Control Theory: a Brief Tutorial

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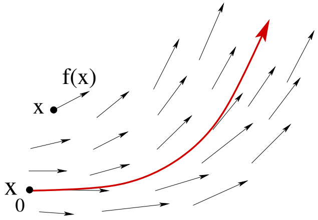
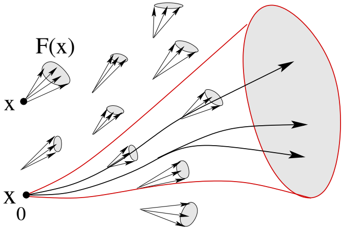
ODE's and control systems $\dot{x}(t) = \frac{d}{dt}x(t)$

$$\dot{x} = f(x) \qquad (ODE)$$





$$\dot{x} = f(x, u(t)), \qquad u(t) \in U \qquad \text{(control system)}$$

$$\dot{x} \in F(x) = \{f(t,u)\,;\ \ u \in U\} \qquad \text{(differential inclusion)}$$

Example 1 - boat on a river

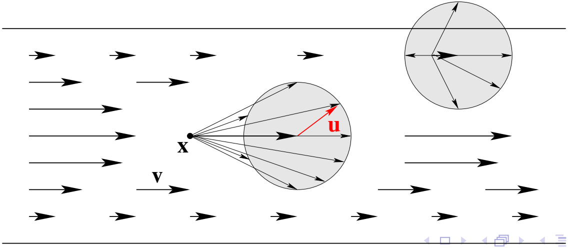
$x(t) =$ position of a boat on a river

$v(x)$ velocity of the water

$M =$ maximum speed of the boat relative to the water

$$\dot{x} \;=\; f(x, u(t)) \;=\; v(x) + u(t) \qquad\qquad u \in U = \{\omega \in \mathbb{R}^2, |\omega| \leq M\} \qquad (CS)$$

$$\dot{x} \in F(x) = \big\{ v(x) + \omega \;; \qquad |\omega| \leq M \big\} \qquad (DI)$$



Example 2 - cart on a rail

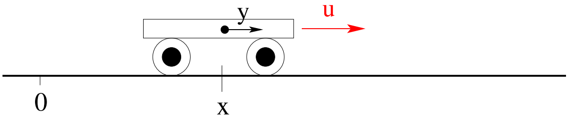
$x(t)$ = position of the cart

$y(t)$ = velocity of the cart

$u(t)$ = force pushing or pulling the cart (control function)

$$m\ddot{x} = u(t), \qquad m = \text{mass of the cart}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m}u(t) \end{cases} \qquad u(t) \in [-1, 1]$$



Example 3 - fishery management

$x(t) =$ amount of fish in a lake, at time t

$M =$ maximum population supported by the habitat

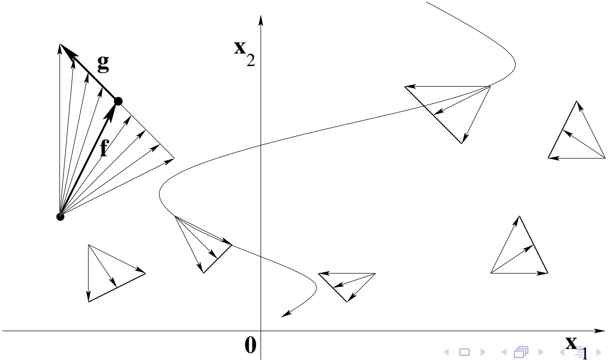
$u(t) =$ harvesting effort (control function)

$$\dot{x} = \alpha x(M - x) - xu, \qquad u(t) \in [0,\ u^{max}]$$

Example 4 - systems with scalar control entering linearly

$$\dot{x} = f(x) + g(x)\,u \qquad\qquad u \in [-1, 1]$$

$$\dot{x} \in F(x) = \{ f(x) + g(x)\,u\ ; \qquad u \in [-1, 1] \}$$



Open-loop controls

If $u = u(t)$ is assigned as a function of time, we say that u is an **open-loop control**.

