[
\begin{align*}
\begin{cases}
ap\Psi_b(U_s(p))h_b\left(\frac{p}{\beta}\right) + b\beta \left[ 1 - \Psi_b(U_s(p))h_b\left(\frac{p}{\beta}\right) \right] = \left[ ap^+ - b\beta \right] h_b\left(\frac{p^+}{\beta}\right) + b\beta \left[ 1 - \Psi^2 h_b\left(\frac{p^+}{\beta}\right) \right], \\
b\beta h_s\left(\frac{p}{\beta}\right) + a \left[ 1 - \Psi_s(U_b(p))h_s\left(\frac{p}{\beta}\right) \right] = \left[ \frac{b\beta}{p} - a \right] \Psi h_s\left(\frac{p^-}{\beta}\right) + a \left[ 1 - \Psi^2 h_s\left(\frac{p^-}{\beta}\right) \right].
\end{cases}
\end{align*}
\]

The functions \(U_s, U_b\) are thus implicitly determined by the relations

\[
\begin{align*}
\Psi_b(U_s(p)) &= \frac{(ap^+ - b\beta)\Psi^2 h_b(p^+ / \beta)}{(ap - b\beta) h_b(p/\beta)} \quad \text{for } p > \beta, \\
\Psi_s(U_b(p)) &= \frac{[(b\beta/p^- - a)\Psi h_s(p^- / \beta)]}{[(b\beta/p - a) h_s(p/\beta)]} \quad \text{for } p < \beta.
\end{align*}
\]

In turn, the minimum ask price \(p_A > \beta\) and the maximum bid price \(p_B < \beta\) are determined by

\[
U_s(p_A) = 0 = U_b(p_B), \quad \Psi_b(U_s(p_A)) = 1 = \Psi_s(U_b(p_B)).
\]

By (2.22), this yields

\[
\begin{align*}
\begin{cases}
( ap - b\beta ) h_b(p_A / \beta) &= (ap^+ - b\beta)\Psi^2 h_b(p^+ / \beta), \\
[(b\beta/p_B - a)\Psi h_s(p_B / \beta)] &= [(b\beta/p^- - a)\Psi h_s(p^- / \beta)].
\end{cases}
\end{align*}
\]

3 Deterministic acceptable prices

The analysis in the previous section can be greatly simplified if we assume that the upper and lower bounds for the prices accepted by external agents are deterministic. More precisely, for some \(\delta_1, \delta_2 > 0\), assume that

\[
h_b\left(\frac{p}{\beta}\right) = \begin{cases} 
1 & \text{if } p \leq (1 + \delta_2)\beta, \\
0 & \text{if } p > (1 + \delta_2)\beta,
\end{cases} \quad
h_s\left(\frac{p}{\beta}\right) = \begin{cases} 
1 & \text{if } p \geq (1 - \delta_1)\beta, \\
0 & \text{if } p < (1 - \delta_1)\beta.
\end{cases}
\]

These assumptions imply

\[
p^+ = (1 + \delta_2)\beta, \quad p^- = (1 - \delta_1)\beta, \quad h_b\left(\frac{p^+}{\beta}\right) = h_s\left(\frac{p^-}{\beta}\right) = 1.
\]
The matrix (2.17) thus reduces to

\[ A = \begin{bmatrix} 
\gamma + \mu(1-\theta)\Psi^\flat & -\mu(1-\theta)\Psi^\flat \cdot (\beta/p^-) \\
-\mu\theta\Psi^\sharp \cdot (p^+/\beta) & \gamma + \mu\theta\Psi^\sharp 
\end{bmatrix}. \tag{3.3} \]

In this case, one can uniquely solve the system (2.16), provided that the discount factor \( \gamma \) in (2.8) is large enough.

**Theorem 3.1** Assume that the exponential discount factor \( \gamma \) satisfies

\[ \gamma > \frac{\delta_1 + \delta_2}{1 - \delta_1} \cdot \frac{\mu}{4}. \tag{3.4} \]

Then the infinite horizon game with exponential discount \( \gamma \) and deterministic acceptable prices (3.1) admits a unique equilibrium solution.

**Proof.** When the matrix \( A \) is given by (3.3), recalling (3.2) we have

\[ \det A = \gamma^2 + \gamma\mu \left( (1-\theta)\Psi^\flat + \theta\Psi^\sharp \right) - \mu^2 (1-\theta)\theta\Psi^\flat \Psi^\sharp \cdot \left( \frac{1 + \delta_2}{1 - \delta_1} - 1 \right). \tag{3.5} \]

Setting \( c = \left( (1-\theta)\Psi^\flat + \theta\Psi^\sharp \right) \)

and observing that

\[(1-\theta)\theta\Psi^\flat \Psi^\sharp \leq \frac{c^2}{4},\]

we obtain

\[ \det A \geq \gamma^2 + c\mu\gamma - \frac{c^2}{4} \mu^2 \cdot \frac{\delta_1 + \delta_2}{1 - \delta_1}. \tag{3.6} \]

Setting the right hand side of (3.6) to be equal to zero and solving for \( \gamma \), we conclude that \( \det A > 0 \) provided that

\[ \gamma \geq -\frac{c\mu}{2} + \frac{1}{2} \sqrt{c^2\mu^2 + 4\mu^2 \cdot \frac{\delta_1 + \delta_2}{1 - \delta_1}} = \frac{c\mu}{2} \left( \sqrt{1 + \frac{\delta_1 + \delta_2}{1 - \delta_1}} - 1 \right). \]

Since \( c \leq 1 \), the above inequality certainly holds if \( \gamma \) satisfies (3.4).

Inverting the matrix \( A \), the explicit solution of (2.16) is computed by

\[ \begin{bmatrix} a \\ b \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \gamma + \mu\theta\Psi^\sharp + \mu(1-\theta)\Psi^\flat/(1-\delta_1) \\ \gamma + \mu\theta\Psi^\sharp (1+\delta_2) + \mu(1-\theta)\Psi^\flat \end{bmatrix}. \tag{3.7} \]

Using (3.2) together with (2.23), the minimum ask price \( p_A \) and the maximum bid price \( p_B \) are found to be

\[ p_A = (1 + \delta_2)\beta\Psi^\sharp + \frac{b\beta}{a}(1 - \Psi^\sharp), \quad p_B = \left( \frac{1}{(1-\delta_1)\beta}\Psi^\flat + \frac{a}{b\beta}(1 - \Psi^\flat) \right)^{-1}. \tag{3.8} \]
Moreover, by (2.22) the shape of the sell portion $U_s(p)$ and the shape of the buy portion $U_b(p)$ are implicitly determined by the relations
\[
\begin{cases}
\Psi_b(U_s(p)) = \frac{ap^* - b\beta}{ap - b\beta} \psi^* & \text{for } p > \beta, \\
\Psi_s(U_b(p)) = \frac{(b\beta/p^*) - a}{(b\beta/p) - a} \psi^* & \text{for } p < \beta.
\end{cases}
\] (3.9)

**Remark 3.1** When $\mu$ is small, very few external orders arrive. Hence the amount of cash or stock held by an agent will undergo little change over time. In particular, as $\mu \to 0$, the expected value of the payoffs in (2.8) will satisfy $a \to 1/\gamma$, $b \to 1/\gamma$.

On the other hand, when $\mu$ is large, the incoming orders will arrive very frequently. In this case, the expected amount of cash and stock held by an agent will increase at a faster rate. The discounting factor $\gamma$ has to be large enough in order that the inequality (3.4) be satisfied. Otherwise the payoff in (2.8) will be unbounded.

**Remark 3.2** A more precise condition that guarantees the positivity of $\det A$ is
\[
\gamma > \frac{\mu}{2} \left( \sqrt{\left((1-\theta)\psi^b - \theta\psi^s\right)^2 + 4\theta(1-\theta)\psi^b\psi^s \left(\frac{1+\delta_2}{1-\delta_1}\right) - \left((1-\theta)\psi^b + \theta\psi^s\right)^2} \right). \tag{3.10}
\]
This is obtained by setting the right hand side of (3.5) to be equal to zero, and solving for $\gamma$.

**Example 3.1** Consider the case where
\[
\psi^b = \psi^s = \psi^*, \quad \theta = 1 - \theta = \frac{1}{2}, \quad 1 + \delta_2 = \frac{1}{1-\delta_1} = 1 + \delta. \tag{3.11}
\]
When the above holds, the inequality (3.4) reduces to
\[
\gamma > \frac{\delta \psi^*}{2} \cdot \mu. \tag{3.12}
\]
Moreover, from (3.7) and (3.8) one obtains the identities
\[
a = b = \frac{1}{\gamma - \mu \delta \psi^*/2}, \quad \frac{p_A}{\beta} = \frac{\beta}{p_B}. \tag{3.13}
\]

**Example 3.2** In the special case where $\Psi_s(s) = \Psi_b(s) = e^{-\alpha s}$, $K_s = K_b = K$, and $\theta = 1/2$, using (3.5)-(3.7) we find
\[
\begin{bmatrix} a \\ b \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\det A(K)} \begin{bmatrix} \gamma + \frac{2-\delta_1}{2-2\delta_1} \mu e^{-\alpha K} \\ \gamma + \frac{2+\delta_2}{2} \mu e^{-\alpha K} \end{bmatrix}, \tag{3.14}
\]
where
\[
\det A(K) = \gamma^2 + \gamma \mu e^{-\alpha K} - \frac{\mu^2}{4} \frac{\delta_1 + \delta_2}{1-\delta_1} e^{-2\alpha K}. \tag{3.15}
\]
3.1 LOB with randomly variable size.

