Flow-Duration Curves. I: New Interpretation and Confidence Intervals

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Abstract: A flow-duration curve (FDC) is simply the complement of the cumulative distribution function of daily, weekly, monthly (or some other time interval of) streamflow. Applications of FDCs include, but are not limited to, hydropower planning, water-quality management, river and reservoir sedimentation studies, habitat suitability, and low-flow augmentation. Although FDCs have a long and rich history in the field of hydrology, they are sometimes criticized because, traditionally, their interpretation depends on the particular period of record on which they are based. If one considers *n* individual FDCs, each corresponding to one of the individual *n* years of record, then one may treat those *n* annual FDCs in much the same way one treats a sequence of annual maximum or annual minimum streamflows. This new annual-based interpretation enables confidence intervals and recurrence intervals to be associated with FDCs in a nonparametric framework.

INTRODUCTION

"It is a capital mistake to theorize before one has data," Sir Arthur Conan Doyle.

A flow-duration curve (FDC) represents the relationship between the magnitude and frequency of daily, weekly, monthly (or some other time interval of) streamflow for a particular river basin, providing an estimate of the percentage of time a given streamflow was equaled or exceeded over a historical period. An FDC provides a simple, yet comprehensive, graphical view of the overall historical variability associated with streamflow in a river basin.

An FDC is the complement of the cumulative distribution function (cdf) of daily streamflow. Each value of discharge Q has a corresponding exceedance probability p, and an FDC is simply a plot of Q_p, the pth quantile or percentile of daily streamflow versus exceedance probability p, where p is defined by

$$p = 1 - P\{Q \leq q\} \tag{1a}$$

$$p = 1 - F_Q(q) \tag{1b}$$

The quantile Q_p is a function of the observed streamflows, and since this function depends upon empirical observations, it is often termed the empirical quantile function. Statisticians term the complement of the cdf the "survival" distribution function. The term survival results from the fact that most applications involve survival data that arise in various fields

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Note. Discussion open until January 1, 1995, To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on January 28, 1993. This paper is part of the *Journal of Water Resources Planning and Management*, Vol. 120, No. 4, July/August, 1994. ©ASCE, ISSN 0733-9496/94/0004-0485/$2.00 + $.25 per page. Paper No. 5474.

such as medicine, manufacturing, and demography [see Anderson and Vaeth (1988) for a review of the literature].

Brief History of Application of Flow-Duration Curves

A sequel to the present paper (Vogel and Fennessey, work in progress) summarizes, in detail, the many applications of FDCs. The first use of an FDC is attributed to Clemens Herschel in about 1880 (Foster 1934). The widespread use of FDCs during the first half of this century is evidenced by many studies that sought to develop FDCs for particular regions of the U.S. For example, Mitchell (1957) developed procedures for estimating FDCs at gaged, partially gaged, and ungaged sites in Illinois. Cross and Bernhagen (1949) summarized FDCs in Ohio, and Saville et al. (1933) summarized FDCs in North Carolina. In the U.S., regional FDC procedures have been developed for ungaged sites in Illinois, New Hampshire, and Massachusetts by Singh (1971), Dingman (1978) and Fennessey and Vogel (1990), respectively. See Fennessey and Vogel (1990) for a review of other recent regional FDC models.

Mitchell (1957), Searcy (1959), and the Institute of Hydrology ("Low" 1980) provide comprehensive manuals on the construction, interpretation, and application of FDCs. Interestingly, most of the important work related to the construction, analysis, and interpretation of FDCs predates the common application of computers (e.g., Foster (1934); Beard (1943); Mitchell (1957); Searcy (1959); Hoyle (1963)].

Since the advent of computer technology, few articles on FDCs have been written, yet, ironically, many recent advances, due to computer technology, can be exploited along with FDC concepts as is shown in the present paper. Fienberg (1979) found that in general, there has been a prolonged decline in the relative use of graphical devices for displaying statistical information ever since the advent of computer technology. Although many of the articles on FDCs were written during the first half of this century, current textbooks still contain discussions pertaining to this important tool [see, for example, Warnick (1984); Gordon et al. (1992)].