Theorem

Assume that the function $f(x, u)$ is differentiable w.r.t. x. Then for every (possibly discontinuous) control function $u(t)$ the Cauchy problem

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(t_0) = x_0$$

has a unique solution.

Feedback controls

If $u = u(x)$ is assigned as a function of the state variable x, we say that u is a **closed-loop (or feedback) control**.

Theorem

Assume that the function $f(x, u)$ is differentiable w.r.t. both x and u, and that the feedback control function $u(x)$ is differentiable w.r.t. x.
Then the Cauchy problem

$$\dot{x}(t) = f(x(t), u(x)), \qquad x(t_0) = x_0$$

has a unique solution.

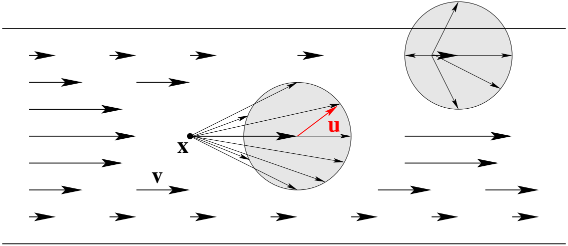
Designing a control function

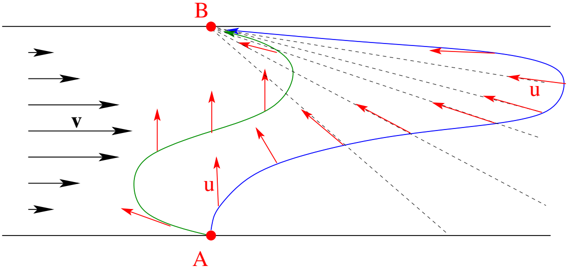
$$\dot{x} = f(x, u), \qquad u(t) \in U$$

Possible goals:

- Reach a target in minimum time
- Construct a feedback control function $u = u(x)$ which stabilizes the system at the origin.
- Construct an open-loop control $u(t)$ which is optimal for a given cost criterion.

Two strategies for crossing a river by boat





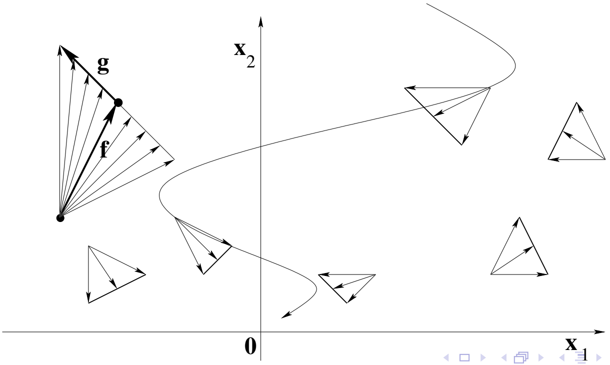
Feedback stabilization

Problem: construct a feedback control $u(x) \in U$ such that all trajectories of the ODE

$$\dot{x} \;=\; f(x, u(x))$$

(which start sufficiently close to the origin) satisfy

asymptotic stability: $\displaystyle\lim_{t\to+\infty} x(t) \;=\; 0$



Asymptotic stabilization by a feedback control

$$\dot{x} \;=\; f(x, u(x))$$

$$x = (x_1, \ldots, x_n)\,, \qquad u = (u_1, \ldots, u_m)\,, \qquad f = (f_1, \ldots, f_n)$$

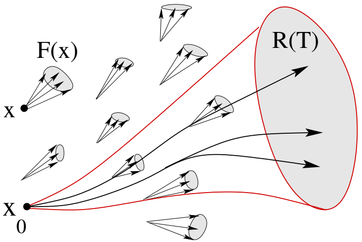
Theorem

Assume that $f(0, u(0)) = 0$, so that $x = 0 \in \mathbb{R}^n$ is an equilibrium point. This equilibrium is **asymptotically stable** *if the $n \times n$ Jacobian matrix $A = (A_{ij})$*

$$A_{ij} \;=\; \left[\frac{\partial f_i}{\partial x_j} + \sum_{k=1}^{m} \frac{\partial f_i}{\partial u_k} \frac{\partial u_k}{\partial x_j} \right]_{x=0}$$

has all eigenvalues with strictly negative real part.