Here and in the remainder of the paper, most of our analysis is concerned with the case where the total amounts \( K_s, K_b \) posted on the LOB are constant in time. At first sight, this appears to be a very restrictive assumption.

However, entirely similar results can be proved also in a case where the sizes of the LOB are randomly varying. More precisely, assume that, after an external order is executed, new bids are posted in the LOB, so that the sizes \( K_s \in [K_s^-, K_s^+] \), \( K_b \in [K_b^-, K_b^+] \) are independent, identically distributed random variables. In place of (2.14), we can thus define the expected values

\[
\Psi^\sharp = E[\Psi_b(K_s)], \quad \Psi^\flat = E[\Psi_s(K_b)],
\]

(3.16)

In the present setting, the equation (2.13) is replaced by

\[
a C_0 + b \beta S_0 = E^{K_s, K_b} \int_0^{+\infty} \mu e^{-\mu \tau} \left\{ \int_0^\tau e^{-\gamma s} (C_0 + \beta S_0) ds \right. \\
+ e^{-\gamma \tau} \left[ \theta a + (1 - \theta)(1 - \Psi_s(K_b))a + (1 - \theta)\Psi_s(K_b) \frac{b\beta}{p^-} \right] C_0 \\
+ e^{-\gamma \tau} \left[ \theta \Psi_b(K_s)p^+ a + \theta (1 - \Psi_b(K_s))b\beta + (1 - \theta)b\beta \right] S_0 \left. \right\} d\tau \]

\[
= \int_0^{+\infty} \mu e^{-\mu \tau} \left\{ \int_0^\tau e^{-\gamma s} (C_0 + \beta S_0) ds \right. \\
+ e^{-\gamma \tau} \left[ \theta a + (1 - \theta)(1 - \Psi^\sharp)a + (1 - \theta)\Psi^\sharp \frac{b\beta}{p^-} \right] C_0 \\
+ e^{-\gamma \tau} \left[ \theta \Psi^\flat p^+ a + \theta (1 - \Psi^\flat)b\beta + (1 - \theta)b\beta \right] S_0 \left. \right\} d\tau.
\]

(3.17)

One thus recovers exactly the same solution as (3.7), with (2.14) replaced by (3.16).

4 Random acceptable prices

In the case where the maximum and minimum prices acceptable by external agents are random, the values \( p^-, p^+ \) are not known a priori, and must be determined as part of the solution. In this section, under suitable assumptions we will show that

- Given the expected payoffs \( a, b\beta \) in (2.10), one can uniquely determine the maximum and minimum prices \( p^+, p^- \).
- In turn, if \( p^+, p^- \) are known, the values of \( a, b \) can be obtained by solving the linear system (2.15).
The equilibrium solution to the bidding game will thus be obtained as a fixed point of a composed map

\[(a, b) \mapsto (p^-, p^+) \mapsto (a, b).\]

The analysis can be somewhat simplified by observing that \(p^-, p^+\) depend on \(a, b\) only through the scalar quantity \(z = b/a\).

As a first step, consider an agent posting a sell order at the maximum price \(p^+\). When an external buy order arrives, there is:

- Probability \(\Psi_b(K_s) \cdot h_b(p^+/\beta)\) of selling the asset. In this case, the expected payoff will be \(ap^+\).
- Probability \(1 - \Psi_b(K_s) \cdot h_b(p^+/\beta)\) of not selling the asset. In this case, the expected payoff will be simply \(b\beta\).

Given \(a, b\), the optimal choice of \(p^+\) is the one that maximizes the quantity

\[\Psi_b(K_s) \cdot h_b\left(\frac{p^+}{\beta}\right) \cdot ap^+ + \left[1 - \Psi_b(K_s) \cdot h_b\left(\frac{p^+}{\beta}\right)\right] b\beta. \tag{4.1}\]

Setting \(s = p^+/\beta\), we observe that the map

\[s \mapsto h_b(s) \cdot (a\beta s - b\beta)\]

has a unique maximum provided that \(h_b\) has \(C^2\) regularity and

\[h''_b(s) \cdot (as - b) + 2h'_b(s) \cdot a < 0, \quad \text{for all } s \in ]1, 1 + \delta_2[. \tag{4.2}\]

Assuming that (4.2) holds, the value of \(p^+\) that maximizes (4.1) is uniquely determined:

\[p^+ = p^+(a, b, \beta) = \arg\max_{p > \beta} \left\{h_b\left(\frac{p}{\beta}\right) \cdot (pa - b\beta)\right\}, \tag{4.3}\]

The analysis of the lowest bid price \(p^-\) is entirely similar. The agent that posts a bid at this price seeks to maximize the expected profit

\[\frac{b\beta}{p^-} \Psi_s(K_b) \cdot h_s\left(\frac{p^-}{\beta}\right) + a \cdot \left[1 - \Psi_s(K_b) \cdot h_s\left(\frac{p^-}{\beta}\right)\right]. \tag{4.4}\]

Setting \(s = p^-/\beta\), we observe that the map

\[s \mapsto h_s(s) \cdot \left(\frac{b\beta}{\beta s} - a\right)\]

has a unique maximum provided that \(h_s\) is continuous and

\[h''_s(s) \cdot \left(\frac{1}{s} - \frac{a}{b}\right) - 2h'_s(s) \cdot \frac{1}{s^2} + 2h_s(s) \cdot \frac{1}{s^3} < 0 \quad \text{for all } s \in ]1 - \delta_1, 1[. \tag{4.5}\]

Assuming that (4.5) holds, the value of \(p^-\) that maximizes (4.4) is uniquely determined:

\[p^- = p^-(a, b, \beta) = \arg\max_{p < \beta} \left\{h_s\left(\frac{p}{\beta}\right) \cdot \left(\frac{b\beta}{p} - a\right)\right\}. \tag{4.6}\]
Factoring out the constant $\beta$, the functions $p^+, p^-$ in (4.3), (4.6) can be written as

$$
p^+ = \beta \sigma^+ \left( \frac{b}{a} \right), \quad p^- = \beta \sigma^- \left( \frac{b}{a} \right),
$$

where $\sigma^+, \sigma^-$ are defined as following maximizers:

$$
\sigma^+(s) = \arg\max_{\sigma \geq 1} \left\{ h_b(\sigma) \cdot (\sigma - s) \right\}, \quad \sigma^-(s) = \arg\max_{\sigma \leq 1} \left\{ h_s(\sigma) \cdot \left( \frac{1}{\sigma} - \frac{1}{s} \right) \right\}.
$$

We now introduce two sets of assumptions, making sure that the functions $\sigma^+, \sigma^-$ in (4.8) are well defined.

**Assumption (A4)** The map $s \mapsto h_b(s)$ is continuous, twice differentiable restricted to $]1, 1 + \delta_2[$, and satisfies

$$
\begin{aligned}
    h_b(s) &= 1 \quad \text{for } s \in [0, 1], \\
    h_b(s) &= 0 \quad \text{for } s \geq 1 + \delta_2, \\
    h_b'(s) &< 0, \quad (\delta_1 + \delta_2) \cdot h_b''(s) < -h_b'(s) \quad \text{for } s \in [1, 1 + \delta_2[.
\end{aligned}
$$

**Assumption (A5)** The map $s \mapsto h_s(s)$ is continuous, twice differentiable restricted to $]1 - \delta_1, 1[$, and satisfies

$$
\begin{aligned}
    h_s(s) &= 0 \quad \text{if } s \in [0, 1 - \delta_1[ , \\
    h_s(s) &= 1 \quad \text{if } s \geq 1, \\
    h_s'(s) &> 0, \quad |h_s''(s)| \cdot \left( \frac{1}{1 - \delta_1} - \frac{1}{1 + \delta_2} \right) < \frac{1}{s^3} \left( h_s'(s)s - 2h_s(s) \right) \quad \text{if } s \in ]1 - \delta_1, 1[.
\end{aligned}
$$

We check that the above assumptions (4.9)-(4.10) imply (4.2) and (4.5). Indeed, setting $z = b/a$, by (2.5) we have

$$
1 - \delta_1 \leq z \leq 1 + \delta_2.
$$

Therefore

$$
-\delta_2 < s - z < \delta_1 + \delta_2 \quad \text{for all } s \in [1, 1 + \delta_2[.
$$

Combined with (A4), the above inequalities imply

$$
h_b''(s) \cdot (s - z) + 2h_b'(s) < h_b''(s) \cdot (s - z) + h_b'(s) < 0.
$$

Hence the condition (4.2) is satisfied.