Streamflow-duration curves have been advocated for use in hydrologic studies such as hydropower, water-supply, and irrigation planning [see, Chow (1964); Warnick (1984)]. Mitchell (1957) and Searcy (1959) describe additional applications to waste-load allocation and other water-quality management problems. Male and Ogawa (1984) show how FDCs can be used to illustrate and evaluate the trade-offs among the variables involved in the selection of a wastewater-treatment-plant capacity. The U.S. Bureau of Reclamation (Strand and Pemberton 1982) use FDCs in river and reservoir sedimentation studies that examine the frequency of suspended sediment loads and determine the long-term average suspended sediment yield for a given site. The U.S. Fish and Wildlife Service (Gordon et al. 1992) use FDCs in their "Instream Flow Incremental Methodology" for determining the suitability of habitats to streamflow of different magnitudes and frequencies. Alaouze (1989) describes the use of FDCs for determining the optimal allocation of water withdrawals from reservoirs, where each withdrawal is to have a unique reliability.

Some Caveats Associated with Flow-Duration Curves

Fig. 1 displays an example of an FDC along with the probability density function (pdf) of average daily streamflow for the Acheron River in Australia for the period 1947–1987. Also depicted are the mean, median, and

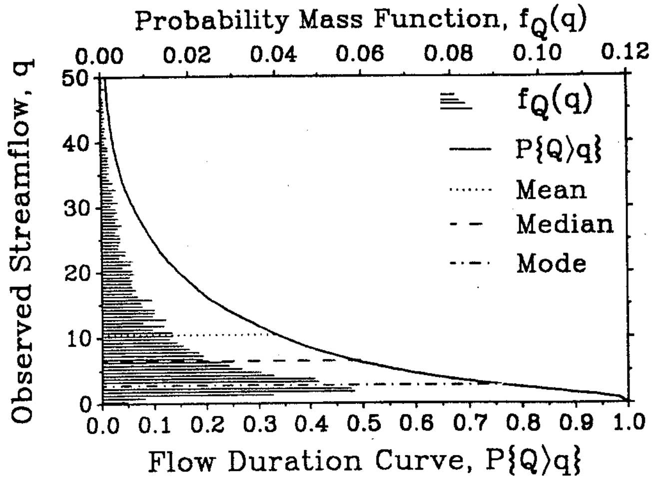

FIG. 1. Comparison of FDC with Probability Density Function of Observed Daily Streamflows (m^3/s) for Acheron River, 1947–1987

modal daily streamflows. Note that only 34.2% of the daily flows exceeded the mean over the period of record. This is not unusual, and this result emphasizes how misleading it can be to use the mean as a measure of central tendency for highly skewed data such as daily stream flow. Daily streamflows are so highly skewed that ordinary product moment ratios such as the coefficient of variation and skewness are remarkably biased and should be avoided, even for samples with tens of thousands of flow observations (Vogel and Fennessey 1993).

Although FDCs are appealing for depicting the hydrologic response of a river basin, they can be misleading because the autocorrelation structure of streamflow series is effectively removed from the plot. To clarify this point, Fig. 2 uses a single graphical image to compare the FDC with the hydrograph of the Acheron River. It should always be understood when viewing an FDC, that streamflow behaves the way it is illustrated in the hydrograph appearing as a dotted line in the background. We recommend plotting FDCs with the complete hydrograph in the background, perhaps using light shading, to reinforce the serial structure of all flow sequences. One could also plot a correlogram, which is a plot of the lag-k serial correlation versus lag k, to expose the significant serial structure associated with daily streamflow.

Although FDCs have a long and rich history in hydrology, they are sometimes criticized because, traditionally, their interpretation depends on the particular period of record on which they are based. If one considers n individual FDCs, each corresponding to one of the individual n years of record, then one may treat those n annual FDCs in much the same way one treats a sequence of annual maximum or annual minimum streamflows. Viewed in that context, the FDC becomes a generalization of the distribution of daily streamflow where the distribution of annual maximum flood flows and annual minimum low flows are simply special cases drawn from either end of the complete annual-based FDC. This new annual-based interpretation of FDCs provides a general approach to streamflow frequency analysis that allows us to derive confidence intervals, recurrence intervals, and quantile-estimation procedures for FDCs in a nonparametric framework.

