Review Notes on Matrix Algebra

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Contents

1 Systems of linear equations

A general system of m linear equations in n variables has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases} \tag{1.1}$$

Here $x_1, \ldots, x_n$ are the unknowns, while the rectangular array of numbers

$$\widetilde{A} = [A \,|\, b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \cdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{1.2}$$

is the matrix of coefficients.

Example. For the system

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 3x_1 - x_2 + x_3 &= 7 \\ 5x_1 + 3x_2 - 4x_3 &= 6 \end{cases} \tag{1.3}$$

the corresponding matrix of coefficients is

$$\widetilde{A} = [A \,|\, b] = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 3 & -1 & 1 & 7 \\ 5 & 3 & -4 & 6 \end{array}\right]. \tag{1.4}$$

1.1 Gaussian elimination

To solve the system of equations (1.1), we observe that the are some special transformations that do not change its solutions. Namely, consider the following elementary operations:

(I) Multiply all terms in one equation by a constant $c \neq 0$.

(II) Switch the position of two equations

(III) Add to one equation a multiple of another equation.

In all cases, we obtain another system of equations which has exactly the same solutions. By performing several of the above operations one after the other, we can eventually reduce the original system (1.1) into another system, in triangular form, which can be readily solved.

Example. Starting with the system (1.3) we perform a series a series of elementary operations. Namely:

- we multiply the first equation by 1/2,
- to the second equation we add the first equation multiplied by -3,
- to the third equation we add the first equation multiplied by -5,
- to the third equation we add the second equation multiplied by -1.

In this way we obtain the equivalent systems

$$\begin{cases} x_1 + 2x_2 - x_3 &= 1 \\ 3x_1 - x_2 + x_3 &= 7 \\ 5x_1 + 3x_2 - 4x_3 &= 6 \end{cases} \qquad \begin{cases} x_1 + 2x_2 - x_3 &= 1 \\ -7x_2 + 4x_3 &= 4 \\ 5x_1 + 3x_2 - 4x_3 &= 6 \end{cases}$$

$$\begin{cases} x_1 + 2x_2 - x_3 &= 1 \\ -7x_2 + 4x_3 &= 4 \\ -7x_2 + x_3 &= 4 \end{cases} \qquad \begin{cases} x_1 + 2x_2 - x_3 &= 1 \\ -7x_2 + 4x_3 &= 4 \\ -3x_3 &= -3 \end{cases}$$

The last system of equations is in triangular form and can be easily solved:

$$x_3 = 1, \qquad x_2 = 0, \qquad x_1 = 2.$$

Observe that the same operations can be performed directly on the matrix $\widetilde{A}$ in (1.4).

More generally, we can transform the matrix $\widetilde{A}$ in (1.2) by performing three elementary operations:

(I) Multiply one row by a constant $c \neq 0$.

(II) Switch two rows.

(III) Add to one row a multiple of another row.

These operations do not change the set of solutions of the corresponding system (1.1). As in the above example, by performing a finite number of these elementary operations we can reduce the matrix $\widetilde{A}$ to a "triangular" form. More precisely:

Definition 1.1 *A matrix is in* **reduced row form** *if each row has a number of initial zeroes* strictly larger *than the previous row (unless all of its entries are already zero).*

Examples. The following matrices are in reduced row form:

$$\begin{bmatrix} 3 & 4 & 7 & -1 \\ 0 & 5 & -3 & 20 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are NOT in reduced row form:

$$\begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 4 & 7 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 1.1 *Every matrix can be put into reduced row form by performing a finite number of elementary operations (I)–(III).*

1.2 Solving a system in reduced row form (by backward substitution)

Given a matrix in reduced row form, the first non-zero element of each row is called its **leading entry**.

To solve a system of linear equations whose matrix is in reduced row form:

(i) Check if the system contains an equation of the form $0 = c$ for some constant $c \neq 0$. If this is the case, the system admits no solution. We then say that it is **inconsistent**.

(ii) Look at the columns which *do NOT contain a leading entry.* Choose arbitrary values for the corresponding variables x_i. Say, $x_i = c_i$.

(iii) Solve the equations for the remaining variables starting from x_n and proceeding backwards.

Example. The system

$$\begin{cases} 3x_1 + 4x_2 \qquad\quad - x_4 &= 3 \\ \qquad\qquad 3x_3 + 2x_4 &= 2 \\ \qquad\qquad\qquad\quad x_4 &= 7 \\ \qquad\qquad\qquad\quad 0 &= 0 \end{cases}$$

corresponds to the matrix in reduced row form

$$\widetilde{A} = [A|b] = \left[\begin{array}{cccc|c} \boxed{3} & 4 & 0 & -1 & 3 \\ 0 & 0 & \boxed{3} & 2 & 2 \\ 0 & 0 & 0 & \boxed{1} & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has solutions. Here the leading entries are shown inside a box. The second column in A does not contain a leading entry. Hence we can choose $x_2 = c_2$ arbitrary. Then we compute

- $x_4 = 7$, solving the third equation,
- $x_3 = -4$, solving the second equation,
- $x_1 = \dfrac{4 - 4c_2}{3}$, solving the first equation.

Note that this system has infinitely many solutions, depending on the choice of the arbitrary constant c_2.

2 Operations on matrices

A matrix is a rectangular array of (real or complex) numbers. We say that A is an $m \times n$ matrix if it has m rows and n columns. The standard notation is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \tag{2.1}$$

Here a_{ij} is the element in the i-th row and j-th column of A.

1 - Multiplication by a scalar number. Multiplying a $m \times n$ matrix A by a scalar number c we obtain the matrix

$$c\,A = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ & & \cdots & \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix},$$

where each entry of A is multiplied by c.

2 - Sum of two matrices. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, their sum is the matrix $C = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$. In other words

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \cdots & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ & & \cdots & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \tag{2.2}$$

3 - Product of two matrices. If A is $m \times n$ and B is $n \times p$, then the product AB is the $m \times p$ matrix $C = [c_{ik}]$, such that

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}\,.$$

Note that this value is obtained by multiplying the entries in the i-th row of A by the corresponding entries in the k-th column of B. This is possible if and only if

$$n = \#\text{ of columns in the matrix } A = \#\text{ of rows in the matrix } B.$$

Example. The product of the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -2 & 5 \end{bmatrix},$$

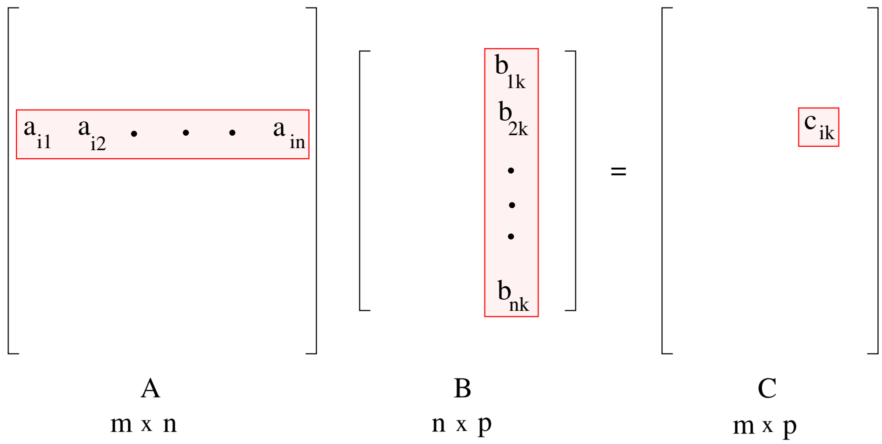

Figure 1: The element c_{ik} in the i-th row and k-th column of the product matrix $C = AB$ is obtained multiplying the i-th row of A with the k-th column of B.

is

$$AB = \begin{bmatrix} 13 & -6 & 18 \\ -4 & 4 & 2 \end{bmatrix}.$$

On the other hand, the product BA is not defined.

Example. Using matrix multiplication, the system of linear equations (1.1) can be written in the concise form

$$A\mathbf{x} = \mathbf{b}, \tag{2.3}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Properties of matrix operations.

- $A + B = B + A$,
- $(A + B) + C = A + (B + C)$,
- $(AB)C = A(BC)$,
- $(A + B)C = AC + BC$.

However, matrix multiplication is not commutative: in general $AB \neq BA$.
Moreover, it is possible that $AB = \mathbf{0}$ (the matrix whose elements are all $= 0$), even if $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.

Matrix multiplication by rows. Consider a product of matrices: $AB = C$. Call $A_1, A_2, \ldots, A_m$ the rows of A, and let $C_1, C_2, \ldots, C_m$ be the rows of C. Then the rows of C are obtained multiplying by B each row of A:

$$AB \;=\; \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \cdots \\ \hline A_m \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ & & \cdots & \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \;=\; \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \cdots \\ \hline A_mB \end{bmatrix} \;=\; C$$

Matrix multiplication by columns. Consider again a product of matrices: $AB = C$. Call $B_1, B_2, \ldots, B_p$ the columns of B, and let $C_1, C_2, \ldots, C_p$ be the columns of C. Then the columns of C are obtained multiplying by A each column of B:

$$AB \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \left[B_1 \middle| B_2 \middle| \cdots \middle| B_p \right] \;=\; \left[AB_1 \middle| AB_2 \middle| \cdots \middle| AB_p \right] \;=\; C$$

3 Matrix multiplication as a linear map

Let A be an $m \times n$ matrix. By matrix multiplication, for any column vector $\mathbf{x} \in \mathbb{R}^n$, we obtain a column vector $\mathbf{y} \in \mathbb{R}^m$, namely

$$\mathbf{y} \;\doteq\; A\mathbf{x}. \tag{3.1}$$

This defines a transformation $\mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$, from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Definition 3.1 *A map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is* **linear** *if, for every vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and any scalar number c one has*

(i) $T(c\mathbf{u}) \;=\; c\,T(\mathbf{u})$,

(ii) $T(\mathbf{u} + \mathbf{v}) \;=\; T(\mathbf{u}) + T(\mathbf{v})$.

NOTE: by the properties of matrix multiplication, one has

$$A(c\mathbf{u}) \;=\; c\,A\mathbf{u}, \qquad\qquad A(\mathbf{u}+\mathbf{v}) \;=\; A\mathbf{u} + A\mathbf{v}.$$

Hence the transformation $\mathbf{u} \mapsto A\mathbf{u}$ is linear.

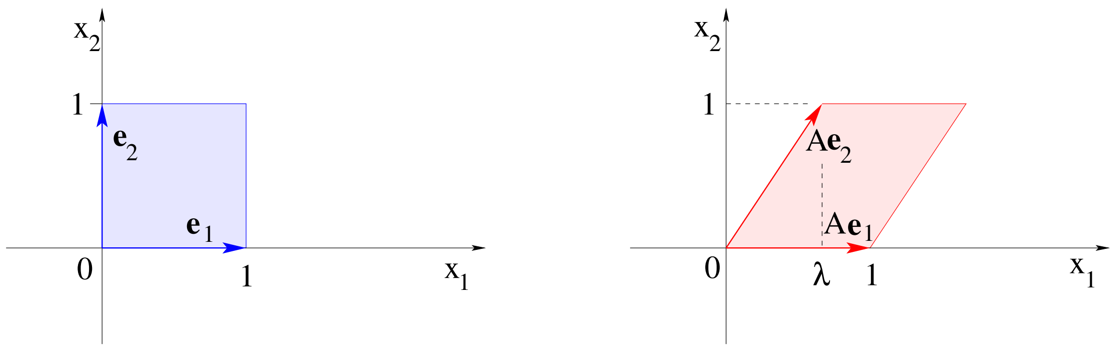

Figure 2: A shear transformation.

Consider the unit vectors in $\mathbb{R}^n$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots \quad , \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{3.2}$$

and write the matrix A in terms of its columns:

$$A \;=\; \Big[A_1 \Big| A_2 \Big| \quad \cdots \quad \Big| A_n \Big].$$

A direct computation shows that *the columns of A are precisely the images of the vectors* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, namely

$$A\mathbf{e}_1 = A_1\,, \qquad A\mathbf{e}_2 = A_2\,, \qquad \cdots \qquad A\mathbf{e}_n = A_n\,.$$

Therefore, knowing these images we completely determine the matrix A.

Examples.

- Taking $A = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ we obtain a **shear transformation,** shown in Fig. 2.
- Taking $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we obtain a counterclockwise **rotation by an angle** θ, as in Fig. 3.

Composition of maps: If $\mathbf{y} = A\mathbf{x}$ and $\mathbf{z} = B\mathbf{y}$, then the composed mapping is $\mathbf{z} = BA\mathbf{x}$. In other words, the composition of linear maps corresponds to the product of matrices.

4 Transposition

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Its transpose is the $n \times m$ matrix $A^T = [a^T_{ij}]$ obtained by switching the rows with the columns, so that

$$a^T_{ij} \;=\; a_{ji}\,.$$

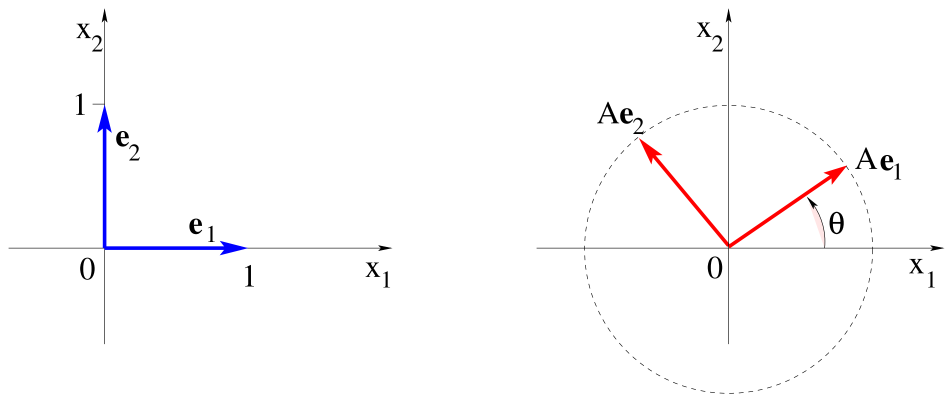

Figure 3: A rotation by an angle θ.

Example:

$$A = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 3 & 5 & 2 \\ 0 & -3 & 1 & 8 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 3 & -3 \\ 0 & 5 & 1 \\ 0 & 2 & 8 \end{bmatrix}.$$

An $n \times n$ matrix A is **symmetric** if $A^T = A$. It is **skew-symmetric** (sometimes also called "antisymmetric") if $A^T = -A$.

Examples. Among the matrices

$$A = \begin{bmatrix} 3 & 4 & 0 \\ 4 & -1 & 5 \\ 0 & 5 & 8 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & -5 \\ -2 & 5 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 3 & 5 \\ 0 & 2 & 8 \end{bmatrix},$$

A is symmetric, B is skew-symmetric, while C is neither symmetric nor skew-symmetric.

Properties of transposition:

- $(A^T)^T = A$,
- $(cA)^T = cA^T$,
- $(A+B)^T = A^T + B^T$,
- $(AB)^T = B^T A^T$.

4.1 Inner product

Given two column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \tag{4.1}$$

their **inner product** is defined as

$$\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n\,. \tag{4.2}$$

Notice that this is a 1×1 matrix. Namely, a single number.

5 Special Matrices

The $n \times n$ **identity matrix** is

$$I_n \doteq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The entries of this matrix are: 1 along the main diagonal and 0 everywhere else. Multiplying an $m \times n$ matrix A by an identity matrix (with the appropriate size), one finds

$$I_mA = A, \qquad AI_n = A.$$

An $n \times n$ matrix $A = [a_{ij}]$ is **diagonal** if all elements outside the main diagonal are zero. This means:

$$i \neq j \qquad \Longrightarrow \qquad a_{ij} = 0.$$

An $n \times n$ matrix $A = [a_{ij}]$ is **upper triangular** if all elements below the main diagonal are zero. This means:

$$i > j \qquad \Longrightarrow \qquad a_{ij} = 0.$$

Examples. Among the following matrices, A is diagonal, A and B are upper triangular, while C is neither diagonal nor upper triangular.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 0 \\ 0 & -5 & 8 \\ 0 & 0 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 9 & 0 & 0 \end{bmatrix}$$

5.1 Elementary matrices

(I) We denote by $E_i(c)$ the $m \times m$ matrix obtained from the identity matrix I_m multiplying the i-th row by c.