Similarly, we have

$$
1 - \frac{1}{1 - \delta_1} < \frac{1}{s} - \frac{1}{z} < \frac{1}{1 - \delta_1} - \frac{1}{1 + \delta_2} \quad \text{for all } s \in ]1 - \delta_1, 1[.
$$

Combining the above inequality with (A5), and recalling that $h_s(s) \geq 0, h_s'(s) \geq 0$, we obtain

$$
h_s''(s) \cdot \left( \frac{1}{s} - \frac{1}{z} \right) \leq |h_s''(s)| \cdot \left( \frac{1}{1 - \delta_1} - \frac{1}{1 + \delta_2} \right) \leq \frac{2}{s^3} \left( h_s'(s)s - h_s(s) \right),
$$

13
showing that the condition (4.5) is also satisfied. As a consequence, \( p^+, p^- \) are uniquely determined.

As soon as the maximum and minimum prices \( p^+ > \beta > p^- \) have been determined, the expected payoffs \( a \) and \( b_\beta \) can be found by solving the linear system (2.16)-(2.17). For convenience, we introduce the notation

\[
\Theta_1 = \mu (1 - \theta) \Psi^b, \quad \Theta_2 = \mu \theta \Psi^b,
\]

observing that \( \Theta_1, \Theta_2 \) are independent of \( a, b \). The matrix \( A \) in (2.17) can now be written as

\[
A = \begin{bmatrix}
\gamma + \Theta_2 h_s(p^-/\beta) & -\Theta_2 h_s(p^-/\beta) \cdot (\beta/p^-) \\
-\Theta_1 h_b(p^+/\beta) \cdot (p^+/\beta) & \gamma + \Theta_1 h_b(p^+/\beta)
\end{bmatrix}. \tag{4.11}
\]

**Theorem 4.1** Assume that the exponential discount factor \( \gamma \) satisfies (3.4), and that the random acceptable prices satisfy (A4)-(A5). Then the bidding game in infinite horizon has at least one equilibrium solution. If, in addition,

\[
\gamma > (1 - \theta) \Psi^b \left( \frac{1 + \delta_2}{(1 - \delta_1)^2} - 1 \right), \tag{4.12}
\]

then the solution is unique.

**Proof.** 1. Let two prices \( p^-, p^+ \) be given, with

\[
(1 - \delta_1) \beta \leq p^- \leq \beta \leq p^+ \leq (1 + \delta_2) \beta. \tag{4.13}
\]

Using the solution formula (3.7) to compute the expected payoffs, we obtain

\[
Z(p^-, p^+) = \frac{b}{a} = \frac{\gamma + \Theta_1 h_b(p^+/\beta) \cdot (p^+/\beta) + \Theta_2 h_s(p^-/\beta)}{\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta) \cdot (\beta/p^-)}. \tag{4.14}
\]

By (4.14) it follows

\[
Z(p^-, p^+) < \frac{(\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)) \cdot (p^+/\beta)}{\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)} = \frac{p^+}{\beta} \leq 1 + \delta_2,
\]

\[
Z(p^-, p^+) > \frac{\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)}{(\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)) \cdot (\beta/p^-)} = \frac{p^-}{\beta} \geq 1 - \delta_1.
\]

In turn, if the distributions \( h_b(s), h_s(s) \) satisfy (A4) and (A5), then for any

\[
z = \frac{b}{a} \in [1 - \delta_1, 1 + \delta_2]
\]

the formulas in (4.7) uniquely determine the couple of prices \( (p^-, p^+) \).

2. We now consider the composite mapping

\[
z = \frac{b}{a} \mapsto (p^-, p^+) \mapsto Z(p^-, p^+). \tag{4.15}
\]
This is a continuous map from the interval \([1 - \delta_1, 1 + \delta_2]\) into itself. Hence it has a fixed point.

Having determined the expected payoffs \(a, b\) and the minimum and maximum prices \(p^-, p^+\), the shape of the two parts of the LOB are then determined by (2.22). This yields a solution to the bidding game, proving the first part of the theorem.

3. Next, assume that the additional inequality (4.12) holds. We claim that in this case the composed map (4.15) is a strict contraction, hence it has a unique fixed point. Indeed, computing the derivative of this map, one finds

\[
\frac{dZ}{dz} = \frac{\partial Z}{\partial p^+} \cdot \frac{\partial p^+}{\partial z} + \frac{\partial Z}{\partial p^-} \cdot \frac{\partial p^-}{\partial z}.
\]

As in (4.8), it is convenient to write \(p^+ = \beta \sigma^+(z), p^- = \beta \sigma^-(z)\). The above derivative can be computed as

\[
\frac{dZ}{dz} = \frac{\partial Z}{\partial \sigma^+} \cdot \frac{\partial \sigma^+}{\partial z} + \frac{\partial Z}{\partial \sigma^-} \cdot \frac{\partial \sigma^-}{\partial z}. \tag{4.16}
\]

A direct computation yields

\[
\frac{\partial Z}{\partial \sigma^+} = \frac{\Theta_1 (h'_b(\sigma^+) (\sigma^+ - Z) + h_b(\sigma^+))}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-},
\]

\[
\frac{\partial Z}{\partial \sigma^-} = \frac{\Theta_2 (h'_s(\sigma^-) (1 - \frac{Z}{\sigma^-}) + Z h_s(\sigma^-) \frac{1}{(\sigma^-)^2})}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-}.
\]

From (4.8) it follows that \(\sigma^+\) satisfies

\[h'_b(\sigma^+)(\sigma^+ - z) + h_b(\sigma^+) = 0.\]

Differentiating w.r.t. \(z\), we obtain

\[h''_b(\sigma^+)(\sigma^+ - z) \frac{\partial \sigma^+}{\partial z} + h'_b(\sigma^+) \frac{\partial \sigma^+}{\partial z} - h'_b(\sigma^+) \frac{\partial \sigma^+}{\partial z} + h'_b(\sigma^+) \frac{\partial \sigma^+}{\partial z} = 0.\]

By the assumption (A4) it follows

\[h''_b(\sigma^+)(\sigma^+ - z) + 2h'_b(\sigma^+) < h'_b(\sigma^+) < 0,\]

which implies

\[0 \leq \frac{\partial \sigma^+}{\partial z} < 1. \tag{4.17}\]

Still from (4.8), it follows that \(\sigma^-\) satisfies

\[h'_s(\sigma^-) \left(\frac{1}{\sigma^-} - \frac{1}{z}\right) - h_s(\sigma^-) \frac{1}{(\sigma^-)^2} = 0\]

Still from (4.8), it follows that \(\sigma^-\) satisfies

\[h''_s(\sigma^-) \left(\frac{1}{\sigma^-} - \frac{1}{z}\right) \frac{\partial \sigma^-}{\partial z} - h'_s(\sigma^-) \frac{2}{(\sigma^-)^2} \frac{\partial \sigma^-}{\partial z} + h'_s(\sigma^-) \frac{1}{z^2} + h'_s(\sigma^-) \frac{2}{(\sigma^-)^2} \frac{\partial \sigma^-}{\partial z} = 0.\]
From the assumption (A5) it now follows
\[ h''_s(\sigma^-) \left( \frac{1}{\sigma^-} - \frac{1}{z} \right) - 2h'_s(\sigma^-) \frac{1}{(\sigma^-)^2} + h_s(\sigma^-) \frac{2}{(\sigma^-)^3} < h'_s(\sigma^-) \frac{1}{(\sigma^-)^2} < 0, \]
which implies
\[ 0 \leq \frac{\partial \sigma^-}{\partial z} = \frac{-h'_s(\sigma^-) \frac{1}{z^2}}{h''_s(\sigma^-) \left( \frac{1}{\sigma^-} - \frac{1}{z} \right) - 2h'_s(\sigma^-) \frac{1}{(\sigma^-)^2} + h_s(\sigma^-) \frac{2}{(\sigma^-)^3}} < 1. \] (4.18)

We remark that all previous computations were made under the assumption that \( \sigma^+, \sigma^- \) lie in the interior of the intervals \([1, 1 + \delta_2]\) and \([1 - \delta_1, 1]\), respectively. If they took values on the extreme points of these intervals, they would be constant. In this trivial case the derivatives \( \partial \sigma^+/\partial z \) or \( \partial \sigma^-/\partial z \) would be zero.