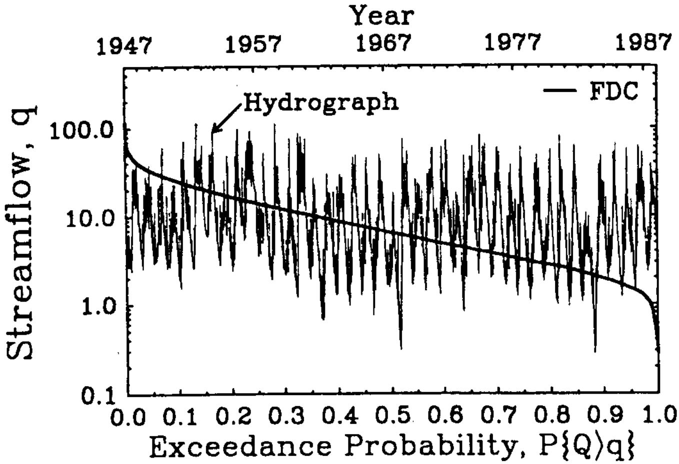

FIG. 2. Comparison of Hydrograph of Acheron River with Its FDC

TRADITIONAL FLOW-DURATION CURVE IS AN OGIVE

Prior to the advent of computer technology, Searcy (1959) and others recommended constructing FDCs by separating observed streamflow into 20–30 well-distributed class intervals, and defining the FDC as the cumulative histogram of streamflow based on those class intervals. Searcy (1959) provides explicit guidelines for the construction of class intervals to be used with his procedure [see Table 1 in Searcy (1959)]. Searcy's approach produces what statisticians term a ogive, which is a plot of the cumulative frequency corresponding to each class interval versus the upper limit of each class interval where straight lines connect consecutive points. An ogive is a grouped data analog of a graph of the empirical cumulative distribution function. Ogives are useful for representing selected percentiles or quantiles of a distribution or for constructing box and whisker plots. However, if one is interested in obtaining an accurate computerized description of FDCs and their associated confidence intervals, the more efficient and smoother quantile-estimation procedures to be described are recommended.

NONPARAMETRIC QUANTILE-ESTIMATION PROCEDURES

Consider the construction of an FDC or empirical quantile function from n observations of streamflow q_i, where $i = 1, \ldots, n$. If the streamflows are ranked, then the set of order statistics $q_{(i)}$, where $i = 1, \ldots, n$, results where $q_{(1)}$ is the largest and $q_{(n)}$ is the smallest observation.

Even before the era of computers, quantiles and associated FDCs could be estimated from one or two order statistics. For example, the simplest empirical quantile function, or quantile estimator, is obtained from a single order statistic using

$$Q_{p,1} = q_{(i)} \qquad \text{if } i = [(n + 1)p] \tag{2a}$$

$$Q_{p,1} = q_{(i+1)} \qquad \text{if } i < [(n + 1)p] \tag{2b}$$

where the quantity in brackets $[(n + 1)p]$ denotes the integer component of $(n + 1)p$ that is always less than or equal to $(n + 1)p$. We recommend setting the smallest possible observation $q_{(n+1)}$ equal to zero, the natural minimum for streamflow. If the observations are not bounded above, the estimator $Q_{p,1}$ is undefined for values of p that lead to $i = [(n + 1)p] = 0$, since $q_{(0)} = \infty$. Essentially, $Q_{p,1}$ is equivalent to plotting the ordered observations $q_{(i)}$ versus an estimate of their plotting positions p_i, where $p_i = i/(n + 1)$ is an estimate of the exceedance probability p in (1) known as the Weibull plotting position. The Weibull plotting position provides an unbiased estimate of $1 - F_Q(q)$, regardless of the underlying probability distribution from which streamflows arise.

The main drawback to the simple quantile estimator $Q_{p,1}$, is that due to the variability of individual order statistics, it is often an inefficient estimator. An efficient quantile estimator is one with low bias, variance, and mean square error. The lack of efficiency associated with $Q_{p,1}$ is particularly significant for small samples ($n < 100$) and for values of p near zero or unity, even for large samples.

One way to improve the efficiency of $Q_{p,1}$ is to reduce its variability by forming a weighted average of two or more adjacent order statistics using an appropriate weighting function. Such quantile estimators, based on a linear combination of the order statistics are termed L-estimators, analogous to L-moment estimators of distributional parameters recently advanced by Hosking (1990) and summarized by Stedinger et al. (1993) for hydrologic applications.

For example, Parzen (1979) introduced a simple quantile estimator based on the weighted average of two adjacent order statistics. One such weighted estimator, which is slightly smoother than $Q_{p,1}$ is

$$Q_{p,2} = (1 - \theta)q_{(i)} + \theta q_{(i+1)} \quad (3)$$

where $i = [(n + 1)p]$ [and the brackets are defined as in (2)]; and $\theta = ((n + 1)p - i)$. The estimator $Q_{p,2}$ is undefined for values of p that lead to $i = [(n + 1)p] = 0$.