$$\dot{x} \;=\; f(x,u), \qquad u(t) \in U, \qquad x(0) \;=\; x_0\,, \qquad t \in [0,T]$$



R(T) = reachable set
at time T

Goal: Choose a control $u(t) \in U$ such that the corresponding trajectory maximizes the payoff

$$J \;=\; \psi(x(T)) - \int_0^T L(x(t),u(t))\,dt$$

$$= \;\; [\text{terminal payoff}] - [\text{running cost}]$$

Existence of optimal controls (with no running cost)

Consider the problem

$$\text{maximize:} \qquad \psi(x(T))$$

$$\text{subject to:} \qquad \dot{x} = f(x,u), \qquad x(0) = x_0\,, \qquad u(t) \in U.$$

Assume that for every x the set of possible velocities

$$F(x) = \{f(x,u)\,; \quad u \in U\}$$

is closed, bounded, and convex.

Than an optimal (open-loop) control $u : [0,T] \mapsto U$ exists.

Existence of optimal controls (with dynamics linear w.r.t. u)

Consider the problem

$$\text{maximize:} \qquad \psi(x(T)) - \int_0^T L(x(t), u(t))\, dt$$

$$\text{subject to:} \qquad \dot{x} \;=\; f(x) + g(x)u, \qquad x(0) = x_0\,, \qquad u(t) \in [a, b].$$

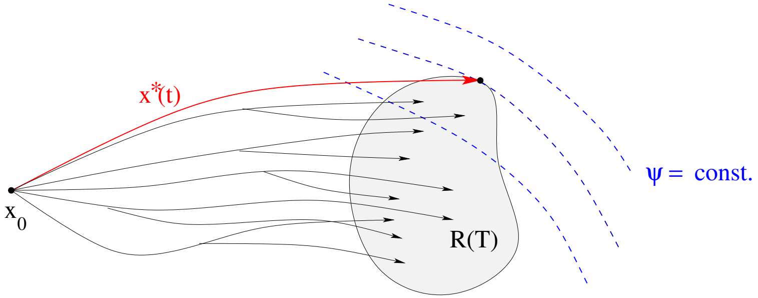
Assume that the cost function L is convex in u, for every fixed x.

Than an optimal (open-loop) control $u : [0, T] \mapsto U$ exists.

Finding the optimal control

$$\text{maximize:} \qquad \psi(x(T))$$

$$\text{subject to:} \qquad \dot{x} = f(x,u), \qquad x(0) = x_0\,, \qquad u(t) \in U$$



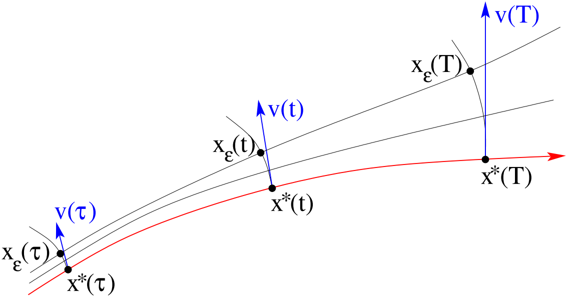
Let $u^*(t)$ be an optimal control and let $x^*(t)$ be the optimal trajectory. Derive necessary conditions for their optimality.

Preliminary: perturbed solutions of an ODE

$$\dot{x}(t) \;=\; g(t, x(t)) \qquad (ODE)$$

Let $x^*(t)$ be a solution, and consider a family of perturbed solutions

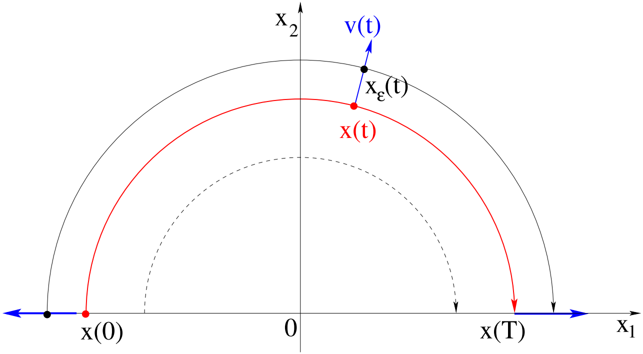
$$x_\varepsilon(t) \;=\; x^*(t) + \varepsilon v(t) + O(\varepsilon^2)$$



How does the "first order perturbation" $v(t)$ evolve in time?