4. Using (4.17)-(4.18) in (4.16), we now obtain
\[
\frac{d}{dz} Z(p^+(z), p^-(z)) < \frac{\partial Z}{\partial \sigma^+} + \frac{\partial Z}{\partial \sigma^-} \\
= \Theta_1 (h'_b(\sigma^+)(\sigma^+ - Z) + h_b(\sigma^+)) + \Theta_2 \left( h'_s(\sigma^-) \left(1 - \frac{Z}{\sigma^-}\right) + Z h_s(\sigma^-) \frac{1}{\sigma^-}\right) \frac{1}{\sigma^-}.
\] (4.19)

Since \( Z \in [\sigma^-, \sigma^+] \), observing that \( h'_b(\sigma^+) \leq 0, h'_s(\sigma^-) \geq 0 \), and then using the assumption (4.12), from (4.19) we obtain
\[
\frac{dZ}{dz} \leq \frac{\Theta_1 h_b(\sigma^+) + \Theta_2 \cdot Z h_s(\sigma^-) \frac{1}{\sigma^-}}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-} \leq \frac{\Theta_1 + \Theta_2 \frac{1 + \delta_2}{(1 - \delta_1)^2}}{\gamma + \Theta_1 + \Theta_2} < 1.
\]

This proves that the map \( z \mapsto Z \) in (4.15) is a strict contraction, and hence it admits a unique fixed point. \( \square \)

5 Agents’ profit depending on the size of the LOB

From now on, we focus on the case of deterministic acceptable prices, as in (3.1).

Based on the formulas (3.5)-(3.7), one can compute how the expected payoffs vary, depending on the sizes \( K_b, K_s \) of the “buy” and “sell” portions of the LOB. We observe that the probabilities that these portions are entirely wiped out by an incoming order depend inversely on the sizes \( K_b, K_s \). Indeed, when \( K_s, K_b \) are large, there is more competition among agents and their profit decreases. In particular, we have
\[
\lim_{K_s \to 0} \Psi^s = \lim_{K_b \to 0} \Psi^b = 1, \quad \lim_{K_s \to +\infty} \Psi^s = \lim_{K_b \to +\infty} \Psi^b = 0.
\]

In view of (3.5), (3.7), this implies
\[
\lim_{K_b, K_s \to 0} \left[ \frac{a}{b} \right] = \left[ \gamma^2 + \gamma \mu - \mu^2 (1 - \theta) \theta \frac{\delta_1 + \delta_2}{1 - \delta_1} \right]^{-1} \cdot \left[ \gamma + \mu \theta + \mu (1 - \theta)/(1 - \delta_1) \right], \quad \lim_{K_b, K_s \to 0} \left[ \frac{a}{b} \right] = \left[ \gamma + \mu \theta + \mu (1 + \delta_2) + \mu (1 - \theta) \right]. \] (5.1)
\[ \lim_{K_b, K_s \to +\infty} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/\gamma \\ 1/\gamma \end{bmatrix}. \]

In turn, using (3.8), we obtain information on the bid-ask spread:

\[ \lim_{K_b, K_s \to 0} \begin{bmatrix} p_A \\ p_B \end{bmatrix} = \begin{bmatrix} (1 + \delta_2)\beta \\ (1 - \delta_1)\beta \end{bmatrix}, \quad \lim_{K_b, K_s \to +\infty} \begin{bmatrix} p_A \\ p_B \end{bmatrix} = \begin{bmatrix} \beta \\ \beta \end{bmatrix}. \] (5.3)

To continue the analysis, as in (3.11) we make some simplifying assumptions.

(A6) The sizes of the two portions of the LOB and the tail distributions (2.3) of the random incoming orders satisfy

\[ K_s = K_b \deq K, \quad \Psi_b(s) = \Psi_s(s) \deq \Psi(s) \quad \text{for all} \ s \geq 0. \] (5.4)

Moreover,

\[ \theta = 1 - \theta = \frac{1}{2}, \quad 1 + \delta_2 = \frac{1}{1 - \delta_1} \deq 1 + \delta. \] (5.5)

Proposition 5.1 In the setting of Theorem 3.1, let the assumptions (A6) hold. Then the payoffs \( a, b \) in (3.7) are monotone decreasing functions of \( K \).

Proof. Under the assumptions (A6), the matrix \( A \) in (3.3) takes the form

\[ A(K) = \begin{pmatrix} \gamma + \mu \Psi(K) & -\mu/2 (1 + \delta) \Psi(K) \\ -\mu/2 (1 + \delta) \Psi(K) & \gamma + \mu \Psi(K) \end{pmatrix}. \] (5.6)

As in Example 3.1, one can immediately check that the solution to the linear system (2.16) is now given by

\[ a = b = \left( \gamma - \frac{\mu \delta}{2} \Psi(K) \right)^{-1}. \] (5.7)

Since \( \Psi'(K) < 0 \), this implies \( a'(K) = b'(K) < 0 \), completing the proof. \( \square \)

Remark 5.1 Consider an agent whose initial holdings in cash and stock have total value

\[ J_0 = C_0 + \beta S_0. \]

In the above setting, his exponentially discounted expected payoff will be

\[ J = \left( \gamma - \frac{\mu \delta}{2} \Psi(K) \right)^{-1} J_0. \] (5.8)

This leads to a natural question. Assume that all agents posting limit orders expect a certain minimum payoff (after the exponential discount), say

\[ J \geq \kappa J_0. \] (5.9)
For any $\kappa$ such that
$$\frac{1}{\gamma} < \kappa < \frac{1}{\gamma - (\mu\delta/2)},$$
by the previous analysis there exists a unique size $K^*$ of the two portions of the LOB such that
$$\left(\frac{\gamma - \mu\delta}{2}\Psi(K^*)\right)^{-1} = \kappa.$$
This determines a maximum viable size $K^*$ of the LOB, implicitly defined by
$$\Psi(K^*) = \frac{2}{\mu\delta} \left(\gamma - \frac{1}{\kappa}\right).$$
When $K > K^*$, trading on the LOB is not sufficiently profitable, and some agents will move away. In the present model, the maximum size of the LOB is thus determined by the profit that the agents want to achieve.

6 An asset with fundamental value modeled by a jump process

In this section we assume that the fundamental value $\beta(t)$ of the asset is piecewise constant in time, and jumps at random times $0 < t_1 < t_2 < \cdots$. These times will be modeled by a Poisson arrival process with rate $\lambda$. Setting
$$\beta(t) = \beta_j \quad t \in [t_j, t_{j+1}[,\quad (6.1)$$
we assume that, at the jump times $t_j$, the ratios
$$Z_j \equiv \frac{\beta_j}{\beta_{j-1}}\quad (6.2)$$
are independent, identically distributed random variables, with distribution function
$$\text{Prob.}\{Z \leq s\} = \eta(s), \quad (6.3)$$
for some absolutely continuous function $\eta(\cdot)$. We assume that, when the fundamental value $\beta(t)$ jumps, this immediately becomes common knowledge to all agents. The LOB thus retains the same shape as before, will all prices being multiplied by the same ratio $Z_j$.

As in the previous models, we assume that external orders arrive with rate $\mu$. Moreover, to simplify the analysis, as in (3.2) we assume that the maximum and minimum acceptable prices $p^-, p^+$ are deterministic functions of $\beta$:
$$p^-(t) = (1 - \delta_1)\beta(t), \quad p^+ = (1 + \delta_2)\beta(t). \quad (6.4)$$
Combining the two independent arrival processes, we obtain a Poisson arrival process with intensity $\lambda + \mu$. Call $t_i$ the random time when the $i$-th event occurs. This event can now be of three types:

1) The fundamental value $\beta$ of the stock has a jump.
2) A buy order arrives,
3) A sell order arrives,

The probabilities of these three events are, respectively,

\[ \theta_1 = \frac{\lambda}{\lambda + \mu}, \quad \theta_2 = \frac{\mu\theta}{\lambda + \mu}, \quad \theta_3 = \frac{\mu(1 - \theta)}{\lambda + \mu}, \]  

(6.5)

Assuming that the two arrival processes are independent, we have

\[ \text{Prob.}\{t_i - t_{i-1} > s\} = e^{-(\lambda + \mu)s}. \]  

(6.6)

Let \( V^C(\beta_0) \) denote the expected payoff of a unit amount of cash, \( V^S(\beta_0) \) denote the expected payoff of one unit amount of stock, given that the initial fundamental value of the stock as \( \beta_0 \).

\[
V^C(\beta_0)C_0 + V^S(\beta_0)S_0 = \int_0^{+\infty} (\lambda + \mu) e^{-(\lambda + \mu)t} \left\{ \int_0^{t_1} e^{-\gamma t}(C_0 + \beta_0 S_0) \, dt + \theta_1 e^{-\gamma t_1} \cdot E\left[ V^C(\beta_1)C_0 + V^S(\beta_1)S_0 \big| \beta_0 \right] \right. \\
+ \theta_2 e^{-\gamma t_1} \cdot \left[ (V^C(\beta_0)C_0 + V^C(\beta_0)\Psi^p p^+ S_0 + (1 - \Psi^p) V^S(\beta_0)S_0) \right. \\
+ \theta_3 e^{-\gamma t_1} \cdot \left[ V^S(\beta_0)\Psi^b p^- C_0 + V^C(\beta_0)(1 - \Psi^b) C_0 + V^S(\beta_0)S_0) \right] \bigg\} \, dt_1. 
\]  

(6.7)

As before, we denoted by \( \Psi^\sharp \equiv \Psi_b(K_s) \) and \( \Psi^\flat \equiv \Psi_s(K_b) \) the probabilities that a bid at the maximum price \( p^+ \) or at the minimum price \( p^- \) at (6.4) will be fulfilled, respectively.