In a comparison of 10 alternative quantile estimators, Parrish (1990) found that (3), with $i = [np]$ and $\theta = (np - i + 0.5)$, performed slightly better than (3), with $i = [(n + 1)p]$ and $\theta = ((n + 1)p - i)$; however, these estimators of i and θ only yield improvements for small samples. Again, one sets the smallest possible observation $q_{(n+1)}$ equal to zero; the natural minimum for streamflow.

The estimators $Q_{p,1}$ and $Q_{p,2}$ are probably adequate for constructing FDCs when thousands of daily streamflows are available; however, if one wishes to construct a series of annual FDCs, each of which is only based on 365 highly correlated daily streamflow observations, it may be wise to use more efficient nonparametric quantile estimators. Similarly, if one wishes to estimate the empirical quantile function associated with a particular quantile estimator, as we do later on, then one requires a reasonably efficient nonparametric quantile estimator for independent samples with n ranging from 10 to 100.

Harrell and Davis (1982), Kaigh and Lachenbruch (1982), Yang (1985), and Sheather and Marron (1990) introduced quantile estimators with smaller mean squared error than either $Q_{p,1}$ or $Q_{p,2}$ for a wide range of distributions and sample sizes. Harrell and Davis (1982) introduced a distribution-free quantile estimator based on linear combinations of all n order statistics, with significantly lower variance than the estimators $Q_{p,1}$ and $Q_{p,2}$. Their estimator is derived from the fact that the cumulative probability $F_Q[q_{(i)}]$ associated with each ranked streamflow $q_{(i)}$ follows a beta distribution [see

Loucks et al. (1981)]. Hence, the expected value of the ith order statistic is given by

$$E[q_{(i)}] = \frac{1}{B[i, n - i + 1]} \int_{-\infty}^{\infty} qF(q)^{i-1}[1 - F(q)]^{n-i}\, dF(q) \quad (4a)$$

$$E[q_{(i)}] = \frac{1}{B[i, n - i + 1]} \int_{0}^{1} F^{-1}(q)q^{i-1}(1 - q)^{n-i}\, dq \quad (4b)$$

where $B[a, b]$ denotes the beta function. Now taking $i = (n + 1)p$, regardless of whether or not i is an integer, $E[q_{(i)}]$ converges to $F^{-1}(p)$ as $n \to \infty$, which leads to the Harrell and Davis estimator

$$Q_{p,3} = \sum_{i=1}^{n} \lambda_i q_{(i)} \quad (5a)$$

with the weights λ_i estimated from

$$\lambda_i = \frac{1}{B[(n + 1)p, (n + 1)(1 - p)]} \int_{(i-1)/n}^{i/n} q^{(n+1)p-1}(1 - q)^{(n+1)(1-p)-1}\, dq \quad (5b)$$

$$\lambda_i = I_{i/n}[p(n + 1), (1 - p)(n + 1)] - I_{(i-1)/n}[p(n + 1), (1 - p)(n + 1)] \quad (5c)$$

where $I_x[a, b]$ = the incomplete beta function and $B[a, b]$ = the beta function. Yang (1985), and Sheather and Marron (1990) describe $Q_{p,3}$ as an analytical form of the bootstrap estimate of the mean of the $i = [(n + 1)p]$th order statistic. In other words, if one were to use simulation to obtain the bootstrap estimate of $E[q_{(i)}]$, for $i = (n + 1)p$, one would obtain the estimate $Q_{p,3}$. Eq. (5a) is the exact analytical version of the computationally intensive bootstrap estimate of $E[q_{(i)}]$ that requires simulated resampling. Efron (1982) provides an introduction to the bootstrap method.