(II) We denote by E_{ij} the $m \times m$ matrix obtained from the identity matrix I_m by switching the i-th and the j-th rows.

(III) We denote by $E_{ij}(c)$ the $m \times m$ matrix obtained from the identity matrix I_m by adding to the i-th row the j-th row multiplied by c.

Examples: Starting with the 3×3 identity matrix, by performing elementary operations on the rows we obtain

$$E_2(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_{31}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}.$$

Theorem 5.1 *Multiplying any $m \times n$ matrix A to the left by an elementary matrix, the same elementary operation is performed on the rows of A.*

(I) The product $E_i(c)A$ is the matrix obtained from A multiplying its i-th row by c.

(II) The product $E_{ij}A$ is the matrix obtained from A switching its i-th and j-th rows.

(III) The product $E_{ij}(c)A$ is the matrix obtained from A adding to the i-th row the j-th row multiplied by c.

Example. Starting with the 3×2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, and multiplying it on the left by various elementary matrices, we obtain

$$E_2(c)A = \begin{bmatrix} 1 & 2 \\ 3c & 4c \\ 5 & 6 \end{bmatrix}, \qquad E_{23}A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 3 & 4 \end{bmatrix}, \qquad E_{31}(c)A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5+c & 6+2c \end{bmatrix}.$$

6 Inverse of a matrix

Let A be a $n \times n$ matrix. Then for any $n \times n$ matrix B we have

$$AB = I_n \qquad \text{if and only if} \qquad BA = I_n\,. \tag{6.1}$$

If this is the case, we say that B is the **inverse** of A, and write $B = A^{-1}$.

Examples. The matrix $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is the inverse of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Indeed,

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2\,,$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2\,.$$

Some matrices do not have an inverse. For example $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ has no inverse. Indeed, if

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$2a + c = 1, \qquad 2b + d = 0, \qquad 4a + 2c = 0, \qquad 4b + 2d = 1.$$

But this system of equations has no solution.

Properties of matrix inverses:

- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = \frac{1}{c} A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- If $AB = AC$ and A is invertible, then $B = C$.
  In particular, if the inverse of a matrix A exists, it must be unique.

To find the inverse of a matrix, we first introduce:

Definition 6.1 *We say that a matrix A is in* **reduced row echelon form** *if*

- *A is in reduced row form. Namely, every row of A has a number of initial zeroes strictly greater than the previous row.*
- *Every leading entry of a row of A is a 1.*
- *For every leading entry, all other elements in the same column are zero.*

Examples. Among the following matrices

$$A = \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}, \qquad B = \begin{bmatrix} \boxed{1} & 4 & 0 & -1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

A and B are in reduced row echelon form, C is not.

By performing elementary operations on the rows, every $n \times n$ matrix A can be put in reduced row echelon form. This yields:

An algorithm for finding the inverse of an $n \times n$ matrix A.

- *Consider the $n \times 2n$ composite matrix $[A|I]$, obtained by writing A and the identity matrix I_n next to each other.*
- *Perform a sequence of elementary operations on the rows, so that the left portion of this matrix is in reduced row echelon form*
- *If this has the form $[I|B]$, (that is: if the reduced echelon form of A is the identity matrix), then the matrix B appearing on the right is the inverse of A.*
- *In all other cases, (that is: if the reduced echelon form of A contains some row which is identically zero), then A is not invertible.*

Justification of the algorithm. Performing several elementary row operations, the left and the right portions of the composite matrix $[A|I]$ take the form

$$\Big[E_k\, E_{k-1} \cdots E_2\, E_1 A \;\Big|\; E_k\, E_{k-1} \cdots E_2\, E_1 I\Big], \tag{6.2}$$

where $E_1, E_2, \ldots, E_k$ are elementary matrices of type (I), (II), or (III).
If we can transform the left portion into the identity, this means

$$(E_k\, E_{k-1} \cdots E_2\, E_1)\, A \;=\; I.$$

Hence $E_k\, E_{k-1} \cdots E_2\, E_1 = A^{-1}$ is the inverse of A.
But this inverse is precisely the matrix $B = E_k\, E_{k-1} \cdots E_2\, E_1\, I$ appearing on the right portion of the composite matrix (6.2).

Theorem 6.1 (invertibility conditions). *Let A be an $n \times n$ matrix. The following statements are equivalent (if one is true, all the others are also true).*

(i) *The matrix A is invertible.*

(ii) *For every vector $\mathbf{b} \in \mathbb{R}^n$, the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution (namely $\mathbf{x} = A^{-1}\mathbf{b}$).*

(iii) *The linear system $A\mathbf{x} = 0$ has only the trivial solution $\mathbf{x} = 0$.*

7 Determinants

To each $n \times n$ matrix A we can associate a number, called the **determinant** of A.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$\det(A) \;=\; ad - bc\,.$$

When $n \geq 3$, the formulas become very complicated.

It is more convenient to compute determinants relying on their properties.

(D1) If $A = [a_{ij}]$ is an upper triangular matrix, then its determinant is the product of all diagonal elements:

$$\det(A) \;=\; a_{11}\, a_{22} \cdots\, a_{nn}\,.$$

(D2) If a row of a matrix is multiplied by a constant c, the determinant gets multiplied by c :

$$\det\bigl(E_i(c)\,A\bigr) \;=\; c\,\det(A).$$

(D3) If a multiple of a row is added to another row of A, the determinant does not change:

$$\det\bigl(E_{ij}(c)\,A\bigr) \;=\; \det(A).$$

(D4) If two rows of A are switched, the determinant changes sign:

$$\det\bigl(E_{ij}\,A\bigr) \;=\; -\det(A).$$

Given an arbitrary $n \times n$ matrix A, we can always put it in reduced row form (hence in upper triangular form) by performing a number of elementary operations on the rows. Using the properties **(D1)–(D4)**, this allows us to compute $\det(A)$.

Example. Let $A \;=\; \begin{bmatrix} 0 & 4 & 1 \\ 1 & 2 & 5 \\ -1 & 2 & 6 \end{bmatrix}$. We can put this matrix in reduced row form by means of three elementary operations:

$$A \;=\; \begin{bmatrix} 0 & 4 & 1 \\ 1 & 2 & 5 \\ -1 & 2 & 6 \end{bmatrix} \xrightarrow{E_{12}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 1 \\ -1 & 2 & 6 \end{bmatrix} \xrightarrow{E_{31}(1)} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 4 & 11 \end{bmatrix} \xrightarrow{E_{32}(-1)} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 10 \end{bmatrix} \;=\; B.$$

The matrix B is upper triangular, hence by **(D1)** its determinant is $\det(B) = 1 \cdot 4 \cdot 10 = 40$.

The elementary operation E_{12} changes the sign of the determinant, while $E_{31}(1)$ and $E_{32}(-1)$ leave the determinant unchanged. Hence $\det(B) = -\det(A)$, and therefore $\det(A) = -40$.

Additional properties of the determinant:

(D5) The matrix A is invertible if and only if $\det(A) \neq 0$.

(D6) The determinant does not change under transposition: $\det(A^T) = \det(A)$.

(D7) The determinant of a product is the product of the determinants:

$$\det(AB) \;=\; \det(A) \cdot \det(B).$$

(D8) If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

(D9) If A is an $n \times n$ matrix, then $\det(cA) = c^n \det(A)$.

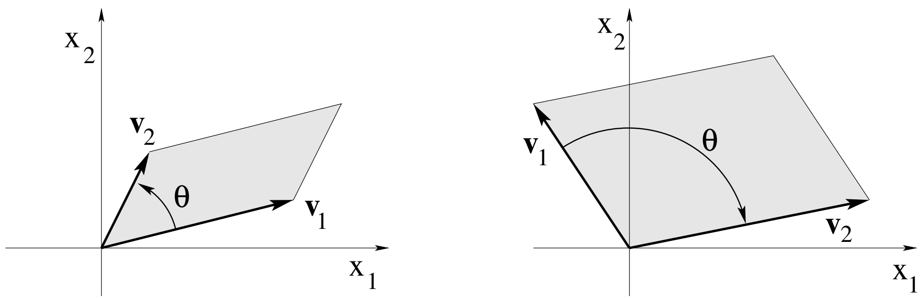

Figure 4: A plot of the column vectors of a 2×2 matrix $A = \left[\mathbf{v}_1 \middle| \mathbf{v}_2\right]$. Left: when the angle θ from $\mathbf{v}_1$ to $\mathbf{v}_2$ is positive (counterclockwise), we have $\det(A) > 0$. Right: when the angle θ from $\mathbf{v}_1$ to $\mathbf{v}_2$ is negative (clockwise), we have $\det(A) < 0$.

7.1 Geometric meaning of the determinant

Consider a 2×2 matrix $A = \left[\mathbf{v}_1 \middle| \mathbf{v}_2\right]$, which we write in terms of its column vectors. Then (see Fig. 4)

- The absolute value $|\det(A)|$ is the area of the parallelogram whose sides are $\mathbf{v}_1, \mathbf{v}_2$.
- The sign of $\det(A)$ is positive if the angle θ from $\mathbf{v}_1$ to $\mathbf{v}_2$ is positive (counterclockwise), and negative otherwise.

Next, consider a 3×3 matrix $A = \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \mathbf{v}_3\right]$, which we write in terms of its column vectors. Then (see Fig. 5)

- The absolute value $|\det(A)|$ is the volume of the parallelepiped whose edges are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
- The sign of $\det(A)$ is positive if the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are oriented according to the first three fingers (thumb, index, middle finger) of the right hand. It is negative otherwise.

7.2 Computing a determinant using cofactors

Given an $n \times n$ matrix $A = [a_{ij}]$, we denote by M_{ij} the smaller matrix obtained from A by removing the i-th row and the j-th column. The **cofactor** of a_{ij} is defined as

$$C_{ij} \;=\; (-1)^{i+j} \det(M_{ij}).$$

The determinant of A can now be computed in several different ways, multiplying the elements of any row (or any column) of A by their cofactors.

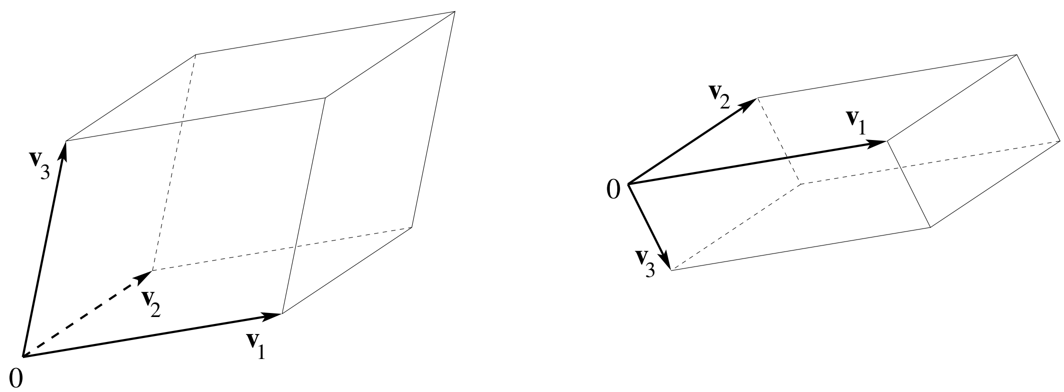

Figure 5: A plot of the column vectors of a 3×3 matrix $A = \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \mathbf{v}_3\right]$. Left: if the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are oriented as the first three fingers of the right hand, then $\det(A) > 0$. Right: if the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are oriented as the first three fingers of the left hand, then $\det(A) < 0$.

For example, consider the i-th row or the j-th column of the matrix A, namely

$$[a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}]\,, \qquad \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}.$$

Then we have

$$\det(A) \;=\; a_{i1}C_{i1} + a_{i2}C_{i2} + \;\cdots\; + a_{in}C_{in}\,, \tag{7.1}$$

and also

$$\det(A) \;=\; a_{1j}C_{1j} + a_{2j}C_{2j} + \;\cdots\; + a_{nj}C_{nj}\,.$$

Example. Consider the 3×3 matrix

$$A \;=\; \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 5 & -1 & 3 \end{bmatrix}.$$

We then have

$$M_{11} \;=\; \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \qquad C_{11} \;=\; (-1)^{1+1}\det M_{11} \;=\; 5,$$

$$M_{23} \;=\; \begin{bmatrix} 2 & 0 \\ 5 & -1 \end{bmatrix}, \qquad C_{23} \;=\; (-1)^{2+3}\det M_{23} \;=\; 2,$$

$$C_{12} \;=\; 10, \qquad C_{13} \;=\; -5, \qquad C_{33} \;=\; 2.$$

Computing the determinant using cofactors of the first row we obtain

$$\det(A) \;=\; a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \;=\; 2 \cdot 5 + 0 \cdot 10 + (-1) \cdot (-5) \;=\; 15.$$

Computing the determinant using cofactors of the third column we obtain

$$\det(A) \;=\; a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \;=\; (-1) \cdot (-5) + 2 \cdot 2 + 3 \cdot 2 \;=\; 15.$$

Remark 7.1 If we multiply the elements of a row times the cofactors of a different row, the sum is always zero.

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0 \qquad \text{if} \quad i \neq k\,. \tag{7.2}$$

Example. For a 3×3 matrix $A = [a_{ij}]$, taking $i = 3$, $k = 1$ we have

$$a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13} = \det \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0$$

Indeed, the matrix on the right hand side is obtained from A replacing the first row with the third row. Since this new matrix has two equal rows, its determinant is zero. When we put this matrix into reduced row form, the last row will be identically zero.

7.3 A formula for the inverse

Given a matrix $A = [a_{ij}]$, let $C = [C_{ij}]$ be the matrix of its cofactors, and let C^T be the transpose. Using (7.1) and (7.2) we compute the product

$$A\,C^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ & & \cdots & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A) \cdot I.$$

This yields a formula for the inverse:

$$\boxed{A^{-1} = \frac{1}{\det(A)}\, C^T}$$

where C is the matrix of cofactors of A.

7.4 Cramer's rule

Consider the system of n linear equations in n unknowns

$$A\mathbf{x} = \mathbf{b}\,.$$

If A is invertible, using the above formula for the inverse we obtain

$$\mathbf{x} \;=\; A^{-1}\mathbf{b} \;=\; \frac{1}{\det(A)}\, C^T\mathbf{b},$$

In components, this means

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;=\; \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ & & \cdots & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In particular, the i-th component is

$$x_i \;=\; \frac{b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}}{a_{1i}C_{1i} + a_{2i}C_{2i} + \cdots + a_{ni}C_{ni}}\,.$$

In this expression

- The denominator is the determinant of A, expanded using the cofactors of the i-th column.
- The numerator is the same as the denominator, with each a_{ki} replaced by b_k, for $k = 1, 2, \ldots, n$.

We can thus interpret this numerator as the determinant of the matrix B_i, obtained by replacing the i-th column of A with $\mathbf{b}$. This yields Cramer's rule:

$$\boxed{x_i \;=\; \frac{\det(B_i)}{\det(A)}}$$

8 Vector spaces

A **vector space** V is a set whose elements can be

- added together,
- multiplied by a scalar number.