We seek a solution in the form

\[
V^C(\beta) = a, \quad V^S(\beta) = b\beta. 
\]  

(6.8)

In view of (6.5), from (6.7) we obtain

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\gamma + \mu + \lambda} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\lambda}{\gamma + \mu + \lambda} \left\{ a E[\beta_1 | \beta_0] / \beta_0 \right\} \\
+ \frac{\mu}{\gamma + \mu + \lambda} \left[ \theta + (1 - \theta)(1 - \Psi^b) (1 - (1 - \theta)\Psi^b (\beta_0/p^-)) \right] \begin{bmatrix} a \\ b \end{bmatrix}. 
\]  

(6.9)

In turn, this yields the linear system

\[
A_\lambda \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_\lambda \doteq \begin{bmatrix} \gamma + \mu(1 - \theta)\Psi^b & -\frac{\mu(1 - \theta)\Psi^b}{1 - \delta_1} \\ -\mu\theta(1 + \delta_2)\Psi^\flat & \gamma + \lambda (1 - E[Z]) + \mu\theta\Psi^\flat \end{bmatrix}. 
\]  

(6.10)

A slight modification of the proof of Theorem 3.1 now yields
Theorem 6.1 Assume that the exponential discount factor $\gamma$ satisfies (3.4) and
\[
E[Z] = \int_0^{+\infty} s \eta'(s) \, ds = 1. \tag{6.11}
\]
Then the infinite horizon game with exponential discount $\gamma$ and deterministic acceptable prices (6.4) admits a unique equilibrium solution.

Indeed, the linear system (6.10) admits a unique solution as long as the matrix $A_\lambda$ has a strictly positive determinant. As shown in Section 3, the assumption (3.4) implies that the determinant of the matrix $A$ at (3.3) is positive. Comparing $A$ with $A_\lambda$, we see that $\det(A_\lambda) > 0$ whenever $\lambda(1 - E[Z]) \geq 0$.

As soon as the constants $a, b$ are determined, the shape of the two sides of the LOB is found in the same way as in the case where the fundamental price $\beta$ is a constant.

7 A model with informed external agents

The model with random jumps in the fundamental value of the stock becomes more interesting if we include the possibility that an external agent gets hold of the information before any of the traders has time to react and change the prices posted on the LOB. As a consequence:

- If the fundamental value of the stock increases, i.e. $\beta_j > \beta_{j-1}$, then all assets put on sale at a price $< \beta_j$ are immediately bought.
- If the value decreases, i.e. $\beta_j < \beta_{j-1}$, then all of the assets that the traders offered to buy at a price $> \beta_j$ are immediately sold.

In this new setting, the agents posting limit orders on the LOB must take this additional possible scenario in consideration. We wish to understand how their strategy should change, and what will be the new shape of the LOB and the expected payoffs.

We consider the same model as in (6.1)–(6.6), assuming that
\[
E[Z] = 1, \tag{7.1}
\]
so that the fundamental value $\beta(t)$ of the stock is a martingale. For example, this happens when the distribution of the jumps is log-normal, such that
\[
\ln \left( \frac{\beta_{j+1}}{\beta_j} \right) \sim \mathcal{N} \left( -\frac{\sigma^2}{2}, \sigma^2 \right). \tag{7.2}
\]

Moreover, at each time when this value jumps, we assume that with positive probability $\varepsilon > 0$ an external agent knows the information in advance.

To study this model, we consider the random times $0 < t_1 < t_2 < \cdots$, where either an incoming buy or sell order arrives, or else the fundamental value of the stock has a jump. Assuming that the external orders come with intensity $\mu$, while jumps in the stock value occur with intensity $\lambda$, the time increments $t_i - t_{i-1}$ between one event and the next are i.i.d. Poisson random variables, with tail distribution (6.6). At each random time $t_i$, four possibilities can now occur:
1) the fundamental value $\beta$ of the stock has a jump, and nothing else happens,
2) $\beta$ has a jump, and an informed external agent wipes out one side of the LOB,
3) a buy order arrives,
4) a sell order arrives.

The corresponding probabilities are

$$
\theta_1 = \frac{(1-\varepsilon)\lambda}{\lambda + \mu}, \quad \theta_2 = \frac{\varepsilon\lambda}{\lambda + \mu}, \quad \theta_3 = \frac{\mu\theta}{\lambda + \mu}, \quad \theta_4 = \frac{\mu(1-\theta)}{\lambda + \mu}.
$$

As before, we call

$$
V^C(\beta) C_0 + V^S(\beta) S_0
$$

the expected (exponentially discounted) payoff to an agent holding an amount $C_0$ of cash and an amount $S_0$ of stock, when the fundamental value of the stock is $\beta$. We seek a solution in the form (6.8), for suitable constants $a,b$.

To determine $a$ and $b$, we proceed in the same way as in (6.7), but now taking into account the possible presence of an informed external agent. Since the LOB is supposed to be in equilibrium, we can compute $V^C, V^S$ in connection with the minimum and maximum acceptable prices $p^-, p^+$ at (6.4). For a trader posting limit orders at $p^-, p^+$, when the fundamental value has a jump and an informed external agent comes, three cases can occur.

(i) $\beta_i/\beta_{i-1} < 1 - \delta_1$. In this case, the (uniformed) trader buys the stock at price $p^- = (1-\delta_1)\beta_{j-1}$, larger than its actual value $\beta_j$.

(ii) $\beta_i/\beta_{i-1} > 1 + \delta_2$. In this case, the (uniformed) trader sells the stock at price $p^+ = (1+\delta_2)\beta_{j-1}$, smaller than its actual value $\beta_j$.

(iii) $\beta_i/\beta_{i-1} \in [1 - \delta_1, 1 + \delta_2]$. In this case, the fundamental value has a jump, but the uniformed trader does not but or sell any stock.

To compute the expected payoff in these three cases, we consider again the random variable $Z$ with the distribution function $\eta(\cdot)$, defined at (6.2)-(6.3). We introduce the constants

$$
\begin{align*}
P_1 &\doteq \text{Prob.}\{Z < 1 - \delta_1\} = \eta(1-\delta_1), \\
P_2 &\doteq \text{Prob.}\{Z > 1 + \delta_2\} = 1 - \eta(1+\delta_2),
\end{align*}
$$

$$
e_1 \doteq E[Z \mid Z < 1 - \delta_1] = \frac{1}{P_1} \int_0^{1-\delta_1} s \eta'(s) \, ds, \\
e_2 \doteq E[Z \mid Z > 1 + \delta_2] = \frac{1}{P_2} \int_{1+\delta_2}^{+\infty} s \eta'(s) \, ds.
$$

If at time $t_1$ the fundamental value of the stock has a jump and an external informed agent arrives, then the expected payoff of a trader initially with $C_0$ in cash and $S_0$ in stock, posting
limit orders at prices $p^-, p^+$, can be computed as

$$E \left[ V^c(\beta_0)C_1 + V^s(\beta_0)S_1 \bigg| \text{an informed agent arrives} \right]$$

$$= \left\{ \text{Prob.}\{\beta_1 < p^-\} \frac{1}{p^-} E \left[ V^s(\beta_1) \bigg| \beta_1 < p^- \right] + \text{Prob.}\{\beta_1 \geq p^-\} E \left[ V^c(\beta_1) \bigg| \beta_1 \geq p^- \right] \right\} C_0$$

$$+ \left\{ \text{Prob.}\{\beta_1 > p^+\} p^+ E \left[ V^c(\beta_1) \bigg| \beta_1 > p^+ \right] + \text{Prob.}\{\beta_1 \leq p^+\} E \left[ V^s(\beta_1) \bigg| \beta_1 \leq p^+ \right] \right\} S_0$$

$$= \left\{ \frac{P_1}{1 - \delta_1} (1 - \delta_1) \beta_0 \right\} b \beta_1 + (1 - P_1) a \right\} C_0 + \left\{ P_2 (1 + \delta_2) \beta_0 a + (1 - P_2) E \left[ b \beta_1 \bigg| \beta_1 \leq (1 + \delta_2) \beta_0 \right] \right\} S_0$$

$$= \left\{ \frac{P_1 e_1}{1 - \delta_1} b + (1 - P_1) a \right\} C_0 + \left\{ P_2 (1 + \delta_2) \beta_0 a + (E[Z] - P_2 e_2) b \right\} S_0 .$$

(7.7)

The identity (6.7) is now replaced by

$$V^c(\beta_0)C_0 + V^s(\beta_0)S_0 = \int_0^{+\infty} (\lambda + \mu) e^{-(\lambda + \mu) t_1} \left\{ \int_0^{t_1} e^{-\gamma t} (C_0 + \beta_0 S_0) \, dt \right\} d t_1$$

$$+ \theta_1 e^{-\gamma t_1} \cdot \left[ V^c(\beta_1)C_0 + V^s(\beta_1)S_0 \right]$$

$$+ \theta_2 e^{-\gamma t_1} \cdot E \left[ V^c(\beta_1)C_1 + V^s(\beta_1)S_1 \bigg| \text{an informed agent arrives} \right]$$

$$+ \theta_3 e^{-\gamma t_1} \cdot \left[ (V^c(\beta_0)C_0 + V^c(\beta_0)\Psi^p p^+ S_0 + (1 - \Psi^p) V^s(\beta_0)S_0) \right]$$

$$+ \theta_4 e^{-\gamma t_1} \cdot \left[ V^s(\beta_0)\Psi^p \frac{1}{p^-} C_0 + V^c(\beta_0)(1 - \Psi^p) C_0 + V^s(\beta_0)S_0 \right] \right\} d t_1 .$$