As long as $n \geq 100$, Harrell and Davis (1982) suggest estimating the weights λ_i using numerical integration with two intervals between $(i - 1)/n$ and i/n. Otherwise they suggest estimating the incomplete beta function exactly. For this purpose one can either use the efficient algorithm suggested by Majumdar and Bhattacharjee (1973) or the approximations given by Abramowitz and Stegun (1972)

$$I_x[a, b] = 1 - \Phi(\chi^2/\eta) \qquad \text{if } (a + b - 1)(1 - x) \leq 0.8 \quad (6a)$$

$$I_x[a, b] = \Phi(y) \qquad \text{if } (a + b - 1)(1 - x) \geq 0.8 \quad (6b)$$

where

$$\chi^2 = (a + b - 1)(1 - x)(3 - x) - (1 - x)(b - 1) \quad (7a)$$

$$\eta = 2b \quad (7b)$$

$$y = 3\left[w_1\left(1 - \frac{1}{9b}\right) - w_2\left(1 - \frac{1}{9a}\right)\right]\left(\frac{w_1^2}{b} + \frac{w_2^2}{a}\right)^{-1/2} \quad (7c)$$

$$w_1 = (bx)^{1/3} \quad (7d)$$

$$w_2 = [a(1 - x)]^{1/3} \quad (7e)$$

and $\Phi(z)$ is equal to $P[Z \leq z]$, where Z = a standard normal random variable.

Kaigh and Lachenbruch (1982) introduced another quantile estimator, which is also an L-estimator with very similar properties to $Q_{p,3}$. The Kaigh and Lachenbruch estimator is termed a generalized sample quantile and is obtained by averaging an appropriate subsample quantile over all possible subsamples of fixed length k. Their estimator is

$$Q_{p,4} = \sum_{i=r}^{r+n-k} \omega_i q_i \tag{8}$$

where

$$\omega_i = \frac{\binom{i-1}{r-1}\binom{n-1}{k-r}}{\binom{n}{k}} \tag{9a}$$

$$r = [(k+1)p] \tag{9b}$$

and

$$\binom{n}{k} = \left(\frac{n!}{(n-k)!k!}\right) \tag{9c}$$

Kaigh and Driscoll (1987) show how (8) can be computed efficiently on a personal computer using a simple recursion formula. If $k = n$, $Q_{p,4}$ reduces to $Q_{p,1}$. The parameter k acts to smooth the quantile estimate, smaller values of k producing smoother estimates of Q_p. The choice of k that minimizes the mean squared error requires knowledge of the underlying distribution. Since precise rules for selecting the subsample size k associated with the estimator $Q_{p,4}$ are unavailable, and since Sheather and Marron (1990) and Parrish (1990) found that $Q_{p,3}$ performed better than $Q_{p,4}$ in small sample Monte Carlo studies, we drop $Q_{p,4}$ from further consideration in spite of the considerable attention given to it in the recent statistics literature.

Another popular class of L-estimators are called kernel quantile estimators. Nonparametric kernel-estimation procedures were introduced to the water-resources literature by Adamowski (1985), Guo (1991), and others for estimating the pdf of annual maximum floodflows. Yang (1985) and Sheather and Marron (1990) summarize the properties of kernel quantile estimators. Sheather and Marron show that for large samples, $Q_{p,4}$ is equivalent to a kernel estimator with a Gaussian kernel. They also found that kernel quantile estimators have surprisingly similar performance to $Q_{p,3}$. Kernel estimators contain a kernel function, which is essentially a smoothing function analogous to the value of k in $Q_{p,4}$.

Sheather and Marron (1990) show that asymptotically (for very large n), all nonparametric quantile estimators developed to date have approximately the same sampling properties. They show that even for small samples, with n ranging from 50 to 100, the estimators $Q_{p,1}$ and $Q_{p,2}$ can perform almost as well as the more complex estimators $Q_{p,3}$, $Q_{p,4}$ and kernel estimators. Since the estimators $Q_{p,1}$ and $Q_{p,2}$ are certainly simpler to compute than either $Q_{p,3}$ or $Q_{p,4}$, one would expect us to drop the more complex estimators entirely, given the fact that they performed only marginally better than the simple estimators in Sheather and Marron's small sample Monte Carlo studies. Nevertheless, the estimators $Q_{p,3}$ and $Q_{p,4}$ provide markedly smoother estimates of the quantile function than the simpler estimators for small samples, and bootstrap and jackknife estimates of the variance of these estimators are available and perform well. One cannot obtain bootstrap or jackknife estimates of the variance of $Q_{p,1}$ since the quantile function is discontinuous.

Each of the above quantile estimators has advantages in particular settings; these situations are described in the following sections within the context of estimating FDCs.