An example

$$\dot{x}_1 \;=\; x_2\,, \qquad \dot{x}_2 \;=\; -\,x_1\,, \qquad\qquad T = \pi$$



A linearized equation for the evolution of tangent vectors

$$\dot{x}(t) \;=\; g(t,\, x(t)) \qquad (ODE)$$

$$x_\varepsilon(t) \;=\; x^*(t) + \varepsilon v(t) + O(\varepsilon^2) \qquad (\dagger)$$

Insert (†) in (ODE), and use a Taylor approximation:

$$\begin{aligned} \dot{x}_\varepsilon(t) &= g(t, x_\varepsilon(t)) \\ \dot{x}^*(t) + \varepsilon \dot{v}(t) + O(\varepsilon^2) &= g\Big(t,\; x^*(t) + \varepsilon v(t) + O(\varepsilon^2)\Big) \\ &= g\big(t, x^*(t)\big) + \frac{\partial g}{\partial x}(t, x^*(t)) \cdot \varepsilon v(t) + O(\varepsilon^2) \end{aligned}$$

$$\Longrightarrow \qquad \dot{v}(t) \;=\; A(t) v(t), \qquad\qquad A(t) \;=\; \frac{\partial g}{\partial x}(t, x^*(t))$$

The adjoint linear system

$$p = (p_1, \ldots, p_n), \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \qquad A(t) \text{ is an } n \times n \text{ matrix}$$

Lemma

Let $v(t)$ and $p(t)$ be any solutions to the linear ODEs

$$\dot v(t) = A(t)v(t), \qquad \dot p(t) = -p(t)A(t)$$

Then the product $p(t)v(t) = \sum_i p_i v_i$ is constant.

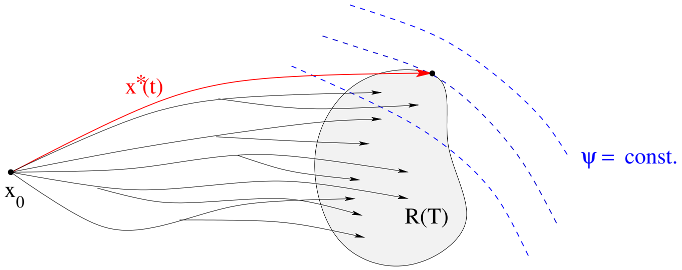
$$\frac{d}{dt}(pv) = \dot p v + p \dot v = (-pA)v + p(Av) = 0$$

Deriving necessary conditions

$$\text{maximize the terminal payoff:} \qquad \psi(x(T))$$

$$\text{subject to:} \qquad \dot{x} \;=\; f(x,u), \qquad x(0) = x_0\,, \qquad u(t) \in U.$$

$u^*(t)$ = optimal control, $x^*(t)$ = optimal trajectory.



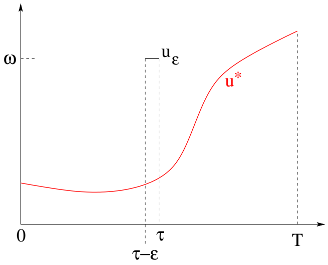
No matter how we change the control $u^*(\cdot)$, the terminal payoff cannot be increased.

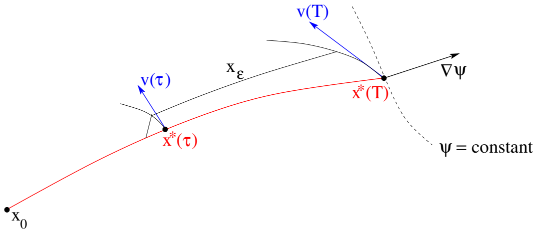
Needle variations

Choose an arbitrary time $\tau \in]0, T]$ and control value $\omega \in U$.