(7.8)

As in (2.14), the constants $\Psi^p, \Psi^v$ denote the probability that, when an external buyer or seller comes, a limit order at the maximum acceptable price $p^+$, or at the minimum acceptable price $p^-$, will be executed, respectively. To simplify the notation, we introduce the constants

$$a \doteq V^c(\beta_0), \quad b \doteq \frac{V^s(\beta_0)}{\beta_0} .$$

Adopting matrix notation, (7.8) leads to

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\gamma + \mu + \lambda} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\mu}{\gamma + \mu + \lambda} \begin{bmatrix} \theta + (1 - \theta)(1 - \Psi^p) & (1 - \theta)\Psi^p \frac{\beta_0}{p^-} \\ \theta \Psi^p \frac{p^+}{\beta_0} & \theta(1 - \Psi^p) + (1 - \theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$+ \frac{\lambda}{\gamma + \lambda + \mu} \begin{bmatrix} 1 - \varepsilon p_1 \\ \varepsilon P_2 (1 + \delta_2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} .$$

(7.9)

This can be written as

$$A_\varepsilon \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

(7.10)
where
\[ A_\varepsilon = A + \varepsilon \lambda B, \] 
with
\[ A = \begin{bmatrix}
\gamma + \mu(1-\theta)\Psi^b & -\mu(1-\theta)\Psi^b \frac{1}{1-\delta_1} \\
-\mu \theta \Psi^z (1+\delta_2) & \gamma + \mu \theta \Psi^z + \lambda (1-E[Z])
\end{bmatrix}, \quad B = \begin{bmatrix}
P_1 & -P_1 e_1 \\
-P_2(1+\delta_2) & P_2 e_2
\end{bmatrix}. \] 

(7.11) (7.12)

In order to prove the existence of a unique LOB profile also in the present setting, two steps are required. First, one needs to check that the matrix equation (7.10) uniquely determines the expected discounted payoffs \(a, b\). Secondly, we have to show that the assumption that the LOB is in Nash equilibrium uniquely determines its shape, as in (3.9).

**Theorem 7.1** Consider the infinite horizon game with exponential discount \(\gamma\) and deterministic acceptable prices (3.1), in the presence of informed external agents. Assume that the exponential discount factor \(\gamma\) satisfies (3.4), while \(E[Z] = 1\) as in (6.11). Then, for any \(\lambda, \varepsilon \geq 0\), the following holds.

(i) The system (7.10) has a unique solution.

(ii) Assume that, in addition, the distribution \(\eta(\cdot)\) of the jumps in (6.3) satisfies
\[ \frac{1 - \eta(s)}{s \eta'(s)} > \delta_1, \quad \text{for every } s \in ]1, 1+\delta_2[, \] 
\[ \int_0^s x \eta'(x) \, dx > \delta_2 \frac{1}{1+\delta_2}, \quad \text{for every } s \in ]1-\delta_1, 1[. \] 

Then there exists a unique shape of the LOB, determined as a Nash equilibrium.

**Proof.** 1. By (7.11) it follows
\[ \det A_\varepsilon = \det A + \varepsilon \lambda C + \varepsilon^2 \lambda^2 \det B, \] 
where, recalling (7.4)–(7.6) and the assumption \(E[Z] = 1\), we set
\[ C = \gamma (P_1 + P_2 e_2) + \mu(1-\theta)\Psi^b P_2 \left( e_2 - \frac{1+\delta_2}{1-\delta_1} \right) + \mu \theta \Psi^z P_1 \left( 1 - e_1 \frac{1+\delta_2}{1-\delta_1} \right). \] 

(7.15) (7.16)

The definitions (7.5) and (7.6) immediately imply
\[ e_1 < 1 - \delta_1, \quad e_2 > 1 + \delta_2. \]

A direct computation yields
\[ \det B = P_1 P_2 \left( e_2 - e_1 \frac{1+\delta_2}{1-\delta_1} \right) > 0. \] 

(7.17)
As observed in Section 6, if the assumption (3.4) holds, since $E[Z] = 1$ we have $\det A > 0$. In addition, $C > 0$. Combining (7.15) with (7.17), we conclude that the determinant of $A_\varepsilon$ is positive. By inverting the matrix $A_\varepsilon$, the explicit solution is found to be

$$
\begin{bmatrix}
a \\
b
\end{bmatrix} = A_\varepsilon^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\det A_\varepsilon} \left[ \begin{array}{c} \gamma + \mu \theta \Psi^2 + \frac{\mu(1-\theta)}{1-\delta_1} \Psi^0 + \lambda \varepsilon \left( P_2 e_2 + \frac{P_1 e_1}{1-\delta_1} \right) \\ \gamma + \mu \theta \Psi^2 (1 + \delta_2) + \mu (1-\theta) \Psi^0 + \lambda \varepsilon (P_1 + P_2 (1 + \delta_2)) \end{array} \right].
$$

(7.18)

Notice that this solution is independent of $\beta$, the current fundamental value of the stock. This proves (i).

The formula (7.18) describes the expected discounted payoffs for a trader who holds a unit amount of cash, or of stock. In the next two steps, relying on the fact that the LOB is in equilibrium (so that bids made at different prices $p$ all yield the same expected payoff), and using the additional assumptions (7.13)-(7.14), we determine the shape of the two sides of the LOB.

2. In this step we analyze the “sell portion” of the LOB. By assumption, the expected profit of an agent posting a sell order at any price $p \in [p_A, p^+]$ is a constant independent of $p$. As in Section 2, we call

$$U_s(p) \equiv \text{[total amount of stock posted for sale at price } \leq p].$$

Compared with (2.19), in the present setting we must also take into account what happens when the fundamental value of the stock has a jump, and an informed external trader may take advantage of this.

To fix ideas, let $\beta_0$ be the original value of the stock, and let $\beta_1$ be the new value, if a jump occurs. Six mutually exclusive scenarios can occur:

(i) The fundamental value $\beta$ does not change. An external buy order arrives and the stock offered at price $p$ is sold.

(ii) The fundamental value $\beta$ does not change. An external buy order arrives but the stock offered at price $p$ is not sold.

(iii) The fundamental value $\beta$ does not change. An external sell order arrives.

(iv) The fundamental value of the stock jumps from $\beta_0$ to $\beta_1$, and nothing else happens.

(v) The fundamental value of the stock jumps from $\beta_0$ to a value $\beta_1 \leq p$, and an informed outside agent comes, buying all stocks offered for sale at price $< \beta_1$.

(vi) The fundamental value of the stock jumps from $\beta_0$ to a value $\beta_1 > p$, and an informed outside agent comes, buying all stocks offered for sale at price $< \beta_1$.

Recalling that $\mu$ is the intensity of arrivals of buy or sell orders, while $\lambda$ is the intensity of jumps in $\beta$, with distribution function (6.2)-(6.3), the probabilities that these events occur are
computed respectively by

\[
\text{Prob. (i) occurs} = \frac{\theta \mu}{\lambda + \mu} \Psi_b(U_s(p))
\]

\[
\text{Prob. (ii) occurs} = \frac{\theta \mu}{\lambda + \mu} \left(1 - \Psi_b(U_s(p))\right)
\]

\[
\text{Prob. (iii) occurs} = \frac{(1 - \theta) \mu}{\lambda + \mu}
\]

\[
\text{Prob. (iv) occurs} = \frac{(1 - \varepsilon) \lambda}{\lambda + \mu}
\]

\[
\text{Prob. (v) occurs} = \frac{\varepsilon \lambda}{\lambda + \mu} \cdot \eta(p/\beta_0),
\]

\[
\text{Prob. (vi) occurs} = \frac{\varepsilon \lambda}{\lambda + \mu} \cdot (1 - \eta(p/\beta_0)).
\]

Multiplying these probabilities by the corresponding expected payoffs, one obtains

\[
ap \cdot \frac{\theta \mu}{\lambda + \mu} \Psi_b(U_s(p)) + b \beta_0 \cdot \frac{\theta \mu}{\lambda + \mu} \left(1 - \Psi_b(U_s(p))\right) + b \beta_0 \cdot \frac{(1 - \theta) \mu}{\lambda + \mu}
\]

\[
+ b \beta_0 \cdot \frac{(1 - \varepsilon) \lambda}{\lambda + \mu} + b \beta_0 \cdot \frac{\varepsilon \lambda}{\lambda + \mu} \cdot \int_0^{p/\beta_0} s \eta'(s) \, ds + ap \cdot \frac{\varepsilon \lambda}{\lambda + \mu} \cdot (1 - \eta(p/\beta_0))
\]

\[= \text{constant}.\] 