These two operations should satisfy the usual commutative, associative, distributive properties. In particular, for any two vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and every real number $c \in \mathbb{R}$, we have

$$\mathbf{u} + \mathbf{v} \;=\; \mathbf{v} + \mathbf{u},$$

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} \;=\; \mathbf{u}+(\mathbf{v}+\mathbf{w}),$$
$$c(\mathbf{u}+\mathbf{v}) \;=\; c\mathbf{u}+c\mathbf{v}.$$

The zero vector will be denoted by $\mathbf{0} \in V$, while the number zero is $0 \in \mathbb{R}$. For every $\mathbf{v} \in V$ we have

$$\mathbf{0}+\mathbf{v} \;=\; \mathbf{v}, \qquad\qquad 0\,\mathbf{v} \;=\; \mathbf{0}.$$

Sometimes we also consider multiplication by complex numbers $c \in \mathbb{C}$, In this case, we say that V is **complex vector space**.

Examples.

1. The set $\mathbb{R}^2$ of all column vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, with v_1, v_2 real numbers, is a vector space with operations

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \;=\; \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}, \qquad\qquad c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \;=\; \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

2. For any fixed m, n, the set of all $m \times n$ matrices is a vector space, with the usual operations of addition of two matrices and multiplication by a scalar number.

3. The set of all polynomials of degree ≤ 2 is a vector space. If $p(x) = a_0+a_1x+a_2x^2$, $q(x) = b_0+b_1x+b_2x^2$ and $c \in \mathbb{R}$, then

$$(p+q)(x) \;=\; (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2, \qquad\qquad (c\,p)(x) \;=\; ca_0+ca_1x+ca_2x^2.$$

4. The set of all matrices (of any size) is NOT a vector space. For example, a 2×2 matrix cannot be added to a 3×4 matrix.

5. The set S of all vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with positive components (that is: with $v_1 \geq 0$, $v_2 \geq 0$), is NOT a vector space. For example,

$$\mathbf{v} \;=\; \begin{bmatrix} 3 \\ 2 \end{bmatrix} \;\in\; S, \qquad\qquad (-1)\mathbf{v} \;=\; \begin{bmatrix} -3 \\ -2 \end{bmatrix} \;\notin\; S.$$

8.1 Subspaces

Let V be a vector space. We say that a set $W \subseteq V$ is a **subspace** if

- For any two vectors $\mathbf{v}, \mathbf{w} \in W$, their sum $\mathbf{v}+\mathbf{w}$ is also in W
- For any vector $\mathbf{w} \in W$ and any number $c \in \mathbb{R}$, the product $c\mathbf{w}$ is still in W.

Theorem 8.1 *If W_1, W_2 are two subspaces of the vector space V, then the intersection $W_1 \cap W_2$ is also a subspace of V.*

However, the union $W_1 \cup W_2$ of two subspaces is NOT a subspace, in general.

Examples. **1.** Fix a number n and let V be the vector space of all $n \times n$ matrices. Then:

(i) The set $W^{sym} \subset V$ of all **symmetric** $n \times n$ matrices is a subspace.

(ii) The set $W^{skew} \subset V$ of all **skew-symmetric** $n \times n$ matrices is a subspace.

(iii) The set $W^{diag} \subset V$ of all **diagonal** $n \times n$ matrices is a subspace.

(iv) The set $W^{triang} \subset V$ of all **upper triangular** $n \times n$ matrices is a subspace.

2. Let V be the vector space of all continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$, and W the space of all polynomial functions (of any degree). Then W is a subspace of V. Indeed, every polynomial is a continuous function, hence $W \subset V$. Moreover, the sum of two polynomials is still a polynomial. Multiplying a polynomial by a real number we obtain another polynomial.

3. Let V be the vector space of all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and consider the set $S \subset V$ of all matrices whose determinant is zero. Then S is NOT a subspace of V. For example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in S, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S, \qquad \text{but} \qquad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S.$$

4. In the vector space $\mathbb{R}^2$ consider the subspaces

$$W_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix}, \ a \in \mathbb{R} \right\} = \text{set of all vectors whose second component is zero.}$$

$$W_2 = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix}, \ b \in \mathbb{R} \right\} = \text{set of all vectors whose first component is zero.}$$

Then the union $W_1 \cup W_2$ is NOT a subspace of $\mathbb{R}^2$. Indeed,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W_1 \cup W_2, \qquad \begin{bmatrix} 0 \\ 2 \end{bmatrix} \in W_1 \cup W_2, \qquad \text{but} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin W_1 \cup W_2.$$

8.2 Linear combinations

Given a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V, for any numbers $c_1, c_2, \ldots, c_n \in \mathbb{R}$ we can form the **linear combination**

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n .$$

The set of *all possible linear combinations* that we can obtain, by choosing different coefficients $c_1, \ldots c_n$, is called the **span** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Namely

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \Big\{ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n ; \quad c_1, c_2, \ldots, c_n \in \mathbb{R} \Big\}.$$

Theorem 8.2 *For any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, the set* $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is always a subspace of V.*

Examples. **1.** In the space $\mathbb{R}^3$ consider the two vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the subspace of all vectors $\mathbf{v} \in \mathbb{R}^3$ whose third component is zero:

$$\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} ; \quad b_1, b_2 \in \mathbb{R} \right\}.$$

2. In the space of all continuous functions, consider the functions $f(x) = 1$ (the constant function always equal to 1), $g(x) = x$, $h(x) = x^2$. Then

$$\mathrm{span}\{f, g, h\} = \mathrm{span}\{1, x, x^2\} = \{a_0 + a_1 x + a_2 x^2 ; \quad a_0, a_1, a_2 \in \mathbb{R}\}$$

is the space of all polynomials of degree ≤ 2.

8.3 Linear independence

Given any set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, by choosing the coefficients $c_1 = c_2 = \cdots = c_n = 0$ we always obtain the zero vector:

$$0\,\mathbf{v}_1 + 0\,\mathbf{v}_2 + \cdots + 0\,\mathbf{v}_n = \mathbf{0}.$$

In general, two cases may occur.

CASE 1: The *only way* to obtain the zero vector as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is to choose $c_1 = c_2 = \cdots = c_n = 0$. We thus have the implication

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \qquad \Longrightarrow \qquad c_1 = c_2 = \cdots = c_n = 0.$$

In this case we say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent**.

CASE 2: It is possible to obtain the zero vector as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ also by some choice of $c_1, c_2, \ldots, c_n$ *NOT all equal to zero.*

In this case we say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly dependent**.

If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then one of them can be written as a linear combination of the others. For example, if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ and $c_1 \neq 0$, then

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \cdots - \frac{c_n}{c_1}\mathbf{v}_n.$$

Examples. **1.** In the space $\mathbb{R}^2$, the three vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are linearly dependent. Indeed

$$0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2. In the space $\mathbb{R}^2$, the two vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are linearly independent.

3. In the space of all continuous functions, the three functions $f(x) = 1$, $g(x) = x$, $h(x) = x^2$ are linearly independent. Indeed, assume that

$$c_1 f + c_2 g + c_3 h \;=\; \mathbf{0} \qquad \text{(the function identically equal to zero)}$$

This means $c_1 + c_2 x + c_3 x^2 = 0$ for every x. This is possible only by choosing $c_1 = c_2 = c_3 = 0$.

4. In the space of all continuous functions, the three functions $f(x) = 1$, $g(x) = \sin^2 x$, $h(x) = \cos^2 x$ are linearly dependent. Indeed, since $-1 + \sin^2 x + \cos^2 x = 0$ for every x, we can obtain the zero function as a linear combination with coefficients $c_1 = -1$, $c_2 = 1$, $c_3 = 1$,

$$(-1)f + g + h \;=\; \mathbf{0}.$$

8.4 Basis of a vector space

We say that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **basis** of the vector space V if

(i) the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, and

(ii) $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V$.

Theorem 8.3 *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a* **basis** *of the vector space V, then every vector $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a* unique *way:*

$$\mathbf{v} \;=\; x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \;\cdots\; + x_n\mathbf{v}_n\,.$$

These coefficients $(x_1, x_2, \ldots, x_n)$ are the **coordinates** *of $\mathbf{v}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.*

Theorem 8.4 *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are two bases of the same vector space V, then they must contain the same number of elements, hence $m = n$. This unique number is called the* **dimension** *of the vector space V.*

Examples. 1. Given the two vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the set $\{\mathbf{e}_2, \mathbf{e}_2\}$ is a basis of the vector space $\mathbb{R}^2$. With respect to this basis, the vector $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has coordinates (b_1, b_2).

2. Given the two vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, the set $\{\mathbf{v}_2, \mathbf{v}_2\}$ is a basis of the vector space $\mathbb{R}^2$. With respect to this basis, the vector $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has coordinates $\left(\frac{b_1 + b_2}{2}, -b_2\right)$. Indeed,

$$\frac{b_1 + b_2}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} - b_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \;=\; \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

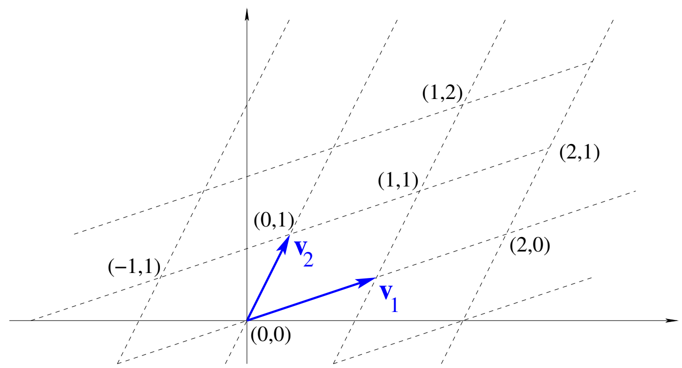

Figure 6: The coordinates of various points of the plane $\mathbb{R}^2$, with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

3. The set of four matrices $\{A_1, A_2, A_3, A_4\}$, with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

is a basis for the space V of all 2×2 matrices. This vector space has dimension $n = 4$.

4. The space of all polynomials of degree ≤ 4 has dimension 5. The set $\{1, x, x^2, x^3, x^4\}$ is a basis of this space.

5. The space of all 3×3 skew-symmetric matrices has dimension 3. A basis for this space is $\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Indeed, the most general 3×3 skew-symmetric matrix has the form

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = aA_1 + bA_2 + cA_3\,.$$

8.5 Linear systems, revisited

Given any $m \times n$ matrix

$$A = \begin{bmatrix} \mathbf{u}_1 \\ \hdashline \cdots \\ \hdashline \mathbf{u}_m \end{bmatrix} = \left[\mathbf{v}_1 \middle| \ldots \middle| \mathbf{v}_n \right], \tag{8.1}$$

we can consider:

- The **row space**, generated by its row vectors: $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$.
- The **column space**, generated by its column vectors: $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$.

- The **null space**, consisting of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$.

Theorem 8.5 *For any $m \times n$ matrix A, the dimension r of its row space is the same as the dimension of its column space. This number r is called the* **rank** *of the matrix A. One always has $r \leq \min\{m, n\}$.*

Moreover, the null space has dimension $n - r$. This number is called the **nullity** *of the matrix A.*

To compute the rank of a matrix A, we can perform elementary operations of its rows, and transform A into another matrix B which is in reduced row form. Then

$$[\text{row space of } A] \;=\; [\text{row space of } B].$$

Hence

$$\text{rank}(A) \;=\; \text{rank}(B) \;=\; \text{number of nonzero rows in } B.$$

Example. Consider the 3×4 matrix

$$A \;=\; \begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 4 & 0 & 1 \\ -1 & -5 & 3 & 1 \end{bmatrix}.$$

Performing elementary row operations, we obtain the matrices

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 2 & 1 \\ -1 & -5 & 3 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & -2 & 2 & 1 \end{bmatrix}, \qquad B \;=\; \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned}[\text{row space of } A] \;&=\; \text{span}\Big\{(1,3,-1,0),\ (2,4,0,1),\ (-1,-5,3,1)\Big\} \\ &=\; \text{span}\Big\{(1,3,-1,0),\ (0,-2,2,1)\Big\} \;=\; [\text{row space of } B].\end{aligned}$$

Hence $\text{rank}(A) = [\text{number of nonzero rows in } B] = 2$.

According to Theorem 8.5, the column space of A also has dimension 2. Indeed

$$\text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \;=\; \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\},$$

because

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \;=\; (-1)\cdot\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \;=\; (-3)\cdot\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The null space of A is the set of all $\mathbf{x} \in \mathbb{R}^4$ such that

$$A\mathbf{x} \;=\; 0, \qquad \text{or equivalently} \qquad B\mathbf{x} \;=\; 0$$

This means
$$\begin{cases} x_1 + 3x_2 - x_3 & = 0, \\ \quad -2x_2 + 2x_3 + x_4 & = 0. \end{cases}$$
Here we can choose x_3, x_4 arbitrarily, and solve for x_1, x_2. The general solution has the form
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3/2 \\ 1/2 \\ 0 \\ 1 \end{pmatrix}.$$
We conclude that the null space of A has dimension $n - r = 4 - 2 = 2$.

Remark 8.1 Consider any $m \times n$ matrix A, and let B a matrix in reduced row form, obtained by performing elementary operations of the rows of A. Then

- The row space of B is the same as the row space of A.
- The null space of B is the same as the null space of A.
- However, the column space of B is NOT the same as the column space of A, in general.

Remark 8.2 As in (8.1), if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the column vectors of A, the equation
$$A\mathbf{x} = \mathbf{b} \qquad \text{is equivalent to:} \qquad x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}.$$
Therefore, a solution exists if and only if we can write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. In other words, the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if
$$\mathbf{b} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

Remark 8.3 If $\widetilde{\mathbf{x}}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$, then every other solution has the form $\widetilde{\mathbf{x}} + \mathbf{z}$, where $\mathbf{z}$ solves the homogeneous problem $A\mathbf{z} = \mathbf{0}$. Indeed,
$$A(\widetilde{\mathbf{x}} + \mathbf{z}) = A\widetilde{\mathbf{x}} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

9 Inner products

Given two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in the space $\mathbb{R}^n$, their **inner product** (also called **dot product**, or **scalar product**) is the number
$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n\,.$$
For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real number c, the following properties hold:

(i) $\mathbf{u} \cdot \mathbf{v} \; = \; \mathbf{v} \cdot \mathbf{u}$,

(ii) $(c\mathbf{u}) \cdot \mathbf{v} \; = \; c(\mathbf{u} \cdot \mathbf{v})$,

(iii) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} \; = \; (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$,

(iv) $\mathbf{v} \cdot \mathbf{v} \; \geq \; 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$.

More generally, for any vector space V over the real numbers, we say that a map $V \times V \mapsto \mathbb{R}$ is an **inner product** if it satisfies the four conditions (i)–(iv).

Example. Let V the vector space of all continuous functions f defined on the interval $[a, b]$. Then

$$f \cdot g \; \doteq \; \int_a^b f(x)\, g(x)\, dx$$

is an inner product on V. Indeed, it satisfies all four properties (i)–(iv).

Using the inner product we can define the **norm** of a vector $\mathbf{v}$, measuring its size:

$$\|\mathbf{v}\| \; = \; \sqrt{(\mathbf{v} \cdot \mathbf{v})}.$$

By the properties (i)–(iv), for every number $c \in \mathbb{R}$ and every two vectors $\mathbf{u}, \mathbf{v}$, the norm satisfies

- $\|c\mathbf{u}\| = |c|\, \|u\|$. In particular: $\| - \mathbf{u}\| = \|\mathbf{u}\|$.
- $\|\mathbf{u}\| \geq 0$
- $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The **distance** between two vectors $\mathbf{v}, \mathbf{w}$ is defined as $\|\mathbf{v} - \mathbf{w}\|$.
Notice that $\|\mathbf{v}\| = \|\mathbf{v} - \mathbf{0}\|$ is the distance of $\mathbf{v}$ to the origin.