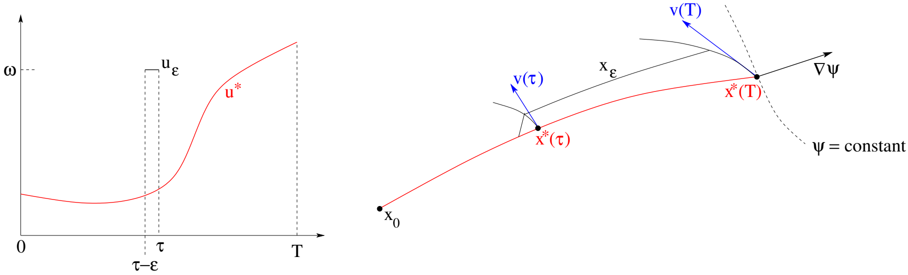
$$\text{needle variation:} \qquad u_\varepsilon(t) = \begin{cases} \omega & \text{if } t \in [\tau - \varepsilon, \tau], \\ u^*(t) & \text{otherwise.} \end{cases}$$

$$\text{perturbed trajectory:} \qquad x_\varepsilon(t) = \begin{cases} x^*(t) & \text{if } t \leq \tau - \varepsilon, \\ x^*(t) + \varepsilon v(t) + \mathcal{O}(\varepsilon^2) & \text{if } t \geq \tau \end{cases}$$





Computing the perturbed trajectory

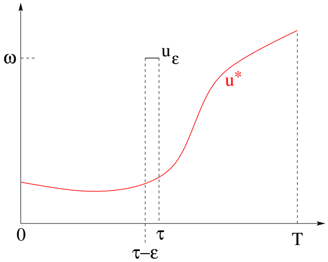


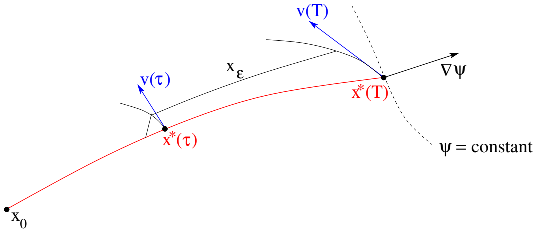
At time τ: $$x_\varepsilon(\tau) = x^*(\tau) + \varepsilon\big[f(x^*(\tau),\omega) - f(x^*(\tau),u^*(\tau))\big] + \mathcal{O}(\varepsilon^2)$$

On the interval $t \in [\tau, T]$: $$x_\varepsilon(t) = x^*(t) + \varepsilon v(t) + \mathcal{O}(\varepsilon^2),$$

$$\begin{cases} \dot v(t) = A(t)v(t), \\ v(\tau) = f(x^*(\tau),\omega) - f(x^*(\tau),u^*(\tau)), \end{cases} \qquad A(t) = \frac{\partial f}{\partial x}(x^*(t),u^*(t))$$

A family of necessary conditions





u^* is optimal $\Longrightarrow$ $\displaystyle \frac{d}{d\varepsilon}\psi(x_\varepsilon(T))\Big|_{\varepsilon=0} = \nabla\psi(x^*(T))\cdot v(T) \leq 0$

Let the row vector $p(t)$ be the solution to

$$\dot{p}(t) = -p(t)A(t), \qquad p(T) = \nabla\psi(x^*(T))$$

$$A(t) = \frac{\partial f}{\partial x}(t, x^*(t))$$

Since $v(t)$ satisfies $\dot{v}(t) = A(t)v(t)$, the product $p(t)v(t)$ is constant in time. Hence

$$p(\tau)v(\tau) = p(T)v(T) = \nabla\psi(x^*(T)) \cdot v(T) \leq 0$$

For every $\tau \in]0, T]$ and $\omega \in U$, we thus have

$$p(\tau)v(\tau) = p(\tau)\Big[f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau))\Big] \leq 0$$