(7.20)

We here used the fact that, when case (iv) occurs, the martingale assumption (7.1) implies

\[b \cdot E[\beta_1|\beta_0] = b \beta_0.\]

Moreover, when case (v) occurs, one has

\[b \cdot E \left[ \beta_1 \bigg| \frac{\beta_1}{\beta_0} \leq \frac{p}{\beta_0} \right] = b \beta_0 \cdot \frac{\eta(p/\beta_0)}{\eta(p/\beta_0)} \cdot \int_0^{p/\beta_0} s \eta'(s) \, ds.\]

After performing some cancellations, we obtain an implicit equation for the function \(U_s\) describing the shape of the “sell” part of LOB:

\[
(ap - b \beta_0) \cdot \theta \mu \Psi_b(U_s(p)) + b \beta_0 \cdot \varepsilon \lambda \cdot \int_0^{p/\beta_0} s \eta'(s) \, ds + ap \cdot \varepsilon \lambda \left(1 - \eta\left(\frac{p}{\beta_0}\right)\right)
\]

\[= \text{constant}.\] 

(7.21)

The constant in (7.21) can be computed by taking

\[p = p^+ = (1 + \delta_2) \beta_0, \quad \Psi_b(U_s(p^+)) = \Psi^2.\]

Introducing the function

\[G_s(p) \equiv \frac{b}{a} \beta_0 \int_0^{p/\beta_0} s \eta'(s) \, ds + p \left(1 - \eta\left(\frac{p}{\beta_0}\right)\right),\]

the identity (7.21) can be written more concisely as

\[
\left(p - \frac{b \beta_0}{a}\right) \theta \mu \Psi_b(U_s(p)) + \varepsilon \lambda \frac{b \beta_0}{a} G_s(p) = \text{constant}.\] 

(7.22)
Observing that \((ap-bβ_0) · θμ > 0\) and \(Ψ'_i(u) < 0\), we determine a unique function \(p \mapsto U_s(p) \in [0, K_s]\) satisfying (7.21), on some interval \([p_A, p^+]\). To check that this function is monotone increasing, we differentiate (7.22) w.r.t. \(p\). After rearranging terms, we obtain

\[
\left( p - \frac{bβ_0}{a} \right) · θμ \Psi'_i(U_s(p)) \frac{U'_s(p)}{U_s(p)} = -θμ \Psi_i(U_s(p)) - ελG'_s(p),
\]

where

\[
G'_s(p) = 1 - η \left( \frac{p}{β_0} \right) - \left( 1 - \frac{b}{a} \right) \frac{p}{β_0} η' \left( \frac{p}{β_0} \right).
\]

This yields

\[
U'_s(p) = - \frac{aθμ(U_s(p))}{Ψ'_i(U_s(p))} \cdot (ap - bβ_0) - \frac{aελG'_s(p)}{μθ · Ψ'_i(U_s(p))} \cdot (ap - bβ_0).
\]

By the definition (6.3) we have \(η, η' \geq 0\). Therefore, if \(b ≥ a\), by (7.24) it immediately follows \(G'_s(p) > 0\). If \(b < a\), recalling the assumption (7.13) and observing that \(b/a > 1 - δ_1\), we obtain

\[
1 - η \left( \frac{p}{β_0} \right) ≥ \left( 1 - \frac{b}{a} \right) \frac{p}{β_0} η' \left( \frac{p}{β_0} \right)
\]

for every \(p \in \]β_0, (1 + δ_2)β_0[,\]

which implies \(G'_s(p) > 0\). We thus conclude that \(U'_s(p) > 0\) on the open interval \([p_A, p^+[^\), hence \(U_s(p)\) is monotone increasing.

3. In this last step we analyze the “buy” portion of the LOB. By assumption, the expected profit of an agent posting a buy order at any price \(p \in [p^-, p_A]\) is a constant independent of \(p\). As in Section 2, we call

\[
U_b(p) = \text{[total amount of stock that agents bid to buy at price } p\text{].}
\]

As in formula (7.20) for the “sell portion”, we multiply the probabilities of the six possible scenarios by the corresponding expected payoffs, and obtain

\[
\frac{(1 - θ)μ}{λ + μ} \Psi_s(U_b(p)) \cdot \frac{bβ_0}{p} + a · \frac{(1 - θ)μ}{λ + μ} \left( 1 - Ψ_s(U_b(p)) \right) + a · \frac{θμ}{λ + μ} + a · \frac{(1 - ε)λ}{λ + μ} + \frac{bβ_0}{p} · \frac{ελ}{λ + μ} · \int_{p/β_0}^{p/p/β_0} sη'(s) ds + a · \frac{ελ}{λ + μ} · (1 - η(p/β_0))
\]

= constant.

Introducing the function

\[
G_b(p) = \frac{bβ_0}{p} · \int_{p/β_0}^{p/p/β_0} sη'(s) ds - a · η \left( \frac{p}{β_0} \right),
\]

the identity (7.26) can be written more concisely as

\[
(1 - θ)μΨ_s(U_b(p)) \left( \frac{bβ_0}{p} - a \right) + ελG_b(p) = \text{constant.}
\]

(7.27)
Differentiating (7.27) w.r.t. $p$, one obtains
\[ U'(b)(p) = \frac{b \beta_0 \Psi_s(U_b(p))}{(b \beta_0 - ap) \Psi_s(U_b(p))} - \frac{\varepsilon \lambda G'_{b}(p)}{(1 - \theta) \mu \Psi_s(U_b(p)) (b \beta_0 / p - a)}, \tag{7.28} \]
where
\[ G'_{b}(p) = -\frac{b \beta_0}{p^2} \int_0^{p / \beta_0} s \eta'(s) \, ds + \frac{b - a}{\beta_0} \eta'\left(\frac{p}{\beta_0}\right). \tag{7.29} \]
If $b \leq a$, it is clear that $G'_{b}(p) < 0$. In the case where $b > a$, recalling the assumption (7.14) and observing that $b/a < 1 + \delta_2$, we obtain
\[ \int_0^{p / \beta_0} s \eta'(s) \, ds > \left(1 - \frac{a}{b}\right) \eta'\left(\frac{p}{\beta_0}\right), \quad \text{for any } p \in \lbrack 1 - \delta_1 b, \beta_0 \rbrack, \]
which implies $G'_{b}(p) < 0$. Since $\Psi_s'(\cdot) < 0$, we conclude that (7.27) determines a unique monotone decreasing function $p \mapsto U_b(p) \in [0, K_b]$, on some interval $[p$, $p_B]$. This completes the proof. \(\Box\)

**Remark 7.1** Comparing (7.18) with (3.7), we see that the main difference is due to the additional terms containing the product $\lambda \varepsilon$, describing the intensity of the jumps times the probability of the arrival of the external informed agent. As this product tends to zero, the solution will converge to the one found in Theorem 3.1.

**Remark 7.2** If $E[Z] = 1$ and the symmetry assumptions (A6) hold, then the matrices $A, B$ at (7.12) take the form
\[ A = \begin{bmatrix} \gamma + \frac{\xi}{2} \Psi & -\frac{\xi}{2} (1 + \delta) \Psi \\ -\frac{\xi}{2} (1 + \delta) \Psi & \gamma + \frac{\xi}{2} \Psi \end{bmatrix}, \quad B = \begin{bmatrix} P_1 & -(1 + \delta) P_1 e_1 \\ -P_2 (1 + \delta) & P_2 e_2 \end{bmatrix}. \tag{7.30} \]
If we further assume that
\[ P_1 - (1 + \delta) P_1 e_1 = -P_2 (1 + \delta) + P_2 e_2, \tag{7.31} \]
then the solution to the linear system at (7.10)-(7.12) satisfies $a = b$, and can thus be easily computed. In this case, retracing the steps 2 and 3 in the proof of Theorem 7.1, the existence of an equilibrium profile for the LOB can be achieved without using the additional assumptions (7.13)-(7.14).