Theorem 9.1 *Given an inner product on a vector space V, for any $\mathbf{v}, \mathbf{w} \in V$ the following inequalities hold:*

$$|\mathbf{v} \cdot \mathbf{w}| \; \leq \; \|\mathbf{v}\|\, \|\mathbf{w}\| \qquad \textit{(Cauchy-Bunyakowsky-Schwarz)},$$

$$\|\mathbf{v} - \mathbf{w}\| \; \leq \; \|\mathbf{v}\| + \|\mathbf{w}\| \qquad \textit{(triangle inequality)}.$$

The angle $\theta \in [0, \pi]$ between two vectors $\mathbf{v}, \mathbf{w}$ is defined by the identity

$$\cos\theta \; = \; \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\, \|\mathbf{w}\|}.$$

Notice that, by the (CBS) inequality, the right hand side is always a number with absolute value ≤ 1. Hence the angle θ is well defined.

We say that the two vectors $\mathbf{v}, \mathbf{w}$ are **orthogonal** (or **perpendicular**) if $\mathbf{v} \cdot \mathbf{w} = 0$.

Pythagora's theorem. *If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then*

$$\|\mathbf{v} - \mathbf{w}\|^2 \;=\; \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

Example. The inner product of the two vectors

$$\mathbf{v} \;=\; \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{w} \;=\; \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

is $\mathbf{v} \cdot \mathbf{w} \;=\; 3$. The angle between these two vectors is computed by

$$\cos\theta \;=\; \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} \;=\; \frac{3}{\sqrt{4} \cdot \sqrt{3}} \;=\; \frac{\sqrt{3}}{2}\,, \qquad \text{hence} \quad \theta \;=\; \frac{\pi}{6}\,.$$

Example. The two vectors $\mathbf{v} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ are perpendicular, because their inner product is zero. We check that Pythagora's theorem holds. Indeed:

$$\|\mathbf{v} - \mathbf{w}\|^2 \;=\; 6^2 + 7^2 \;=\; 85, \qquad \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \;=\; (2^2 + 8^2) + \big((-4)^2 + 1^2\big) \;=\; 68 + 17 \;=\; 85.$$

9.1 Orthogonal bases

Theorem 9.2 *In a vector space V, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be nonzero vectors which are mutually orthogonal, so that*

$$\mathbf{v}_i \cdot \mathbf{v}_j \;=\; 0 \qquad \text{for} \;\; i \neq j. \tag{9.1}$$

Then these vectors are linearly independent.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V. We want to find the coordinates of a given vector $\mathbf{v}$ with respect to this basis. That means: find numbers $x_1, \ldots, x_n$ such that

$$\mathbf{v} \;=\; x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n\,. \tag{9.2}$$

In general, this requires solving a system of n equations for the n unknowns $x_1, \ldots, x_n$.

However, if the vectors $\mathbf{v}_i$ are mutually orthogonal, our task becomes much easier. Indeed, taking the inner product of (9.2) with $\mathbf{v}_i$ we obtain

$$\mathbf{v} \cdot \mathbf{v}_i \;=\; (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) \cdot \mathbf{v}_i \;=\; x_i(\mathbf{v}_i \cdot \mathbf{v}_i).$$

This quickly yields the solution

$$x_i \;=\; \frac{\mathbf{v} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\,. \tag{9.3}$$

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called **orthonormal** if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

That means: all vectors $\mathbf{v}_i$ have length 1, and they are mutually orthogonal. In the case of an orthonormal basis, the formula (9.3) for the coordinates of the vector $\mathbf{v}$ simplifies to

$$x_i = \mathbf{v} \cdot \mathbf{v}_i\,.$$

10 Orthogonal projections

Given a vector $\mathbf{u}$, and a second vector $\mathbf{v}$, as shown in Fig. 7 we want to decompose $\mathbf{u}$ as a sum:

$$\mathbf{u} = \mathbf{p} + \mathbf{q}\,,$$

where $\mathbf{p}$ is parallel to $\mathbf{v}$ while $\mathbf{q}$ is perpendicular to $\mathbf{v}$. This problem has the explicit solution

$$\mathbf{p} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}\,, \qquad \mathbf{q} = \mathbf{u} - \mathbf{p}\,. \tag{10.4}$$

Indeed, by the first identity $\mathbf{p}$ is a scalar multiple of $\mathbf{v}$. On the other hand, the second identity yields

$$\mathbf{q} \cdot \mathbf{v} = (\mathbf{u} \cdot \mathbf{v}) - (\mathbf{p} \cdot \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} \cdot \mathbf{v} = 0,$$

showing the $\mathbf{q}$ is perpendicular to $\mathbf{v}$.

We observe that, among all vectors parallel to $\mathbf{v}$, the vector $\mathbf{p}$ is the one closest to $\mathbf{u}$. Indeed, for every $x \in \mathbb{R}$ consider the square of the distance

$$\|\mathbf{u} - x\mathbf{v}\|^2 = (\mathbf{u} - x\mathbf{v}) \cdot (\mathbf{u} - x\mathbf{v}) = (\mathbf{u} \cdot \mathbf{u}) - 2x(\mathbf{u} \cdot \mathbf{v}) + x^2(\mathbf{v} \cdot \mathbf{v}).$$

Differentiating with respect to the variable x and setting this derivative to be zero we conclude that the distance is minimum when

$$-2(\mathbf{u} \cdot \mathbf{v}) + 2x(\mathbf{v} \cdot \mathbf{v}) = 0, \qquad \text{hence} \qquad x = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\,.$$

The vector $\mathbf{p}$ in (10.4) is called the **perpendicular projection** of $\mathbf{u}$ on the line parallel to $\mathbf{v}$.

10.1 Perpendicular projection on a subspace

In a vector space V, consider a subspace, say $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$. We can then define the **orthogonal subspace**

$$W^\perp = \Big\{\mathbf{v} \in V\,; \quad \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all} \quad \mathbf{w} \in W\Big\}.$$

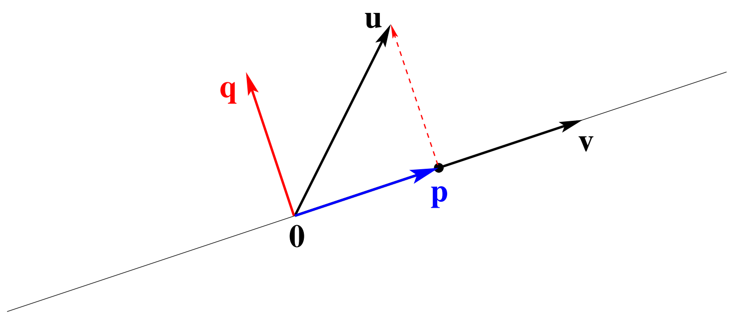

Figure 7: Here $\mathbf{p}$ is the perpendicular projection of $\mathbf{u}$ on the line parallel to $\mathbf{v}$.

In other words, a vector $\mathbf{v}$ is in $W^\perp$ if it is orthogonal to all vectors of W.

Next, given a vector $\mathbf{v} \in V$, we seek a vector $\mathbf{p} \in W$ which which has minimum distance from $\mathbf{v}$. This vector $\mathbf{p}$ is called the **perpendicular projection** of $\mathbf{v}$ on the space W. It is the unique vector such that (see Fig. 8)

$$\mathbf{p} \in W, \qquad\qquad \mathbf{v} - \mathbf{p} \ \in \ W^\perp.$$

Writing $\mathbf{p} \ = \ x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_m\mathbf{w}_m$, this leads to a system of m equations for the

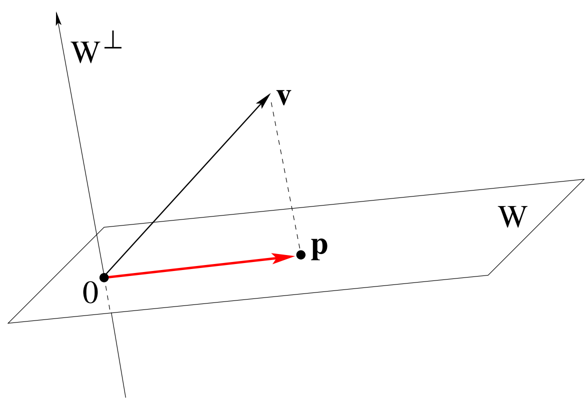

Figure 8: The perpendicular projection of a vector $\mathbf{v}$ on the subspace W. Here $\mathbf{v} - \mathbf{p}$ lies in the orthogonal space $W^\perp$.

m unknowns $x_1, \ldots, x_m$:

$$\begin{cases} (\mathbf{v}-\mathbf{p})\cdot\mathbf{w}_1 &= 0, \\ (\mathbf{v}-\mathbf{p})\cdot\mathbf{w}_2 &= 0, \\ \cdots \\ (\mathbf{v}-\mathbf{p})\cdot\mathbf{w}_m &= 0, \end{cases} \qquad \text{that is:} \qquad \begin{cases} (x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_m\mathbf{w}_m)\cdot\mathbf{w}_1 &= \mathbf{v}\cdot\mathbf{w}_1\,, \\ (x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_m\mathbf{w}_m)\cdot\mathbf{w}_2 &= \mathbf{v}\cdot\mathbf{w}_2\,, \\ \cdots \\ (x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_m\mathbf{w}_m)\cdot\mathbf{w}_m &= \mathbf{v}\cdot\mathbf{w}_m\,. \end{cases}$$

In matrix notation, this can be written as

$$\begin{bmatrix} (\mathbf{w}_1\cdot\mathbf{w}_1) & (\mathbf{w}_1\cdot\mathbf{w}_2) & \cdots & (\mathbf{w}_1\cdot\mathbf{w}_m) \\ (\mathbf{w}_2\cdot\mathbf{w}_1) & (\mathbf{w}_2\cdot\mathbf{w}_2) & \cdots & (\mathbf{w}_2\cdot\mathbf{w}_m) \\ & & \cdots & \\ (\mathbf{w}_m\cdot\mathbf{w}_1) & (\mathbf{w}_m\cdot\mathbf{w}_2) & \cdots & (\mathbf{w}_m\cdot\mathbf{w}_m) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} (\mathbf{v}\cdot\mathbf{w}_1) \\ (\mathbf{v}\cdot\mathbf{w}_2) \\ \vdots \\ (\mathbf{v}\cdot\mathbf{w}_m) \end{bmatrix}. \tag{10.5}$$

To determine the coefficients $x_1, \ldots, x_m$ we thus need to solve a system of m equations in m unknowns.

In the special case where the vectors $\mathbf{w}_i$ are orthogonal to each other, so that $\mathbf{w}_i \cdot \mathbf{w}_j = 0$ for $i \neq j$, the above system simplifies to

$$\left\{ \begin{array}{rcl} (\mathbf{w}_1 \cdot \mathbf{w}_1) x_1 & = & \mathbf{v} \cdot \mathbf{w}_1 \,, \\ (\mathbf{w}_2 \cdot \mathbf{w}_2) x_2 & = & \mathbf{v} \cdot \mathbf{w}_2 \,, \\ & \cdots & \\ (\mathbf{w}_m \cdot \mathbf{w}_m) x_m & = & \mathbf{v} \cdot \mathbf{w}_m \,. \end{array} \right.$$

This can be immediately solved:

$$x_i \;=\; \frac{\mathbf{v} \cdot \mathbf{w}_i}{\mathbf{w}_i \cdot \mathbf{w}_i} \,, \qquad\qquad i = 1, \ldots, m \,.$$

10.2 The Gram-Schmidt orthonormalization algorithm

Definition 10.1 *A basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of a vector space V is called an* **orthogonal basis** *if the vectors $\mathbf{v}_i$ are perpendicular to each other, so that (9.1) holds. If, in addition, all vectors have length $\|\mathbf{v}_i\| = 1$, then we say it is an* **orthonormal basis***.*

In many situations, it is a big advantage to work with an orthonormal basis. For example, if $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is an orthonormal basis of W, then the $m \times m$ matrix in (10.5) is the identity matrix. In this case, we immediately find the solution

$$x_i \;=\; \mathbf{v} \cdot \mathbf{w}_i \,, \qquad\qquad i = 1, \ldots, m.$$

Given any basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of a vector space V, we now show how to construct an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Gram-Schmidt orthonormalization algorithm. *Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a basis of a vector space V.*

FIRST STEP: *Define* $\mathbf{v}_1 = \dfrac{\mathbf{w}_1}{\|\mathbf{w}_1\|}$.

INDUCTIVE STEP: *For $1 < k \leq n$, assume that $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ have already been constructed. Consider the vector*

$$\begin{aligned} \widetilde{\mathbf{v}}_k \;&=\; \mathbf{w}_k - \Big[\textit{perpendicular projection of } \mathbf{w}_k \textit{ on span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}\Big] \\ &=\; \mathbf{w}_k - \Big((\mathbf{w}_k \cdot \mathbf{v}_1)\mathbf{v}_1 + \;\cdots\; + (\mathbf{w}_k \cdot \mathbf{v}_{k-1})\mathbf{v}_{k-1} \Big). \end{aligned}$$

Then define $\mathbf{v}_k \;=\; \dfrac{\widetilde{\mathbf{v}}_k}{\|\widetilde{\mathbf{v}}_k\|}$.

By induction, this algorithm generates n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, all of length 1. Moreover, by construction each vector $\widetilde{\mathbf{v}}_k$ (and hence $\mathbf{v}_k$ as well) is perpendicular to all previous vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. Therefore $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of the space V.

Example. Given the three vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

an application of the Gram-Schmidt algorithm yields

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\widetilde{\mathbf{v}}_2 = \mathbf{w}_2 - (\mathbf{w}_2 \cdot \mathbf{v}_1)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{(-2)}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \frac{\widetilde{\mathbf{v}}_2}{\|\widetilde{\mathbf{v}}_2\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\widetilde{\mathbf{v}}_3 = \mathbf{w}_3 - (\mathbf{w}_3 \cdot \mathbf{v}_1)\mathbf{v}_1 - (\mathbf{w}_3 \cdot \mathbf{v}_2)\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_3 = \frac{\widetilde{\mathbf{v}}_3}{\|\widetilde{\mathbf{v}}_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

10.3 Orthogonal matrices

An $n \times n$ matrix A is an **orthogonal matrix** if its transpose coincides with its inverse. That means: $A^T = A^{-1}$. Writing A in terms of its column vectors, this implies

$$A^T A = \begin{bmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \cdots \\ \hline \mathbf{v}_n^T \end{bmatrix} \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \ldots \middle| \mathbf{v}_n \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In other words, *A is an orthogonal matrix if and only if its column vectors are an orthonormal basis of $\mathbb{R}^n$,* namely

$$\mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{v}_i^T \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

If A is an orthogonal matrix, then for every vector $\mathbf{x} \in \mathbb{R}^n$ we have $\|A\mathbf{x}\| = \|\mathbf{x}\|$. Indeed: $\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$.

Example. For every angle θ, the matrix $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

As suggested by this example, orthogonal matrices typically describe a rigid rotation of the space.

11 Least squares

Several approximation problems lead to a system of m linear equations in n unknowns

$$A\mathbf{x} = \mathbf{b}, \tag{11.1}$$

with $m >> n$. When the number of equations is larger than the number of variables, the system (11.1) usually does not have any solution. But here is the main idea:

- *If we cannot find any vector* $\mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} - \mathbf{b} = \mathbf{0}$, *we can search for a vector* $\mathbf{x}$ *which makes the square of the distance* $\|A\mathbf{x} - \mathbf{b}\|^2$ *is as small as possible.*

Toward this goal, we write A in terms of its column vectors, so that

$$A\mathbf{x} = \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \ldots \middle| \mathbf{v}_n\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n .$$

Consider the subspace spanned by the columns of A:

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{A\mathbf{x}\,;\;\; \mathbf{x} \in \mathbb{R}^n\}.$$

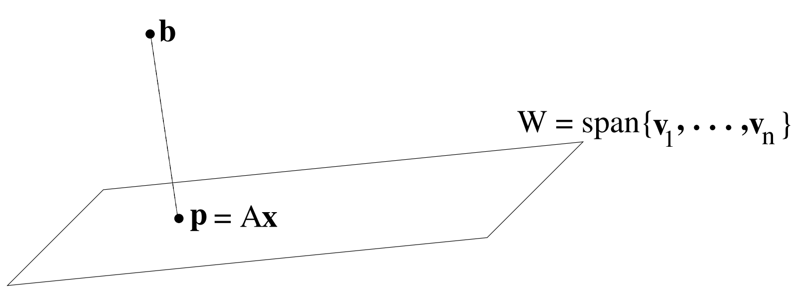

Figure 9: If $A\mathbf{x} = \mathbf{b}$ has no solution, we can seek a point $\mathbf{p} = A\mathbf{x}$ that has minimum distance from $\mathbf{b}$. This point $\mathbf{p}$ is the perpendicular projection of $\mathbf{b}$ on the vector space W spanned by the columns of A.