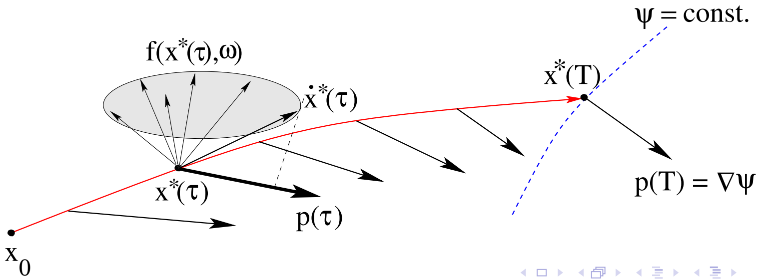
For every $\tau \in]0, T]$, the inequality

$$p(\tau)\big[f(x^*(\tau),\omega) - f(x^*(\tau),u^*(\tau))\big] \ \leq \ 0 \qquad \text{for all } \omega \in U$$

implies

$$p(\tau)\cdot\dot x^*(\tau) \ = \ p(\tau)\cdot f\big(x^*(\tau),\, u^*(\tau)\big) \ = \ \max_{\omega\in U}\ \Big\{p(\tau)\cdot f\big(x^*(\tau),\, \omega\big)\Big\} \quad (PMP)$$

For every time $\tau \in]0, T]$, the speed $\dot x^*(\tau)$ corresponding to the optimal control $u^*(\tau)$ is the one maximizing the product with $p(\tau)$.



Statement of the Pontryagin Maximum Principle

maximize the terminal payoff: $\psi(x(T))$

subject to: $\dot{x} = f(x,u), \qquad x(0) = x_0\,, \qquad u(t) \in U.$

Theorem

Let $t \mapsto u^*(t)$ be an optimal control and $t \mapsto x^*(t)$ be the corresponding optimal trajectory.

Let the row vector $t \mapsto p(t)$ be the solution to the linear adjoint system

$$\dot{p}(t) = -p(t)\,A(t), \qquad A_{ij}(t) \doteq \frac{\partial f_i}{\partial x_j}\big(x^*(t),\, u^*(t)\big)$$

with terminal condition $p(T) = \nabla\psi\big(x^*(T)\big)$.

Then, at every time $\tau \in [0,T]$ where $u^*(\cdot)$ is continuous, one has

$$p(\tau)\cdot f\big(x^*(\tau),\, u^*(\tau)\big) \;=\; \max_{\omega\in U}\,\Big\{p(\tau)\cdot f\big(x^*(\tau),\,\omega\big)\Big\}$$

Computing the Optimal Control

STEP 1: solve the pointwise maximixation problem, obtaining the optimal control u^* as a function of p, x, i.e.

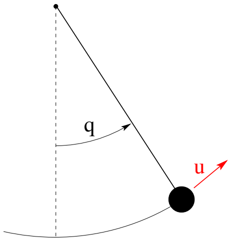
$$u^*(x,p) \;=\; \operatorname*{argmax}_{\omega \in U} \Big\{ p \cdot f(x,\omega) \Big\} \tag{1}$$

STEP 2: solve the two-point boundary value problem

$$\begin{cases} \dot x &= f\big(x,\ u^*(x,p)\big) \\ \dot p &= -p \cdot \frac{\partial}{\partial x} f\big(x,\ u^*(x,p)\big) \end{cases} \qquad\qquad \begin{cases} x(0) &= x_0 \\ p(T) &= \nabla \psi\big(x(T)\big) \end{cases} \tag{2}$$

- In general, the function $u^* = u^*(p,x)$ in (1) is highly nonlinear. It may be multivalued or discontinuous.
- The two-point boundary value problem (2) can be solved by a *shooting method*: Guess an initial value $p(0) = p_0$ and solve the corresponding Cauchy problem. Try to adjust the value of p_0 so that the terminal values $x(T),\ p(T)$ satisfy the given conditions.