**Example 7.1** Assume that the fundamental value $\beta(t)$ of the stock is a martingale, whose jumps have log-normal distribution as in (7.2). Introducing the notation
\[ \Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{s^2}{2}\right\} \, ds, \quad \phi(x) \equiv \Phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{s^2}{2}\right\}, \tag{7.32} \]
the distribution function (6.3) takes the form
\[ \text{Prob.}\{Z \leq s\} = \eta(s) = \Phi\left(\frac{\ln s}{\sigma} + \frac{\sigma}{2}\right). \tag{7.33} \]
From the definitions (7.4)–(7.6) we now obtain

\[
\begin{align*}
P_1 &= \eta(1 - \delta_1) = \Phi\left(\frac{\ln(1 - \delta_1)}{\sigma} + \frac{\sigma}{2}\right), \\
P_2 &= 1 - \eta(1 + \delta_2) = \Phi\left(-\frac{\ln(1 + \delta_2)}{\sigma} - \frac{\sigma}{2}\right), \\
P_1 e_1 &= \int_0^{1-\delta_1} s\eta'(s) \, ds = \Phi\left(\frac{\ln(1 - \delta_1)}{\sigma} - \frac{\sigma}{2}\right), \\
P_2 e_2 &= \int_{1+\delta_2}^{+\infty} s\eta'(s) \, ds = \Phi\left(-\frac{\ln(1 + \delta_2)}{\sigma} + \frac{\sigma}{2}\right).
\end{align*}
\]

(7.34)

Assuming that \(\delta_1, \delta_2\) satisfy (5.5), we further obtain

\[
P_2 e_2 + (1 + \delta)P_1 e_1 = \Phi\left(-\frac{\ln(1 + \delta)}{\sigma} + \frac{\sigma}{2}\right) + (1 + \delta)\Phi\left(-\frac{\ln(1 + \delta)}{\sigma} - \frac{\sigma}{2}\right) = P_1 + P_2(1 + \delta).
\]

(7.35)

This provides an example where the identity (7.31) holds.

8 Dependence on parameters

In the same setting considered in Theorem 7.1, we wish to understand how the parameters \(\varepsilon, \lambda, \text{ and } \sigma\) affect the shape of the LOB. Two specific issues will be investigated.

- The increase in the bid-ask spread \(p_A - p_B\).
- The decrease in the total volume of the LOB, if agents who cannot achieve a minimum expected payoff move away, i.e., they stop posting limit orders.

To simplify the main computations, throughout the following we assume that the “buy” and “sell” portions of the LOB satisfy the symmetry assumptions (A6), while the random jumps in the fundamental value of the stock, considered at (7.4)–(7.6), satisfy the identity (7.31). As shown in Example 7.1, this is indeed the case when the jumps follow a log-normal distribution. Under these assumptions, the \(2 \times 2\) matrices \(A, B\) in (7.12) take the simpler form (7.30). As a consequence, the system

\[
(A + \varepsilon\lambda B) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

(8.1)

has the explicit solution

\[
a = b = \left(\gamma - \frac{\mu \delta}{2} \Psi(K) + \varepsilon\lambda\left(1 - (1 + \delta)e_1\right)P_1\right)^{-1}.
\]

(8.2)

We recall that here \(\lambda\) is the intensity of jumps in the fundamental value, and can be regarded as a measure of volatility. Moreover, \(\varepsilon > 0\) is the probability that, immediately before a jump occurs, an informed external agent appears.
8.1 Growth of the bid-ask spread.

Computing the constant in (7.22) at $p = p_A$ and at $p = p^+$, one obtains
\[
\left(p_A - \frac{b\beta_0}{a}\right) \theta \mu \Psi_b(U_s(p_A)) + \varepsilon \lambda \frac{b\beta_0}{a} G_s(p_A) = \left(p^+ - \frac{b\beta_0}{a}\right) \theta \mu \Psi_b(K) + \varepsilon \lambda \frac{b\beta_0}{a} G_s(p^+). \tag{8.3}
\]

Observing that $\Psi_b(U_s(p_A)) = 1$, and recalling that under the previous assumptions we have $\theta = 1/2$ and $a = b$, we obtain
\[
(p_A - \beta_0) \frac{\mu}{2} + \varepsilon \lambda \beta_0 [G_s(p_A) - G_s(p^+)] = (p^+ - \beta_0) \frac{\mu}{2} \Psi(K). \tag{8.4}
\]

Since $p^+ = (1 + \delta_2)\beta_0$ is independent of $\varepsilon$, differentiating (8.4) w.r.t. $\varepsilon$ we obtain
\[
\left[\frac{\mu}{2} + \varepsilon \lambda \beta_0 \cdot G'(p_A)\right] \cdot \frac{\partial p_A}{\partial \varepsilon} + \lambda \beta_0 [G_s(p_A) - G_s(p^+)] = 0. \tag{8.5}
\]

Therefore, the rate of change of the minimum ask price $p_A$, depending on the probability $\varepsilon > 0$ of the presence of an informed external agent, is computed by
\[
\frac{\partial p_A}{\partial \varepsilon} = \frac{\lambda \beta_0 \cdot (G_s(p^+) - G_s(p_A))}{(\mu/2) + \varepsilon \lambda \beta_0 \cdot G'(p_A)} > 0. \tag{8.6}
\]

Indeed, by the analysis in the previous section it follows that $G'(p) > 0$, i.e., $G_s(p)$ is increasing w.r.t $p$. Since $p_A < p^+ = (1 + \delta_2)\beta_0$, we conclude that the above inequality (8.6) holds.

The analysis of the “buy” portion is similar. Indeed, (7.26) implies
\[
\frac{\mu}{2} \left(\frac{b\beta_0}{p_B} - a\right) + \varepsilon \lambda \cdot G_b(p_B) = \frac{\mu}{2} \Psi(K) \cdot \left(\frac{b\beta_0}{p} - a\right) + \varepsilon \lambda \cdot G_b(p^+).
\]

Differentiating the above equation w.r.t. $\varepsilon$ and recalling that $a = b$, we obtain
\[
\frac{\partial p_B}{\partial \varepsilon} = \frac{\lambda (G_b(p^-) - G_b(p_B))}{(\mu/2)(-b\beta_0/P_B^2) + \varepsilon \lambda G'_b(p_B)} < 0, \tag{8.7}
\]
because $G_b(p)$ is a decreasing function.

According to (8.6)-(8.7), if external better informed agents show up with increasing probability, then the bid-ask spread $p_A - p_B$ will also increase. This will reduce the potential loss to the traders posting limit orders, generated by the external agents.

**Remark 8.1** The two variables $\varepsilon, \lambda$ enter in the identity (8.3) only within the product $\varepsilon \lambda$. Therefore, if $\varepsilon > 0$ is kept constant, as the intensity $\lambda$ of the jumps increases the same computations as above yield
\[
\frac{\partial p_A}{\partial \lambda} > 0, \quad \frac{\partial p_B}{\partial \lambda} < 0. \tag{8.8}
\]
In other words, if the jumps in the fundamental value occurred more frequently, the LOB will have a wider bid-ask spread.
8.2 Liquidity reduction.

Here the analysis follows the same lines as in Section 5. Let $\beta_0$ be the fundamental value of the stock at the initial time $t = 0$. Consider an agent whose initial holdings in cash and stock have total value

$$J_0 = C_0 + \beta_0 S_0.$$  

In view of (8.2), his exponentially discounted expected payoff will be

$$J = \left( \gamma - \frac{\mu \delta}{2} \Psi(K) + \varepsilon \lambda \left( 1 - (1 + \delta)e_1 \right) P_1 \right)^{-1} J_0. \quad (8.9)$$

We recall that $P_1, e_1$ are the quantities defined at (7.4)-(7.5). Since $\Psi'(K) < 0$, from the above formula we obtain:

(i) By increasing probability $\varepsilon > 0$ of the presence of an informed external agent, or the volatility $\lambda$ of the stock, the expected profit of agents posting orders on the LOB decreases.

(ii) As the size $K$ of the LOB decreases, the expected profit of agents posting orders on the LOB increases.

As a consequence, given $\varepsilon, \lambda > 0$, for any $\kappa$ such that

$$\left( \gamma + \varepsilon \lambda \left( 1 - (1 + \delta)e_1 \right) P_1 \right)^{-1} < \kappa < \left( \gamma - \frac{\mu \delta}{2} + \varepsilon \lambda \left( 1 - (1 + \delta)e_1 \right) P_1 \right)^{-1}, \quad (8.10)$$

there exists a unique size $K$ of the LOB that allows $J = \kappa J_0$. This size decreases as the product $\varepsilon \lambda$ increases.

We observe that, for an agent that stops doing any trade, the exponentially discounted payoff is simply

$$J = \frac{1}{\gamma} J_0.$$

A comparison with (8.10) reveals that, if the volatility $\lambda$ is large enough so that

$$\varepsilon \lambda \left( 1 - (1 + \delta)e_1 \right) P_1 \geq \frac{\mu \delta}{2},$$

staying away from the LOB becomes the best possible strategy.

9 Concluding remarks

In this paper we analyzed some models of the limit order book in infinite time horizon, in a “mean field limit”. Namely, the amount of stock held by every agent is small, compared with the size of the entire LOB. This assumption greatly simplifies the analysis, and allows us to derive some explicit formulas. The more general case, where some agents post large amount of stock for sale on the LOB, may be studied by the techniques in [5].
Another modeling assumption which we have used is that agents posting limit orders are indistinguishable from each other. Their payoff is calculated by the same exponential discount factor, and they all share the same information. It would be of interest to analyze models including different types of agents. Namely: (i) “fast traders”, who can instantly access information about the change in the fundamental value of the stock, and (ii) “slow traders”, whose bids may be wiped out by better informed external agents, at times where the fundamental value of the stock jumps, as described in Section 7. We expect that the relative size of fast versus slow traders should be an important factor in determining the overall shape of the LOB. These issues are left for future investigation.

References