Among all points in W, the one closest to $\mathbf{b}$ is the perpendicular projection of $\mathbf{b}$ on W. As shown in Fig. 9, we thus seek a vector $\mathbf{p} = A\mathbf{x} \in W$ such that

$$\mathbf{b} - \mathbf{p} \in W^\perp.$$

In particular, $\mathbf{b}-\mathbf{p}$ has to be perpendicular to all vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$. This leads to the system of n equations

$$\mathbf{v}_i^T(A\mathbf{x}-\mathbf{b}) \;=\; 0 \qquad\qquad i=1,2,\ldots,n.$$

Using matrix notation, this system can be written as

$$\begin{bmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \cdots \\ \hline \mathbf{v}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \big| \mathbf{v}_2 \big| \ldots \big| \mathbf{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \cdots \\ \hline \mathbf{v}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{b} \end{bmatrix}.$$

In a more compact form, this means

$$A^TA\mathbf{x} \;=\; A^T\mathbf{b}\,. \tag{11.2}$$

Formally, this equation is obtained from (11.1) by multiplying both sides on the left by A^T.

Notice that (11.1) requires $A\mathbf{x}-\mathbf{b}=\mathbf{0}$. On the other hand, (11.2) only requires that $A\mathbf{x}-\mathbf{b}$ should be perpendicular to all vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$.

It is important to observe that (11.2) is a system of n equations in n variables. One can prove that this system always has a solution, even if (11.1) does not.

11.1 Application: least squares interpolation

Consider a function that depends on n unknown coefficients $c_1,c_2,\ldots,c_n$:

$$f(x) \;=\; c_1f_1(x)+c_2f_2(x)+\;\cdots\;+c_nf_n(x). \tag{11.3}$$

To determine these coefficients, several measurements are made:

$$\begin{cases} f(x_1) &= y_1 \\ f(x_2) &= y_2 \\ \quad\cdots \\ f(x_m) &= y_m \end{cases} \qquad \text{that is:} \qquad \begin{cases} c_1f_1(x_1)+c_2f_2(x_1)+\;\cdots\;+c_nf_n(x_1) &= y_1 \\ c_1f_1(x_2)+c_2f_2(x_2)+\;\cdots\;+c_nf_n(x_2) &= y_2 \\ \qquad\cdots \\ c_1f_1(x_m)+c_2f_2(x_m)+\;\cdots\;+c_nf_n(x_m) &= y_m\,. \end{cases}$$

This leads to a system of m linear equations for the coefficients $c_1,\ldots,c_m$. In matrix notation:

$$\begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ & & \cdots & \\ f_1(x_m) & f_2(x_m) & \cdots & f_n(x_m) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

which we write as

$$Ac \;=\; \mathbf{y}\,.$$

In several applications $m >> n$, and this system has no solution. However, we can try to find coefficients c_i so that the square of the distance

$$\|A\mathbf{c} - \mathbf{y}\|^2 \;=\; \sum_{i=1}^{m} \big|f(x_i) - y_i\big|^2$$

is as small as possible. This leads to the system

$$A^T A\mathbf{c} \;=\; A^T\mathbf{y}\,. \tag{11.4}$$

Examples. 1. An unknown function $f(x)$ has been measured at four points:

$$\begin{cases} f(-1) &= 0\,,\\ f(0) &= 2\,,\\ f(1) &= 3\,,\\ f(2) &= 5\,. \end{cases} \tag{11.5}$$

Assuming that $f(x) = c_1 + c_2 x$ is a polynomial of degree 1, we want to determine the coefficients c_1, c_2 that best fit the data, in the sense of least squares. Notice that we are in the situation described above, with $f_1(x) = 1$, $f_2(x) = x$.

According to (11.5) we should have

$$\begin{cases} c_1 - c_2 &= 0\\ c_1 &= 2\\ c_1 + c_2 &= 3\\ c_1 + 2c_2 &= 5 \end{cases} \qquad \text{that is:} \qquad \begin{bmatrix} 1 & -1\\ 1 & 0\\ 1 & 1\\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1\\ c_2 \end{bmatrix} = \begin{bmatrix} 0\\ 2\\ 3\\ 5 \end{bmatrix},$$

This is a system of 4 equations in 2 unknowns, and has no solution. However, we can solve the corresponding system (11.4), namely

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1\\ 1 & 0\\ 1 & 1\\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1\\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1\\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0\\ 2\\ 3\\ 5 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 2\\ 2 & 4 \end{bmatrix} \begin{bmatrix} c_1\\ c_2 \end{bmatrix} = \begin{bmatrix} 10\\ 13 \end{bmatrix}, \qquad \text{hence} \quad c_1 = \frac{7}{6}, \quad c_2 = \frac{8}{3}.$$

The polynomial of degree 1 that best interpolates the given data is thus

$$f(x) \;=\; \frac{7}{6} + \frac{8}{3}x\,.$$

2. Next, assume that f is a polynomial of degree 2, say

$$f(x) \;=\; c_1 + c_2 x + c_3 x^2\,.$$

We seek coefficients c_1, c_2, c_3 so that f best interpolates the same data as in (11.5). We now should have

$$\begin{cases} c_1 - c_2 + c_3 &= 0\,,\\ c_1 &= 2\,,\\ c_1 + c_2 + c_3 &= 3\,,\\ c_1 + 2c_2 + 4c_3 &= 5\,, \end{cases} \qquad \text{that is:} \qquad \begin{bmatrix} 1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix} = \begin{bmatrix} 0\\ 2\\ 3\\ 5 \end{bmatrix},$$

This system of 4 equations in 3 unknowns still has no solution. However, we can provide a "best approximation" in the sense of least squares, by solving the corresponding system (11.4). In this case we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 18 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 23 \end{bmatrix}.$$

Remark 11.1 Least squares interpolation can also be used to determine the coefficients of functions of the form

$$f(x) = c_1 + c_2 \sin x + c_3 \cos x$$

because this is a linear combination of the form (11.3), with $f_1(x) = 1$, $f_2(x) = \sin x$, $f_3(x) = \cos x$.

However, it does NOT apply to functions of the form

$$f(x) = e^{c_1 x} + \sin(c_2 x) + \cos(c_3 x),$$

because this is NOT a linear combination, with c_1, c_2, c_3 as coefficients.

12 Review of complex numbers

In the following chapters we shall need to work with complex numbers $z \in \mathbb{C}$. These can be written in the form

$$z = a + ib,$$

where a is the real part and b is the imaginary part.

- The *absolute value* of the complex number z is defined as $|z| = \sqrt{a^2 + b^2}$.
- The *complex conjugate* of z is defined as $\overline{z} = a - ib$.

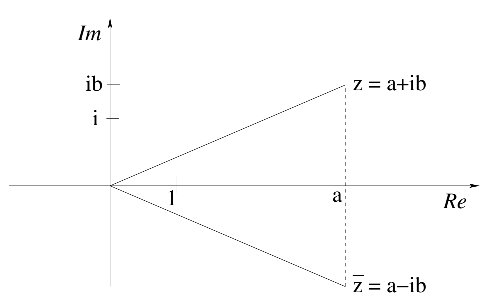

Figure 10: A complex number $z = a + ib$ and its conjugate $\overline{z} = a - ib$.

By the identity $i^2 = -1$, the sum and the product of two complex numbers are

$$(a+ib)+(c+id) = (a+c)+i(b+d),$$

$$(a+ib)(c+id) = (ac-bd)+i(bc+ad).$$

If z_1, z_2 are complex numbers, their complex conjugates satisfy

$$\overline{z}_1+\overline{z}_2 = \overline{(z_1+z_2)}, \qquad \overline{z}_1\overline{z}_2 = \overline{(z_1 z_2)}. \tag{12.1}$$

A major motivation for using complex numbers is:

Fundamental Theorem of Algebra. *Let*

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

be a polynomial of degree n, with coefficients $a_0, a_1, \ldots a_{n-1} \in \mathbb{C}$.
Then $p(\cdot)$ has n (possibly coinciding) roots $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. As a consequence, it can be factored as

$$p(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n).$$

Remark 12.1 If the coefficients $a_0, a_1, \ldots, a_{n-1}$ are real numbers, the roots of p can still be complex. However, the following useful property holds:

If z is a root of a polynomial p with real coefficients, then its complex conjugate $\overline{z}$ is also a root of p.

Indeed, since the coefficients are real, for every k we have $a_k = \overline{a}_k$. If $p(z) = 0$, using the properties (12.1) we thus obtain

$$p(\overline{z}) = \overline{z}^n + \overline{a}_{n-1}\overline{z}^{n-1} + \cdots + \overline{a}_1\overline{z} + \overline{a}_0 = \overline{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} = \overline{p(z)} = 0.$$

Example. The polynomial $p(\lambda) = \lambda^2 - 4\lambda + 13$ has real coefficients but it does not have any real root. Its two complex roots are

$$\lambda_1 = 2+3i, \qquad \lambda_2 = 2-3i.$$

Notice that one is the complex conjugate of the other.

13 Eigenvalues and eigenvectors

Consider an $n \times n$ matrix A, possibly with complex entries. This determines a map

$$\mathbf{x} \mapsto A\mathbf{x}.$$

A nonzero vector $\mathbf{v}$ (in $\mathbb{R}^n$ or in $\mathbb{C}^n$) is called an **eigenvector** of A if there exists a (real or complex) number λ such that

$$A\mathbf{v} \;=\; \lambda \mathbf{v}\,. \tag{13.1}$$

In this case, we say that λ is the corresponding **eigenvalue** of A.

We can write (13.1) as

$$(A - \lambda I)\mathbf{v} \;=\; \mathbf{0} \tag{13.2}$$

where I is the $n \times n$ identity matrix. This fact can be stated in various equivalent ways:

- The linear map $\mathbf{x} \mapsto (A - \lambda I)\mathbf{x}$ is not one-to-one, because it sends a nonzero vector $\mathbf{v}$ into the origin.
- The matrix $A - \lambda I$ is not invertible.
- Computing the determinant, one has

$$\det(A - \lambda I) \;=\; 0. \tag{13.3}$$

To find the eigenvalues and eigenvectors of A we proceed as follows:

STEP 1: Find the (real or complex) numbers λ which satisfy

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; 0.$$

If A is an $n \times n$ matrix, then $p(\lambda)$ is a polynomial of degree n in the variable λ. It is called the **characteristic polynomial** of A. By the Fundamental Theorem of Algebra, this polynomial has n roots (not necessarily distinct)

$$\lambda_1, \lambda_2, \ldots, \lambda_n \;\in\; \mathbb{C}\,.$$

These are the eigenvalues of A.

STEP 2: For each eigenvalue λ_k, we solve the system of equations

$$(A - \lambda_k I)\mathbf{x} \;=\; \mathbf{0}.$$

This is a system of n equations in n unknowns. However, since the rank of the matrix $A - \lambda_k I$ is $< n$, this system admits a nonzero solution $\mathbf{v}_k$.

This solution $\mathbf{v}_k$ is an eigenvector of A, corresponding to the eigenvalue λ_k.

Example 1. Let $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$. We compute

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; \det \begin{bmatrix} 7-\lambda & 4 \\ 3 & 6-\lambda \end{bmatrix} \;=\; (7-\lambda)(6-\lambda) - 4\cdot 3 \;=\; \lambda^2 - 13\lambda + 30.$$

The roots of this second degree polynomial are $\lambda_1 = 3$, and $\lambda_2 = 10$.

Choosing $\lambda = 3$ we obtain the system $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where c is an arbitrary constant. Hence $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A, with corresponding eigenvalue $\lambda_1 = 3$.

Choosing $\lambda = 10$ we obtain the system $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A nonzero solution of this system is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Hence $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is an eigenvector of A, with corresponding eigenvalue $\lambda_2 = 10$.

Example 2. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. We compute

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 1 \cdot (-1) = \lambda^2 - 2\lambda + 2.$$

The roots of this second degree polynomial are complex numbers

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2}, \qquad \begin{cases} \lambda_1 = 1 + i\,, \\ \lambda_2 = 1 - i\,. \end{cases}$$

Choosing $\lambda = 1 + i$ we obtain the system $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} i \\ 1 \end{bmatrix}$, where c is an arbitrary constant. Hence $\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector of A, with corresponding eigenvalue $\lambda_1 = 1 + i$.

Choosing $\lambda = 1 - i$ we obtain the system $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A nonzero solution of this system is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, where c is an arbitrary constant. Hence $\mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector of A, with corresponding eigenvalue $\lambda_2 = 1 - i$.

Remark 13.1 Let A be a matrix with real entries. As shown in Example 2, it can have a complex eigenvalue λ and a complex eigenvector $\mathbf{v}$. Taking the complex conjugates, we obtain that $\overline{\lambda}$ is another eigenvalue, with $\overline{\mathbf{v}}$ as corresponding eigenvector.

Indeed, if $A\mathbf{v} = \lambda\mathbf{v}$, then

$$A\overline{\mathbf{v}} = \overline{A}\overline{\mathbf{v}} = \overline{(A\mathbf{v})} = \overline{(\lambda\mathbf{v})} = \overline{\lambda}\overline{\mathbf{v}}.$$

Remark 13.2 When A is a 2×2 matrix, the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has degree 2 and we have a simple formula for computing its roots.

However, for an $n \times n$ matrix A, the polynomial $p(\lambda) = \det(A - \lambda I)$ has degree n. Finding the roots is a difficult task.

In the special case where A is a triangular matrix, the eigenvalues are easy to find. Indeed, they are precisely the diagonal elements of A.

Example 3. Let $A = \begin{bmatrix} 7 & 4 & -5 \\ 0 & 3 & 6 \\ 0 & 0 & -2 \end{bmatrix}$. Then the characteristic polynomial is

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; \det \begin{bmatrix} 7-\lambda & 4 & -5 \\ 0 & 3-\lambda & 6 \\ 0 & 0 & -2-\lambda \end{bmatrix} \;=\; (7-\lambda)(3-\lambda)(-2-\lambda).$$

Its roots are $\lambda_1 = 7$, $\lambda_2 = 3$, $\lambda_3 = -2$.

14 Bases of eigenvectors

Given an $n \times n$ matrix A, our next goal is to construct a basis of the space $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A.

That means: find n linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ which are eigenvectors of the matrix A. In this direction, we have

Theorem 14.1 *If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of A corresponding to* distinct *eigenvalues, then they are linearly independent.*

- If the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has n distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and provide a basis of $\mathbb{C}^n$.

- However, if the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has repeated roots, then a set of n linearly independent eigenvectors may not exist.

Example 4. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$. Its characteristic polynomial is

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; (1-\lambda)(2-\lambda)^2.$$

The eigenvalues are $\lambda_1 = 1$ (with multiplicity 1) and $\lambda_2 = 2$ (with multiplicity 2).

In connection with the eigenvalue $\lambda_1 = 1$, we solve the system

$$(A - I)\mathbf{x} \;=\; \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and obtain the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

In connection with the eigenvalue $\lambda_2 = 2$, we solve the system

$$(A - 2I)\mathbf{x} \;=\; \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and obtain two linearly independent eigenvectors $\mathbf{v}_2 \;=\; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 \;=\; \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

For this example, we have been successful in constructing a basis of $\mathbb{R}^3$ consisting of eigenvectors of A, namely

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

Example 5. Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Its characteristic polynomial is:

$$p(\lambda) = \det(A - \lambda I) = (1-\lambda)(3-\lambda)^2.$$

The eigenvalues are $\lambda_1 = 1$ (with multiplicity 1) and $\lambda_2 = 3$ (with multiplicity 2).