Example: a linear pendulum



$q(t)$ = position of a linearized pendulum, controlled by an external force with magnitude $u(t) \in [-1, 1]$.

$$\ddot{q}(t) + q(t) = u(t), \qquad q(0) = \dot{q}(0) = 0, \qquad u(t) \in [-1, 1]$$

We wish to maximize the terminal displacement $q(T)$.

$$\ddot{q}(t) + q(t) = u(t), \qquad q(0) = \dot{q}(0) = 0, \qquad u(t) \in [-1, 1]$$

Equivalent control system: $x_1 = q$, $x_2 = \dot{q}$

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2, u) &= x_2 \\ \dot{x}_2 &= f_2(x_1, x_2, u) &= u - x_1 \end{cases} \qquad \begin{cases} x_1(0) &= 0 \\ x_2(0) &= 0 \end{cases}$$

Goal: maximize $\psi(x(T)) \doteq x_1(T)$

Let $u^*(t)$ be an optimal control, and let $x^*(t)$ be the optimal trajectory.

The adjoint vector $p = (p_1, p_2)$ is found by solving the linear system of ODEs

$$\dot{p} = -p(t)A(t), \qquad p(T) = \nabla\psi(x^*(T))$$

$$A_{ij}(t) = \frac{\partial f_i}{\partial x_j}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\psi(x_1, x_2) = x_1, \qquad (p_1(T), p_2(T)) = \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}\right)_{x = x^*(T)} = (1, 0)$$

$$(\dot p_1, \dot p_2) \;=\; -(p_1, p_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (p_1, p_2)(T) \;=\; (1, 0) \tag{3}$$

In this special case, we can explicitly solve the adjoint equation (3) without needing to know x^*, u^*, namely

$$(p_1, p_2)(t) = \big(\cos(T-t), \quad \sin(T-t)\big) \tag{4}$$

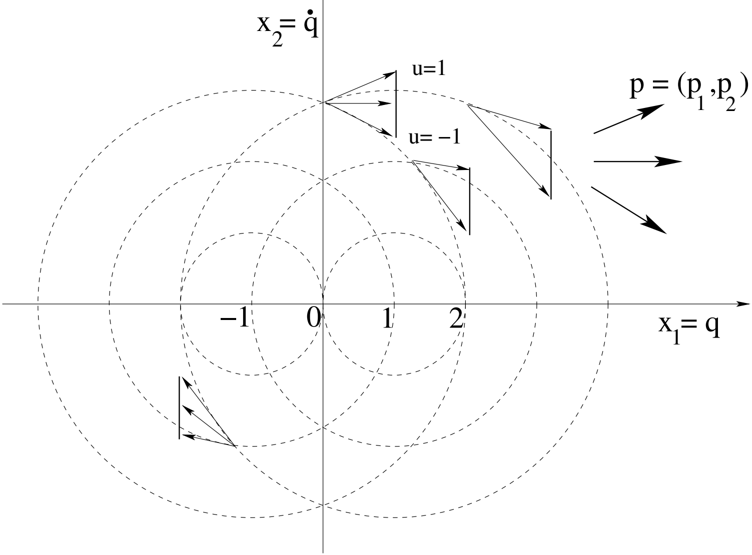
$$\begin{cases} \dot x_1 \;=\; f_1(x_1, x_2) \;=\; x_2 \\ \dot x_2 \;=\; f_2(x_1, x_2) \;=\; u - x_1 \end{cases}$$

Given $p = (p_1, p_2)$, the optimal control is

$$u^*(x,p) \;=\; \arg\max_{\omega\in[-1,1]} \{p\cdot f(x,\omega)\} \;=\; \arg\max_{\omega\in[-1,1]} \Big\{p_1x_2+p_2(-x_1+\omega)\Big\} \;=\; \mathrm{sign}(p_2)$$

By (4), the optimal control is

$$u^*(t) \;=\; \mathrm{sign}\big(p_2(t)\big) \;=\; \mathrm{sign}\big(\sin(T-t)\big) \qquad t \in [0, T]$$



$$p_2(t) > 0 \quad \Longrightarrow \quad u^*(t) \;=\; 1$$
$$p_2(t) < 0 \quad \Longrightarrow \quad u^*(t) \;=\; -1$$

References

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