PERIOD-OF-RECORD FLOW-DURATION CURVE

Previous investigators, for example, Searcy (1959), Mitchell (1957), Beard (1943), and Foster (1934), have suggested constructing FDCs as ogives. Those studies, which were published prior to the advent of computers, advocated plotting the empirical cumulative histogram or ogive since only approximately 30 points needed to be plotted. When the complete period of record is used to construct an FDC, the quantile estimators described here are based on 365*n* daily streamflows or between 3,650 and 36,500 observations for records ranging from 10 to 100 years, respectively! With so many observations, one need not fit a curve through the points, since the points, themselves, could be used to create a curve. Another equivalent approach is to plot a hundred or so points and use spline curve-fitting procedures to draw a smooth curve through the points. Furthermore, with such large samples, the differences among the estimators $Q_{p,1}$, $Q_{p,2}$, $Q_{p,3}$, and $Q_{p,4}$ are negligible, and the simplest estimators $Q_{p,1}$ or $Q_{p,2}$ suffice.

Comparison of Quantile Estimators

Fig. 3 uses lognormal probability paper to compare the period-of-record (1947–1987) FDC of daily streamflow constructed using an empirical cumulative histogram [Searcy's (1959) recommended approach] with the estimator $Q_{p,1}$ for the Acheron River. Lognormal probability paper is constructed by plotting the logarithms of Q_p versus the inverse of the standard

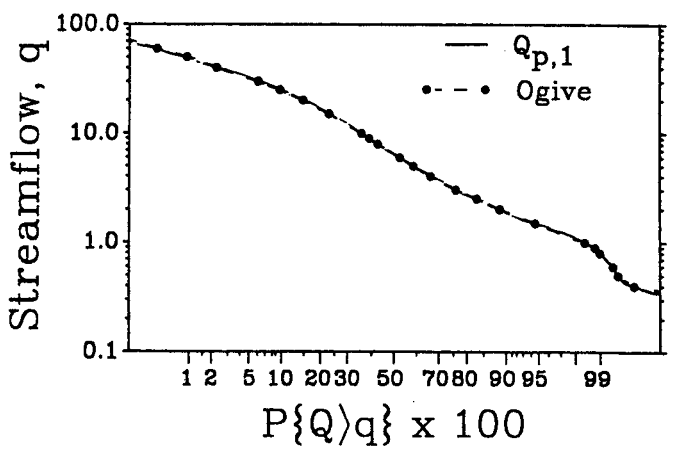

FIG. 3. Period-of-Record FDC (m³/s) Based On Searcy's (1959) Ogive Method Compared with Estimator $Q_{p,1}$ for Acheron River

normal cdf $z_p = \Phi^{-1}(p)$. See Stedinger et al. (1993) for a review of procedures for constructing probability plots.

The estimator $Q_{p,1}$ yields a slightly smoother and more representative FDC than the piecewise linear empirical cumulative histogram advocated by Searcy (1959) and others, even for a large sample such as this one. One could argue that both curves in Fig. 3 are almost equivalent, but that the estimator $Q_{p,1}$ has the advantage of being easily implemented on a computer and leads to significantly smoother quantile functions than the traditional ogive for small samples.

Fig. 4 uses lognormal probability paper to compare the period-of-record FDC for the Acheron River using the estimators $Q_{p,1}$, $Q_{p,2}$, and $Q_{p,3}$ based on the complete 40-year period-of-record 1947–1987 in Fig. 4(*a*) and based on only the single year 1987 in Fig. 4(*b*). Here one observes that the three quantile estimators are almost indistinguishable for the 40-year sample of