In connection with the eigenvalue $\lambda_1 = 1$, we solve the system

$$(A - I)\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and obtain the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

In connection with the eigenvalue $\lambda_2 = 3$, we consider the system

$$(A - 2I)\mathbf{x} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Its general solution is $\mathbf{x} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$, where c is an arbitrary constant. In this case we find the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Every other eigenvector with eigenvalue $\lambda_2 = 3$ is a multiple of $\mathbf{v}_2$.

Observe that, in this example, we cannot construct a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

If λ_i is an eigenvalue of A, we define:

- The **algebraic multiplicity** of λ_i is the multiplicity of λ_i as a root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- The **geometric multiplicity** of λ_i is the dimension of the null space

  $$\big\{\mathbf{x} \in \mathbb{C}^n\,; \quad (A - \lambda_i I)\mathbf{x} = \mathbf{0}\big\}.$$

  This is the maximum number of linearly independent eigenvectors having λ_i as corresponding eigenvalue.

Theorem 14.2 *Given a matrix A, the geometric multiplicity of each eigenvalue λ_i is always less or equal to its algebraic multiplicity.*

If for every eigenvalue λ_i of A these multiplicities coincide, then there exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of A.

Notice that in Example 4 the eigenvalue $\lambda_2 = 2$ has algebraic multiplicity 2 and geometric multiplicity 2.

On the other hand, in Example 5, the eigenvalue $\lambda_2 = 3$ has algebraic multiplicity 2 and geometric multiplicity 1.

15 Diagonalization

- We say that a matrix A is **similar** to a matrix B if there exists an invertible matrix P such that
$$B = P^{-1}AP.$$
- We say that A is **diagonalizable** if A is similar to a diagonal matrix.

To understand whether an $n \times n$ matrix A is diagonalizable, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. In matrix notation, this means

$$A \cdot \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \cdots \middle| \mathbf{v}_n \right] = \left[\lambda_1\mathbf{v}_1 \middle| \lambda_2\mathbf{v}_2 \middle| \cdots \middle| \lambda_n\mathbf{v}_n \right] = \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \cdots \middle| \mathbf{v}_n \right] \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

We write the above identity in the form

$$AP = PD, \tag{15.1}$$

where P is the $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and D is the diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ along the diagonal.

If the n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, the matrix P has an inverse P^{-1}. Multiplying both sides of (15.1) on the left by P^{-1} we obtain

$$P^{-1}AP = D.$$

We thus conclude:

Theorem 15.1 *If the $n \times n$ matrix A has n linearly independent eigenvectors, then A is diagonalizable.*

Example 6. The matrix $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$ considered in Example 1 is diagonalizable, because it has the two linearly independent eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 10$.

Calling $P = \begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix}$ the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2$, we compute

$$P^{-1}AP = \begin{bmatrix} -3/7 & 4/7 \\ 1/7 & 1/7 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix} = D.$$

15.1 Powers of a matrix

Given an $n \times n$ matrix A, we want to compute the power $A^k = A \cdot A \cdots A \quad (k \text{ times})$.

Computing the powers of a diagonal matrix is easy:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \Longrightarrow \qquad D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}.$$

Now consider any matrix A that can be diagonalized, so that $P^{-1}AP = D$ for some invertible matrix P. Then $A = PDP^{-1}$, and hence

$$A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^kP^{-1}.$$

This provides an efficient way to compute A^k, for every integer $k \geq 1$.

Example 7. For the matrix $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$ in Example 7, for every integer k we have the formula

$$A^k = PD^kP^{-1} = \begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 10^k \end{bmatrix} \begin{bmatrix} -3/7 & 4/7 \\ 1/7 & 1/7 \end{bmatrix}.$$

16 Diagonalization of symmetric matrices

When we try to diagonalize a general $n \times n$ matrix A, two difficulties may arise:

(i) Even if all the entries of A are real numbers, the eigenvalues and eigenvectors of A may be complex valued,

(ii) If A has multiple eigenvalues, we are not guaranteed to find a basis of eigenvectors.

When A is symmetric, these two "bad" situations (i)-(ii) do not occur:

Theorem 16.1 *Let A be an $n \times n$* symmetric *matrix (with real entries). Then*

- *All the eigenvalues are real.*
- *Eigenvectors corresponding to distinct eigenvalues are perpendicular to each other.*
- *There exists an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A.*

If A is symmetric, we can thus find an orthogonal matrix $Q = \left[\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \cdots \middle| \mathbf{v}_n\right]$, whose columns are eigenvectors of A, such that

$$Q^{-1}AQ \;=\; D \;=\; \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

is diagonal. Its diagonal elements are precisely the eigenvalues of A.

Notice that Q is orthogonal because $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis, hence $Q^TQ = I$ and $Q^T = Q^{-1}$.

Example. Consider the symmetric matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Computing the roots of the polynomial $p(\lambda) = \det(A - \lambda I) = \big(-\lambda(3-\lambda) - 4\big)(4-\lambda)$, one finds the eigenvalues

$$\lambda_1 \;=\; -1, \qquad\qquad \lambda_2 \;=\; \lambda_3 \;=\; 4.$$

In connection with the eigenvalue $\lambda = -1$, we find the eigenvector $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

For the eigenvalue $\lambda = 4$, we find the two eigenvectors $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

By normalizing these three eigenvectors, we obtain a matrix Q whose columns form an orthonormal basis of $\mathbb{R}^3$, namely $Q \;=\; \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A direct computation yields $Q^TQ = I$, while

$$Q^TAQ \;=\; \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \;=\; D.$$

17 Quadratic forms

A **quadratic form** is a homogeneous polynomial $q(x_1, x_2, \ldots, x_n)$ of degree 2 in the variables $x_1, \ldots, x_n$.

$$q(\mathbf{x}) \;=\; \sum_{i,i=1}^{n} a_{ij}x_ix_j \;=\; \mathbf{x}^TA\mathbf{x}\,. \tag{17.1}$$

Here $A = (a_{ij})_{i,j=1,\ldots,n}$. We can assume that $a_{ij} = a_{ij}$, so that the matrix A is symmetric. Otherwise, we replace both a_{ij} and a_{ji} with the average value $\frac{a_{ij}+a_{ji}}{2}$.

Notice that in the quadratic form $q(x_1, x_2, \ldots, x_n)$

- the coefficient of x_i^2 is a_{ii} ,
- for $i \neq j$, the coefficient of $x_i x_j$ is $a_{ij} + a_{ji}$.

Example 1. The quadratic form

$$q(x_1, x_2, x_3) \ = \ 3x_1^2 - x_2^2 + 5x_3^2 + x_1x_2 + 2x_1x_3 - 8x_2x_3$$

can be written as

$$q(x_1, x_2, x_3) \ = \ [x_1 \ x_2 \ x_3] \begin{bmatrix} 3 & 1/2 & 1 \\ 1/2 & -1 & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

Given a symmetric matrix A, by Theorem 16.1 there exists an orthogonal matrix Q such that $D = Q^{-1}AQ$ is diagonal. Its diagonal elements are precisely the eigenvalues of A. Making the change of variables

$$\mathbf{x} \ = \ Q\mathbf{y}\,, \qquad\qquad \mathbf{y} \ = \ Q^T\mathbf{x}\,, \tag{17.2}$$

from (17.1) one obtains

$$q(\mathbf{x}) \ = \ \mathbf{x}^T A \mathbf{x} \ = \ (Q\mathbf{y})^T A (Q\mathbf{y}) \ = \ \mathbf{y}^T Q^T A Q \mathbf{y} \ = \ \mathbf{y}^T D \mathbf{y}\,.$$

Theorem 17.1 *Given the quadratic form (17.1), one can always make a change of variables* $\mathbf{x} = Q\mathbf{y}$ *so that, in terms of the new variables* $\mathbf{y} = (y_1, \ldots, y_n)$*, the quadratic form is diagonal, namely*

$$q(\mathbf{x}) \ = \ \mathbf{y}^T D \mathbf{y} \ = \ \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2\,. \tag{17.3}$$

Example 2. Consider the quadratic form

$$q(\mathbf{x}) \ = \ 3x_1^2 + 4x_1x_2 + 6x_2^2 \ = \ [x_1 \ \ x_2] \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$ is symmetric. Its eigenvalues and (normalized) eigenvectors are

$$\lambda_1 = 7, \qquad \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \qquad\qquad \lambda_2 = 2, \qquad \mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$

Introducing the orthogonal matrix $Q \ = \ \left[\mathbf{v}_1 \middle| \mathbf{v}_2\right] \ = \ \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$, we obtain

$$D \ = \ \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \ = \ Q^{-1}AQ\,.$$

We can now perform the change of variable $\mathbf{x} = Q\mathbf{y}$, so that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ = \ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ = \ \frac{1}{\sqrt{5}} \begin{bmatrix} y_1 - 2y_2 \\ 2y_1 + y_2 \end{bmatrix}. \tag{17.4}$$

In terms of the new variables, the quadratic form becomes

$$\begin{aligned} q(\mathbf{x}) &= 3x_1^2 + 4x_1x_2 + 6x_2^2 \;=\; \frac{1}{5}\Big(3(y_1 - 2y_2)^2 + 4(y_1 - 2y_2)(2y_1 + y_2) + 6(2y_1 + y_2)^2\Big) \\ &= \frac{1}{5}\big(35y_1^2 + 10y_2^2\big) \;=\; 7y_1^2 + 2y_2^2\,. \end{aligned}$$

Notice that the change of variables (17.4) represents a rotation of the coordinate axis, as shown in Fig. 11.

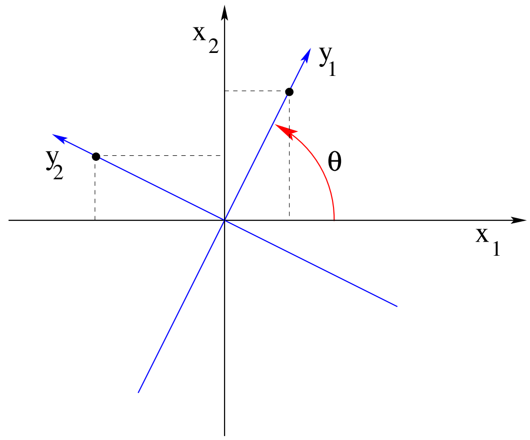

Figure 11: The change of coordinates in Example 2 corresponds to a rotation of the axes by an angle θ, with $\cos\theta = 1/\sqrt{5}$.

17.1 Positive definite matrices

- A quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is **positive semidefinite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for every $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

  In this case we also say that the symmetric matrix A is **positive semidefinite**.

- A quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$.

  In this case we also say that the symmetric matrix A is **positive definite**.

- A quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is **indefinite** if it takes both positive and negative values.

Example 3. The quadratic form $q(\mathbf{x}) = x_1^2 - x_1x_2 + x_2^2$ is positive definite. Indeed, we can write

$$q(\mathbf{x}) \;=\; \frac{1}{2}\Big(x_1^2 + x_2^2 + (x_1 - x_2)^2\Big) \;>\; 0 \qquad \text{whenever} \quad (x_1, x_2) \neq (0, 0).$$

On the other hand, the quadratic form $q(\mathbf{x}) = x_1^2 + 3x_1x_2 + x_2^2$ is indefinite. Indeed taking $\mathbf{x} = (x_1, x_2) = (1, 1)$ we obtain $q(\mathbf{x}) = 5$, while taking $\mathbf{x} = (x_1, x_2) = (1, -1)$ we obtain $q(\mathbf{x}) = -1$.

To decide whether the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is positive definite, it suffices to find the eigenvalues of the matrix A.

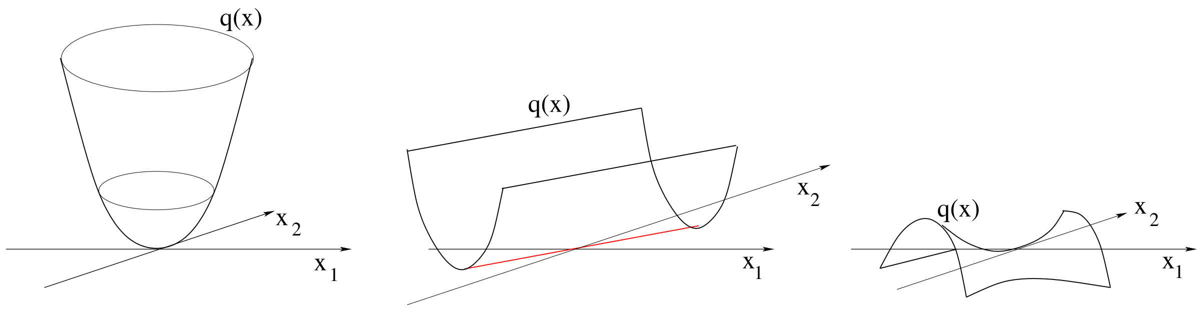

Figure 12: Left: a positive definite quadratic form. Center: a positive semidefinite quadratic form. Right: a quadratic form which is indefinite.

Theorem 17.2 *Consider the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.*

(i) If all eigenvalues of A are ≥ 0, then the quadratic form $q(\mathbf{x})$ is positive semidefinite.

(ii) If all eigenvalues of A are > 0, then the quadratic form $q(\mathbf{x})$ is positive definite.

(iii) If some eigenvalues of A are > 0 and other eigenvalues are < 0, then the quadratic form $q(\mathbf{x})$ is indefinite.

Indeed, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Performing the change of variables $\mathbf{x} = Q\mathbf{y}$, as in (17.2)-(17.3) we obtain

$$q(\mathbf{x}) \;=\; \mathbf{y}^T D\mathbf{y} \;=\; \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2, . \tag{17.5}$$

- If $\lambda_k \geq 0$ for every k, then the right hand side of (17.5) is ≥ 0 for every $\mathbf{y} = (y_1, \ldots, y_n)$.
- If $\lambda_k > 0$ for every k, then the right hand side of (17.5) is strictly positive for every $\mathbf{y} = (y_1, \ldots, y_n) \neq (0, \ldots, 0)$.
- If some eigenvalues are positive and other are negative, then the right hand side of (17.5) can be sometimes positive, sometimes negative, depending on the choice of $(y_1, y_2, \ldots, y_n)$.

Example 3 (continued). The quadratic form

$$q(\mathbf{x}) \;=\; x_1^2 - x_1 x_2 + x_2^2 \;=\; \mathbf{x}^T A\mathbf{x} \qquad \text{where} \quad A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

is positive definite. Indeed, the matrix A has eigenvalues $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{2}$, which are both strictly positive.

On the other hand, the quadratic form

$$q(\mathbf{x}) \;=\; x_1^2 + 3x_1 x_2 + x_2^2 \;=\; \mathbf{x}^T B\mathbf{x} \qquad \text{where} \quad B = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 1 \end{bmatrix}$$

is indefinite. Indeed, the matrix B has eigenvalues $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{5}{2}$. One of these is positive, the other is negative.

Remark 17.1 Let A be any $m \times n$ matrix. Then the $n \times n$ matrix $B = A^T A$ is symmetric and positive semidefinite. Indeed, for $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x}^T B \mathbf{x} \;=\; \mathbf{x}^T A^T A \mathbf{x} \;=\; (A\mathbf{x})^T (A\mathbf{x}) \;=\; \|A\mathbf{x}\|^2 \;\geq\; 0.$$

Here is another way to check whether an $n \times n$ symmetric matrix A is positive definite, without computing the eigenvalues. As in Fig. 13, for every $k = 1, 2, \ldots, n$, let A_k be the $k \times k$ submatrix containing the elements in the first k rows and k columns of A. These are called the *principal submatrices* of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & a_{33} & & \\ & & & & \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix}$$

Figure 13: The principal submatrices of the $n \times n$ matrix $A = (a_{ij})$.

Theorem 17.3 *(i) A symmetric matrix A is positive semidefinite if and only if all the principal submatrices A_k have determinant ≥ 0.*

(ii) A symmetric matrix A is positive definite if and only if all the principal submatrices A_k have determinant > 0.

Example 5. Consider the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 4 \\ 0 & 4 & 16 \end{bmatrix}$. We compute the determinants of the principal submatrices:

$$\det[2] = 2, \qquad \det \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} = 2, \qquad \det \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 4 \\ 0 & 4 & 16 \end{bmatrix} = 0.$$

We conclude that the matrix A is positive semidefinite, but not positive definite.

18 Singular Value Decomposition

Given an $n \times n$ symmetric matrix A, one can always find an orthogonal matrix Q such that $D = Q^T A Q$ is diagonal.

In this section we show how to transform a general $m \times n$ matrix A into diagonal form. This can be achieved by means of two orthogonal matrices U, V.

Definition 18.1 *An $m \times n$ matrix $D = [d_{ij}]$ is* **diagonal** *if there exists numbers $\sigma_1, \ldots, \sigma_p$ with $p = \min\{m, n\}$ such that*

$$\begin{cases} d_{ii} = \sigma_i & \text{if } 1 \leq i \leq p, \\ d_{ij} = 0 & \text{if } i \neq j\,. \end{cases}$$

In this case we write $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$.

A diagonal $m \times n$ matrix thus has the form

$$D = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{or} \qquad D = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_p & 0 & \cdots & 0 \end{bmatrix}.$$

Theorem 18.1 (Singular Value Decomposition). *Let A be any $m \times n$ matrix. Set $p = \min\{m, n\}$. Then there exist:*

- *an orthogonal $m \times m$ matrix $U = \Big[\mathbf{u}_1 \Big| \cdots \Big| \mathbf{u}_m\Big]$,*
- *an orthogonal $n \times n$ matrix $V = \Big[\mathbf{v}_1 \Big| \cdots \Big| \mathbf{v}_n\Big]$,*
- *non-negative numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ (called the* **singular values** *of the matrix A),*

such that the product $U^T A V = D = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$ is a diagonal matrix.

We now describe a procedure to construct the orthogonal matrices U, V, and the diagonal matrix D. To fix ideas, consider the case $n \leq m$ (If $n > m$, one can apply the same procedure to the transposed matrix A^T).

STEP 1: We begin by computing the $n \times n$ matrix $B = A^T A$. This matrix is symmetric and positive semidefinite, hence it has eigenvalues $\lambda_i \geq 0$, $i = 1, \ldots, n$. Assume

$$\lambda_1 \;\geq\; \lambda_2 \;\;\geq\;\; \cdots \;\geq\; \lambda_n \;=\; 0. \tag{18.1}$$

STEP 2: The diagonal elements σ_i and the column vectors $\mathbf{v}_i, \mathbf{u}_j$ are determined as follows.

- For $i = 1, \ldots, n$, the diagonal entries are $\sigma_i \;=\; \sqrt{\lambda_i}$. These are called the **singular values** of the matrix A.

- The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are an orthonormal basis of $\mathbb{R}^n$, given by the corresponding eigenvectors of the symmetric matrix $B = A^T A$.

- For every $j \in \{1, \dots, m\}$ such that $\sigma_j > 0$, we define $\mathbf{u}_j = \frac{1}{\sigma_j} A\mathbf{v}_j$. This construction already guarantees that these vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ have length 1 and are perpendicular to each other.

  If $r < m$, we can then find additional vectors $\mathbf{u}_{r+1}, \dots, \mathbf{u}_m$ such that the set $\{\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$ is an orthonormal basis of $\mathbb{R}^m$.

Since U, V are orthogonal matrices, $U^T = U^{-1}$, $V^T = V^{-1}$ and we have

$$U^T A V \;=\; D \qquad \Longrightarrow \qquad A \;=\; U D V^T$$

$$A \;=\; \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{array}\right] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{bmatrix} \;=\; \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T$$

Notice that here each product $\mathbf{u}_k \mathbf{v}_k^T$ is an $m \times n$ matrix with rank 1. The above formula yields a representation of an arbitrary $m \times n$ matrix A as the sum of n matrices of rank one.

- The diagonal values $\sigma_1, \sigma_2, \dots, \sigma_n \geq 0$, are called the **singular values** of the matrix A.

  They are uniquely determined because

  $$\sigma_1^2 \;\geq\; \sigma_2^2 \;\geq\; \cdots \;\geq\; \sigma_n^2 \;\geq\; 0$$

  are the eigenvalues of the symmetric matrix $A^T A$.

- The column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p \in \mathbb{R}^n$ are called the **right singular vectors** of the matrix A.

- The column vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in \mathbb{R}^m$ are called the **left singular vectors** of the matrix A.

Remark 18.1 In some applications, one needs to store the data contained in a very large matrix, say with $m, n >> 1$. To compress this data, we can choose an integer q much smaller than m, n, and consider the approximation

$$A \;=\; \sum_{j=1}^{n} \sigma_j \mathbf{u}_j \mathbf{v}_j^T \;\approx\; \sum_{j=1}^{q} \sigma_j \mathbf{u}_j \mathbf{v}_j^T \,,$$

by removing all terms with $j > q$.

Since the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ are given in decreasing order, we expect that the coefficients σ_j with $j > q$ will be small, and can thus be neglected. This procedure often yields very good results in image reconstruction.

Example. Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix}$. To construct a singular value decomposition, we first compute the symmetric matrix

$$A^T A \;=\; \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} \;=\; \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}.$$

Its eigenvalues and normalized eigenvectors are

$$\lambda_1 = 9, \quad \lambda_2 = 4\,, \qquad \mathbf{v}_1 \;=\; \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 \;=\; \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Notice that these vectors are perpendicular to each other. Therefore, the matrix

$$V = \begin{bmatrix} \mathbf{v}_1 \big| \mathbf{v}_2 \end{bmatrix} \;=\; \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

is an orthogonal matrix. The singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} = 3\,, \qquad \sigma_2 = \sqrt{\lambda_2} = 2.$$

We now compute

$$\mathbf{u}_1 \;=\; \frac{1}{\sigma_1} A \mathbf{v}_1 \;=\; \frac{1}{3} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \;=\; \frac{1}{\sqrt{45}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix},$$

$$\mathbf{u}_2 \;=\; \frac{1}{\sigma_2} A \mathbf{v}_2 \;=\; \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; \frac{1}{\sqrt{20}} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}.$$

Notice that the two vectors $\mathbf{u}_1$, $\mathbf{u}_2$ have length 1 and are are perpendicular to each other. The decomposition of A is now

$$\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \;=\; 3\,\frac{1}{\sqrt{45}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \end{bmatrix} + 2\,\frac{1}{\sqrt{20}} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

$$= \; \frac{1}{5} \begin{bmatrix} 8 & -4 \\ -4 & 2 \\ 10 & -5 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \;=\; \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} \;=\; A.$$

19 Some applications

19.1 Maxima and minima of functions of several variables

To find local maxima and minima of a function $f : \mathbb{R} \mapsto \mathbb{R}$, following rules of basic Calculus:

- We first find a point $\bar{x}$ where $f'(\bar{x}) = 0$.
- We then apply the second derivative test:

$$
\begin{aligned}
f''(\bar{x}) > 0 \qquad &\Longrightarrow \qquad f \text{ has a local minimum at } \bar{x} \\
f''(\bar{x}) < 0 \qquad &\Longrightarrow \qquad f \text{ has a local maximum at } \bar{x}
\end{aligned}
$$

  If $f''(\bar{x}) = 0$, then the second derivative test yields no information.

A similar approach applies to functions of several variables $f : \mathbb{R}^n \mapsto \mathbb{R}$.

To find local maxima and minima of a function $f = f(x_1, x_2, \ldots, x_n)$, we proceed as follows.

- We first find a point $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ where the gradient vanishes

$$\nabla f(\bar{x}) \;=\; \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \;\ldots\;, \frac{\partial f}{\partial x_n} \right) \;=\; (0, 0, \ldots, 0).$$

- We then apply the second derivative test, computing the $n \times n$ symmetric matrix of second order partial derivatives

$$A \;=\; \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ & & \cdots & \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \;=\; \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,n}$$

$$
\begin{aligned}
A \text{ positive definite} \qquad &\Longrightarrow \qquad f \text{ has a local minimum at } \bar{x} \\
A \text{ negative definite} \qquad &\Longrightarrow \qquad f \text{ has a local maximum at } \bar{x} \\
A \text{ indefinite} \qquad &\Longrightarrow \qquad f \text{ has neither a local maximum nor a local minimum at } \bar{x}
\end{aligned}
$$

In the remaining cases, where A is only semi-definite, the second derivative test yields no information.

Example. Consider the function of two variables

$$f(x_1, x_2) \;=\; (x_1^2 - 2x_1)\sin x_2\,.$$

Its gradient is

$$\nabla f(x_1, x_2) \;=\; \left(\frac{\partial f}{\partial x_1}\,,\;\frac{\partial f}{\partial x_2}\right) \;=\; \Big((2x_1 - 2)\sin x_2\,,\;(x_1^2 - 2x_1)\cos x_2\Big).$$

Moreover, the 2×2 symmetric matrix of second order partial derivatives of f is

$$D^2 f(x_1, x_2) \;=\; \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix} \;=\; \begin{pmatrix} 2\sin x_2 & (2x_1 - 2)\cos x_2 \\[2ex] (2x_1 - 2)\cos x_2 & -(x_1^2 - 2x_1)\sin x_2 \end{pmatrix}.$$

At the point $\bar{x} = (\bar{x}_1, \bar{x}_2) = (1, \pi/2)$, the gradient $\nabla f(\bar{x})$ vanishes, while the matrix of second derivatives is

$$A \;=\; D^2 f(\bar{x}) \;=\; \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix is positive definite, hence f has a local minimum at the point $(1, \frac{\pi}{2})$.

At the point $\bar{x} = (\bar{x}_1, \bar{x}_2) = (2, 0)$, the gradient $\nabla f(\bar{x})$ also vanishes, while the matrix of second derivatives is now

$$A \;=\; D^2 f(\bar{x}) \;=\; \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

This matrix is indefinite, hence f has neither a local minimum nor a local maximum at the point $(2, 0)$.

19.2 Discrete dynamical systems

Consider a population which keeps growing: every year, its size gets multiplied by a factor λ. Calling

$$x_k \;=\; \text{size of the population on year } k\,,$$

for every $k = 0, 1, 2, \ldots$ we thus have the relation

$$x_{k+1} \;=\; \lambda x_k\,. \tag{19.1}$$

If the initial size of the population is x_0, by (19.1) it follows

$$x_1 = \lambda x_0, \qquad x_2 = \lambda x_1 = \lambda^2 x_0, \qquad \ldots \qquad x_k \;=\; \lambda^k x_0 \qquad \text{for all } k \geq 1.$$

Next, consider two populations. For example, think of predators (wolves) and preys (rabbits). Call

$$x_k \;=\; \text{number of predators, on year } k,$$

$$y_k \;=\; \text{number of preys, on year } k.$$

We now write an equation, similar to (19.1), describing how these populations change from one year to the next. For example

$$\begin{cases} x_{k+1} &= ax_k + by_k\,, \\ y_{k+1} &= cx_k + dy_k\,. \end{cases} \tag{19.2}$$

for some constants a, b, c, d.

Having observed the initial sizes of the two populations, say x_0 and y_0, we wish to understand how these populations will evolve in future years. For this purpose we rewrite (19.2) using vector notation:

$$\mathbf{w}^{(k+1)} = A\mathbf{w}^{(k)}, \tag{19.3}$$

where

$$\mathbf{w}^{(k)} = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{19.4}$$

By (19.3) it follows

$$\mathbf{w}^{(1)} = A\mathbf{w}^{(0)}, \qquad \mathbf{w}^{(2)} = A\mathbf{w}^{(1)} = A^2\mathbf{w}^{(0)}, \qquad \ldots \qquad \mathbf{w}^{(k)} = A^k\mathbf{w}^{(0)} \quad \text{for all } k \geq 1.$$

In other words, our problem can be solved by computing all the powers of the matrix A. If we can write $A = PDP^{-1}$ for some invertible matrix P and a diagonal matrix D, the solution is expressed by

$$\mathbf{w}^{(k)} = PD^kP^{-1}\mathbf{w}^{(0)}.$$

Example. Assume that the populations of two cooperating species evolve according to

$$\begin{cases} x_{k+1} &= 8y_k\,, \\ y_{k+1} &= \frac{3}{2}x_k + y_k\,. \end{cases}$$

In matrix notation, this means

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 3/2 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}.$$

The eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 8 \\ 3/2 & 1 \end{bmatrix}$ are found to be

$$\lambda_1 = -3, \quad \mathbf{v}_1 = \begin{bmatrix} -8 \\ 3 \end{bmatrix}, \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We thus have the decomposition

$$A = PDP^{-1} \qquad \text{with} \quad P = \begin{bmatrix} -8 & 2 \\ 3 & 1 \end{bmatrix}, \quad P^{-1} = \frac{-1}{14} \begin{bmatrix} 1 & -2 \\ -3 & -8 \end{bmatrix}, \qquad D = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}.$$

This yields the solution

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \frac{-1}{14} \begin{bmatrix} -8 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 \\ 0 & 4^k \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

19.3 Second order difference equations

We now consider a model where the population size on a given year depends on the sizes during the previous two years:

$$x_{k+1} = ax_k + bx_{k-1}, \qquad k = 2, 3, 4, \ldots$$

for some constants a, b.

Introducing the additional variable $y_k = x_{k-1}$, the above equation can be rewritten as a system:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}.$$

Knowing the initial data $y_0 = x_{-1}$ and x_0, this can be solved by applying the previous methods.

Example. The **Fibonacci numbers** $x_0, x_1, x_2, \ldots$ are defined by setting

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 1, \qquad \text{and inductively} \quad x_{k+1} = x_k + x_{k-1}.$$

Introducing the variable $y_k = x_{k-1}$ (with $y_1 = 0$, $y_2 = x_1 = 1$, etc...), we obtain the inductive equations

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \qquad \begin{bmatrix} x_k \\ y_k \end{bmatrix} = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{for all } k \geq 1.$$

The eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are

$$\lambda_1 = \frac{1-\sqrt{5}}{2}, \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ \dfrac{\sqrt{5}+1}{2} \end{bmatrix}, \qquad \lambda_2 = \frac{1+\sqrt{5}}{2}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \dfrac{\sqrt{5}-1}{2} \end{bmatrix}.$$

Standard computations now yield

$$A^k = PD^kP^{-1} = \begin{bmatrix} -1 & 1 \\ \dfrac{\sqrt{5}+1}{2} & \dfrac{\sqrt{5}-1}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^k \end{bmatrix} \cdot \frac{1}{2\sqrt{5}} \begin{bmatrix} 1-\sqrt{5} & 2 \\ 1+\sqrt{5} & 2 \end{bmatrix}$$

Taking the first component, we conclude that the k-th Fibonacci number is

$$x_k = \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^k - \left(\dfrac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}.$$

Since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, we have $\lim_{k\to\infty} \left(\frac{1-\sqrt{5}}{2}\right)^k = 0$. Therefore, for k large the Fibonacci numbers are well approximated by

$$x_k \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k.$$

The number $\lambda = \dfrac{1+\sqrt{5}}{2}$ satisfies the equation $\lambda = 1 + \dfrac{1}{\lambda}$. It is called the **golden ratio.**

19.4 Linear systems of differential equations

Every $n \times n$ matrix A determines a **vector field**: to each point $\mathbf{x} \in \mathbb{R}^n$ we associate the vector $A\mathbf{x}$.