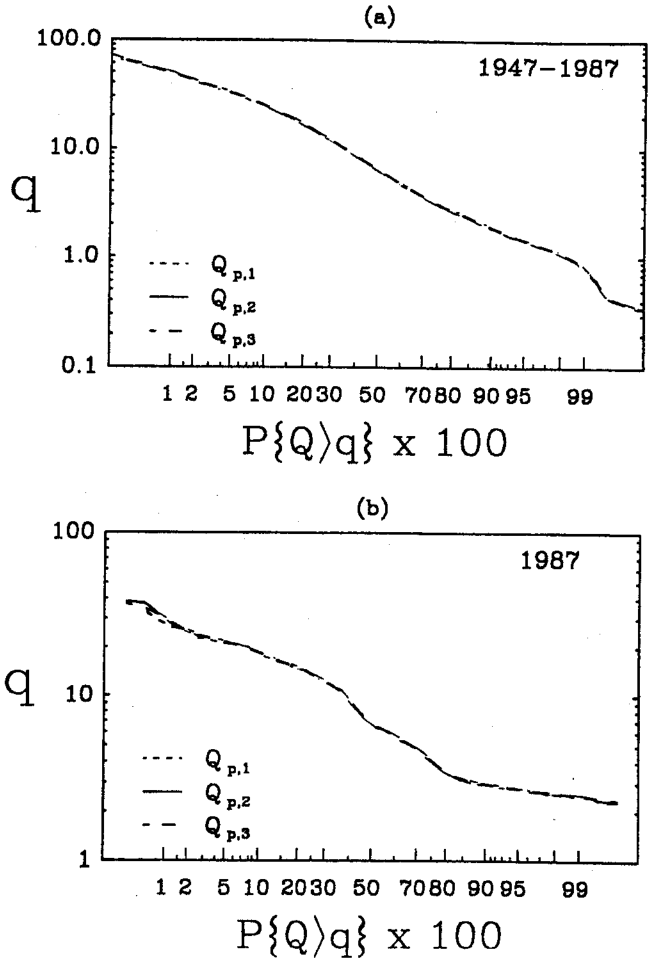

FIG. 4. Comparison of Period-of-Record FDC for Acheron River: (*a*) Based on Complete Record 1947–1987; and (*b*) Based on Single Year 1987

daily observations in Fig. 4(a), whereas $Q_{p,3}$ provides a slightly smoother estimate of the quantile function than either $Q_{p,2}$ or $Q_{p,1}$ for the shorter sample of only 365 correlated observations in Fig. 4(b), particularly in the upper tail of the distribution.

Interpretation of Traditional Period-of-Record Flow-Duration Curve

Historically, FDCs have been interpreted only on the basis of the period of record used in their development. For example, Searcy (1959, p. 21) cautions that, "To say that a flow-duration curve based on a 15-year record represents the distribution of the yearly flow is incorrect." A quantile Q_p, obtained from a period-of-record FDC, is defined as the discharge that was exceeded p percent of the time over the entire period of record on which it is based. This interpretation may be quite useful, as long as the period of record used to construct the FDC is long enough to provide the "limiting" distribution of streamflow or if the particular period of record corresponds to a particular design life or planning horizon.

In flood and low-flow frequency analysis, the interpretation of the frequency curve does not depend upon the period of record. For example, when computing the 100-year flood flow, interpretation of the quantile remains the same whether a 10-year or 50-year record was used to estimate the design-event quantile. Surely longer periods of record lead to less sampling error in quantile estimates, and hence more precise frequency curves. The interpretation of the design event, however, is unaffected by the length of record. This results from the fact that flood flow and low-flow frequency analysis procedures use an annual-event–based framework. The benefits of using an annual framework are that it allows one to define annual reliability and the average return-period concept, useful in hydrologic planning and design.

ANNUAL INTERPRETATION OF FLOW-DURATION CURVES

In this section we perform an experiment to document that other interpretations and methods for constructing FDCs are possible. In our first experiment, the dashed lines in Fig. 5 illustrate the period-of-record FDC for Moss Brook at Wendell Depot, in Massachusetts based on a 67-year period of record [Fig. 5(a)] and the Acheron River based on a 40-year period of record [Fig. 5(b)]. For example, in Fig. 5(a), the dashed (period-of-record) curve is constructed by plotting the estimator $Q_{p,2}$ based on approximately (ignoring leap years) (365 days/yr)$\cdot$(67 years) = 24,455 ordered daily streamflows $q_{(i)}$ versus exceedance probability p. In addition, we compute an FDC for each of the 67 individual years. For this purpose, we estimate each annual FDC using $Q_{p,2}$. Each of the 67 individual annual FDCs look rather different from the period-of-record FDC due to the expected year-to-year hydrologic variability. Some portions of the 67 individual annual FDCs fall above the period-of-record curve, and some below. To summarize this year-to-year variability in the annual FDCs, we compute two measures of central tendency of the annual FDCs, the mean and the median annual FDC. For each exceedance probability p, the mean and median values of discharge are computed using the 67 and 40 individual annual FDCs for Moss Brook and the Acheron River, respectively. Fig. 5 compares the mean annual and the median annual FDC with the period-of-record FDC for both Moss Brook and the Acheron River. The mean and median annual FDCs tend to approximate the period-of-record FDC,