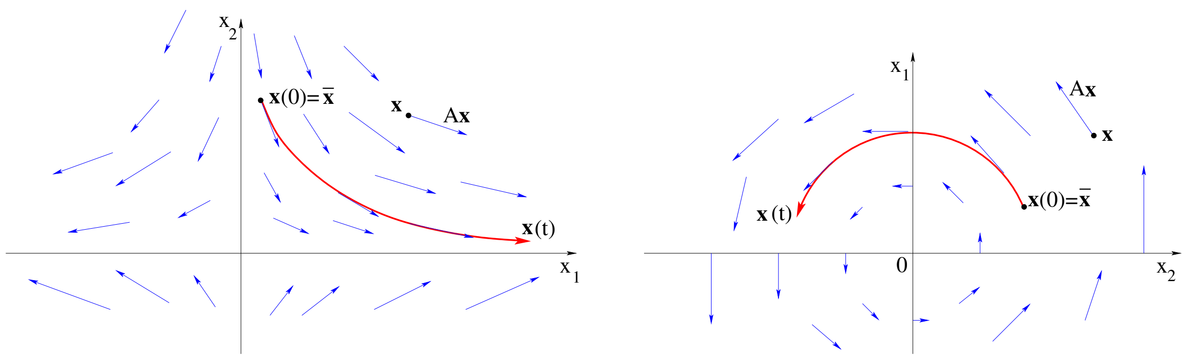

Figure 14: Left: the vector field in Example 1, and a trajectory of the ODE starting at the point $\bar{\mathbf{x}}$. Right: the vector field in Example 2, and a trajectory of the ODE starting at the point $\bar{\mathbf{x}}$.

Example 1. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ yields the vector field $A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$.

Example 2. The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ yields the vector field $A\mathbf{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.

In the following, we think of these vector fields as **velocities** of a moving point. Namely, consider a point $\mathbf{x}(t) \in \mathbb{R}^n$ which moves in time. Assume that at every time t its velocity is given by

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t). \tag{19.5}$$

Moreover, at the initial time $t = 0$, assume that the point is located at

$$\mathbf{x}(0) = \bar{\mathbf{x}}. \tag{19.6}$$

The system of ODEs (19.5) together with the initial data (19.6) completely determine the trajectory of the moving point. This solution is given by the formula

$$\mathbf{x}(t) = e^{tA}\bar{\mathbf{x}}, \tag{19.7}$$

where the exponential of the matrix is defined by

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \cdots$$

To check that (19.7) provides the correct solution, we compute

$$\mathbf{x}(0) = e^{0A}\bar{\mathbf{x}} = I\,\bar{\mathbf{x}} = \bar{\mathbf{x}},$$

$$\frac{d}{dt}e^{tA} = \frac{d}{dt}\sum_{k=0}^{\infty}\frac{t^k A^k}{k!} = \sum_{k=1}^{\infty}\frac{kt^{k-1}A^k}{k!} = \sum_{k=1}^{\infty}\frac{t^{k-1}A\,A^{k-1}}{(k-1)!}$$

$$\text{(setting } j=k-1\text{)} \qquad = A\sum_{j=0}^{\infty}\frac{t^j A^j}{j!} = A\,e^{tA},$$

$$\frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}e^{tA}\bar{\mathbf{x}} = Ae^{tA}\bar{\mathbf{x}} = A\mathbf{x}(t).$$

To compute the matrix exponential e^{tA}, we diagonalize the matrix A. Let $A = PDP^{-1}$, for some invertible matrix P and a diagonal matrix D. Then

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \Longrightarrow \qquad e^{tD} = \begin{bmatrix} e^{t\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{t\lambda_n} \end{bmatrix}.$$

This yields the explicit formula

$$e^{tA} = Pe^{tD}P^{-1}.$$

We say that the system of differential equations (19.5) is **asymptotically stable** if every solution approaches the origin as time increases:

$$\lim_{t\to+\infty}\mathbf{x}(t) = \mathbf{0}.$$

To check if stability holds, we first observe that:

- If λ is a real number with $\lambda < 0$, then $\lim_{t\to+\infty} e^{t\lambda} = 0$.
- If $\lambda = a + ib$ is a complex number, with real part $a < 0$, then again

$$e^{t\lambda} = e^{ta}e^{itb} = e^{ta}(\cos tb + i\sin tb), \qquad \lim_{t\to+\infty} e^{t\lambda} = 0.$$

Using the representation

$$\mathbf{x}(t) = Pe^{tD}P^{-1}\bar{\mathbf{x}},$$

we see that

- *every solution will approach the origin as $t \to +\infty$ provided that all eigenvalues of the matrix A have strictly negative real part.*

Indeed, in this case all entries of the diagonal matrix e^{tD} approach zero.

We study the case of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in more detail. The characteristic polynomial can be written as

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \;=\; \lambda^2 - \tau\lambda + \Delta,$$

where

$$\tau \;=\; a+d \;=\; \mathrm{trace}(A), \qquad\qquad \Delta \;=\; ad-bc \;=\; \det(A).$$

The eigenvalues of A are

$$\lambda_1,\ \lambda_2 \;=\; \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}\,.$$

Depending on the values of τ and Δ, various cases arise, as shown in Fig. 15. The system is asymptotically stable if $\tau < 0$ and $\Delta > 0$. Indeed, in this case both eigenvalues have negative real part.

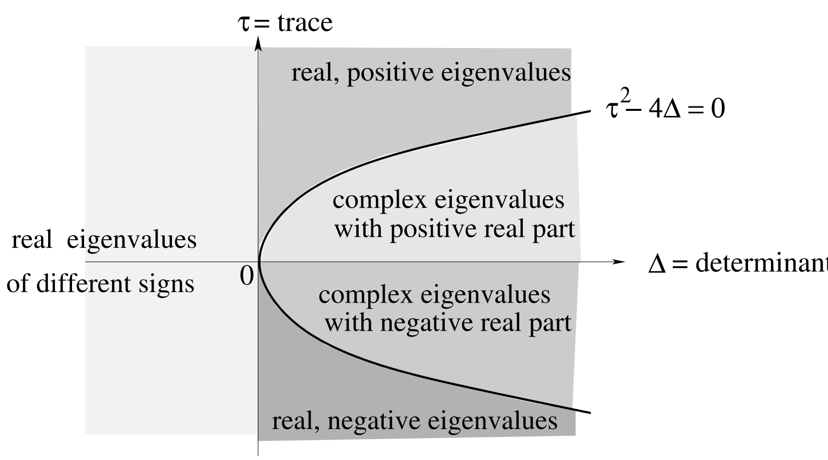

Figure 15: The eigenvalues λ_1, λ_2 of the matrix A, depending on the trace τ and on the determinant Δ. When $\Delta > \tau^2/4$, these eigenvalues are complex numbers.

Example. Two populations of bacteria evolve according to the system of ODEs

$$\begin{cases} \dfrac{d}{dt}x_1(t) &=& -x_1(t) + 5\,x_2(t) \\[2ex] \dfrac{d}{dt}x_2(t) &=& x_1(t) - c\,x_2(t) \end{cases}$$

Here c is a constant that can be increased by using an antibiotic. In matrix notation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \;=\; \begin{bmatrix} -1 & 5 \\ 1 & -c \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We wish to understand which values of c guarantee that both populations shrink to zero as time $t \to +\infty$.

This will be the case if all eigenvalues of the matrix $A = \begin{bmatrix} -1 & 5 \\ 1 & -c \end{bmatrix}$ have strictly negative real part. We thus require

$$\tau = \text{trace}(A) = -1 - c < 0, \qquad\qquad \Delta = \det(A) = c - 5 > 0.$$

Both of these inequalities are satisfied when $c > 5$.

Depending on the eigenvalues λ_1, λ_2 of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, trajectories of the system (19.5) are shown in Figures 16 and 17.

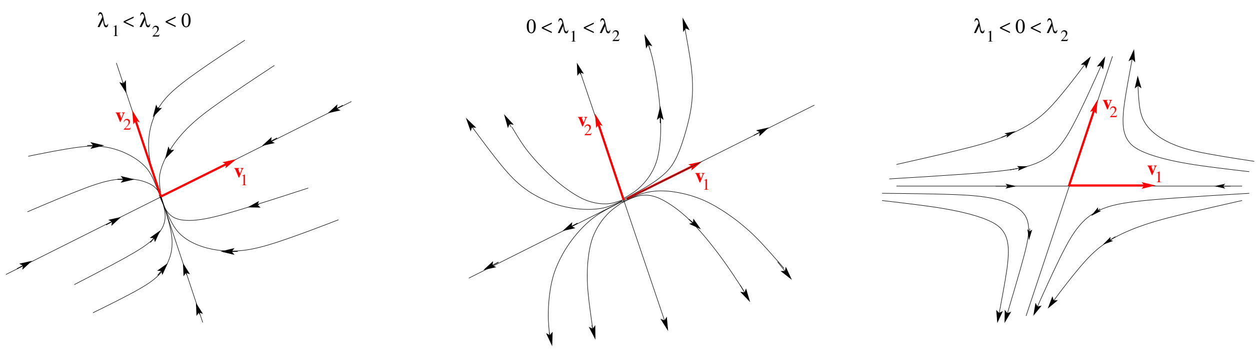

Figure 16: Trajectories of the system (19.5), when the 2×2 matrix A has real eigenvalues. Left: a stable case where both eigenvalues are negative and all trajectories approach the origin as $t \to +\infty$. Center and right: two unstable cases.

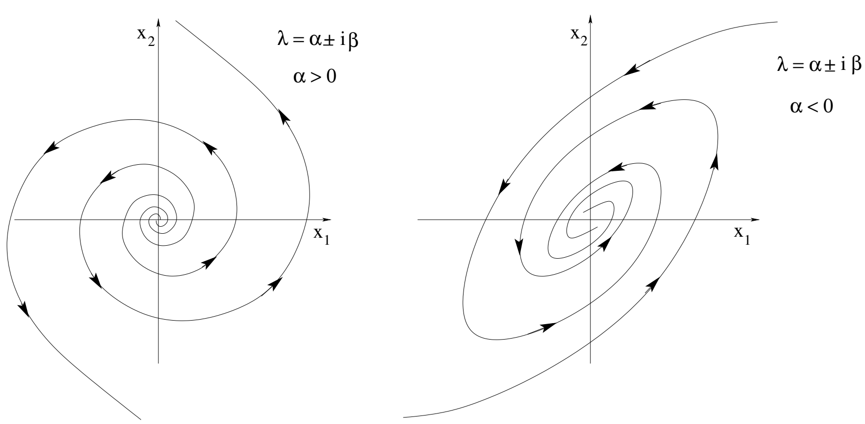

Figure 17: Trajectories of the system (19.5), when the 2×2 matrix A has complex eigenvalues. Right: a stable case where the eigenvalues have negative real part. Left: an unstable case.

19.5 Graphs

A **graph** is a set S, whose elements are called **vertices** (or **nodes**), together with a set E of (unordered) pairs of vertices, called **edges**.

The set E accounts for the couples of nodes (i,j) which are linked together by an edge. It is understood that if node i is linked to node j, then node j is linked to node i as well.

On the other hand, in a **directed graph** (or **digraph**) we specify the orientation of each edge. We write $(i,j) \in E$ if there is a link from node i to node j. This does not necessarily imply that there is a link from j to i.

Figure 18: Left: the set of nodes and edges in a graph. Right: nodes and edges in a digraph.

Example. Referring to the graph in Fig. 17, left, the set of nodes is $S = \{1,2,3,4,5,6\}$, while the set of edges is $E = \Big\{(1,2),\ (1,3),\ (2,3),\ (2,4),\ (3,4),\ (3,5),\ (3,6),\ (5,6)\Big\}$.

For the digraph in Fig. 17, right, the set of nodes is $S = \{1,2,3,4,5\}$ while the set of edges is $E = \Big\{(1,3),\ (1,4),\ (2,3),\ (3,1),\ (3,4),\ (3,5),\ (4,1),\ (4,5),\ (5,2)\Big\}$

Every graph with n vertices can be represented by an $n \times n$ symmetric matrix $A = [a_{ij}]$, by setting

$$a_{ij} \;=\; \begin{cases} 1 & \text{if the nodes } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

Every digraph with n vertices can be represented by a (possibly not symmetric) $n \times n$ matrix $A = [a_{ij}]$, by setting

$$a_{ij} \;=\; \begin{cases} 1 & \text{if there is an edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

Example. Referring to the graph in Fig. 17, left, the corresponding 6×6 symmetric matrix is

$$A \;=\; \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

For the digraph in Fig. 17, right, the corresponding 5×5 matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

If A is the matrix that represents a digraph, we can compute the matrix $B = A^2$. The entries of this matrix $B = [b_{ij}]$ have a nice interpretation:

b_{ij} = *number of paths of length 2, starting from node i and ending at node j.*

More generally, if $C = A^k$, the entries of the matrix $C = [c_{ij}]$ satisfy the property:

c_{ij} = *number of paths of length k, starting from node i and ending at node j.*

Example. For the digraph in Fig. 17, right, one has

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad B = A^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

One can check, for example, that $b_{15} = 2$ = number of paths of length 2 starting from node 1 and ending at node 5.

19.6 Web page ranking

Consider a directed graph, whose nodes are a set $S = \{1, 2, \ldots, n\}$ of web pages on the internet, and the edges are the links from one page to another. Our goal is to rank these web pages. Roughly speaking, page i should rank high if there are many other pages j pointing to i.

An algorithm for page ranking. *Define*

- L_i = *set of nodes j that have a link pointing to node i,*
- n_j = *number of links originating from page j.*

To each node j we assign a score x_j, in such a way that

$$x_i = \sum_{j \in L_i} \frac{x_j}{n_j}, \qquad j = 1, 2, \ldots, n. \tag{19.8}$$

Notice that the system of linear equations (19.8) has the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. We claim that it always has a nontrivial solution. Indeed, in matrix notation the system (19.8) takes the form

$$\mathbf{x} = B\mathbf{x}, \qquad \text{or equivalently} \qquad (B - I)\mathbf{x} = \mathbf{0}, \tag{19.9}$$

where B is an $n \times n$ matrix with the key property:

- *in every column of B, the entries add up to 1.*

This implies that $\lambda = 1$ is an eigenvalue of B. If $\mathbf{x}$ is a corresponding eigenvector, this provides the desired solution to (19.9).

Example. Four pages on the internet are linked according to the digraph in Fig. 19.

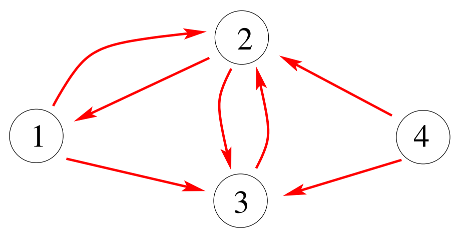

Figure 19: Four pages linked on the internet.

The matrix which represents the digraph is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We have

$$L_1 = \{2\}, \qquad L_2 = \{1,3\}, \qquad L_3 = \{1,3,4\}, \qquad L_4 = \emptyset,$$
$$n_1 = 2, \qquad n_2 = 2, \qquad n_3 = 1, \qquad n_4 = 2.$$

According to (19.8), the scores x_i of the four pages are found by solving

$$\begin{cases} x_1 &= \frac{x_2}{2}\,, \\ x_2 &= \frac{x_1}{2} + x_3 + \frac{x_4}{2} \\ x_3 &= \frac{x_1}{2} + \frac{x_2}{2} + \frac{x_4}{2} \\ x_4 &= 0 \end{cases}$$

In matrix notation

$$\mathbf{x} = B\mathbf{x}, \qquad B = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that B is obtained by first taking the transpose A^T, and then dividing each column j by the sum n_j of its entries. In this way, in each column of B the entries add up to 1. A nontrivial solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}.$$

Indeed, this is an eigenvector of B, corresponding to the eigenvalue $\lambda = 1$.

In conclusion: page 2 has the highest score and is ranked first, page 3 is ranked second, page 1 third, and page 4 last.