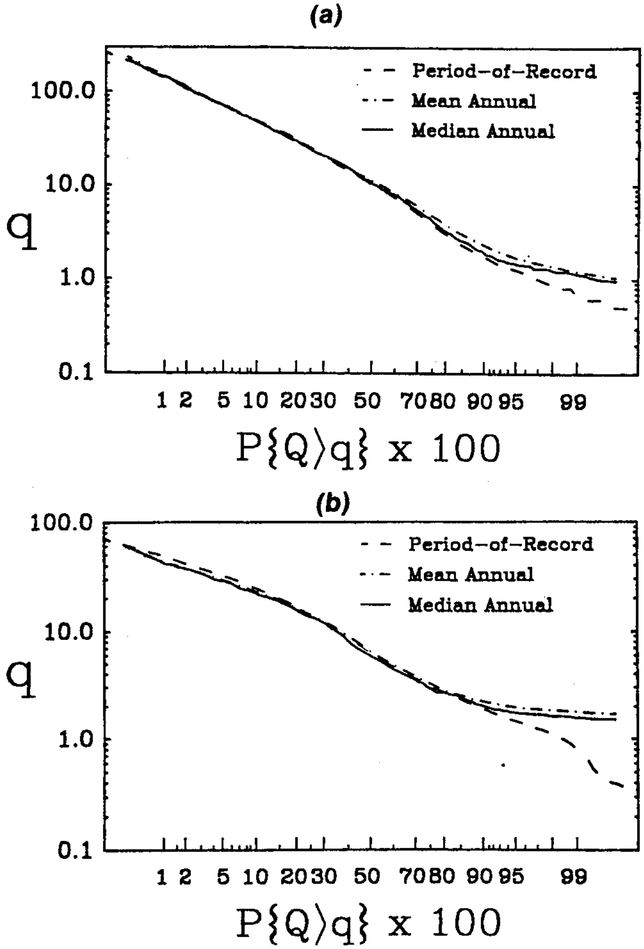

FIG. 5. Comparison of Period-of-Record FDC with Mean and Median Annual FDCs: (*a*) Moss Brook; and (*b*) Acheron River

except for exceedance probabilities above about 0.8 (low-flows), in which case the period-of-record FDC is always significantly lower than either the mean or median annual FDC. Similar results were found at other sites in Massachusetts. The significant differences between the period-of-record FDC and either the mean or median annual FDC occurs because the period-of-record FDC is highly sensitive to the hydrologic extremes associated with the particular period of record chosen, whereas the mean and median annual FDCs are not nearly as sensitive. This effect is explored in Fig. 6.

Fig. 6 compares the median annual FDC at Moss Brook with the period-of-record FDC using two different periods of record. Fig. 6 illustrates how sensitive the lower tail of an FDC can be to the chosen period of record. The period of record 1950–1981 contains the 1960s' drought that was more severe than any drought experienced over the 1917–1949 period, hence the FDC for these two periods are significantly different. Fig. 6 makes it obvious

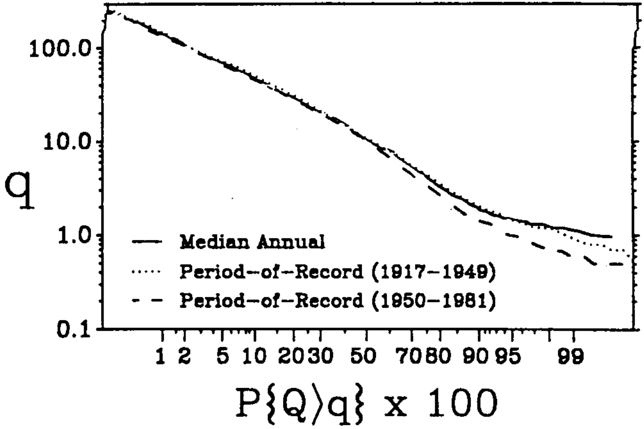

FIG. 6. Comparison of 1917–1981 Period-of-Record FDC for Moss Brook; Computed Over Periods 1917–1949 and 1950–1981 with Median Annual FDC

why hydrologists are often reluctant to use period-of-record FDCs: their interpretation depends so heavily upon the selected period of record. Annual FDCs provide a solution to this dilemma. The median annual FDC represents the distribution of daily streamflow in a "typical" or median hypothetical year and its interpretation is not affected by the observation of abnormally wet or dry periods during the period of record.

CONFIDENCE INTERVALS FOR QUANTILES OF FLOW-DURATION CURVES

When one focuses on a particular quantile or percentile of an FDC, the FDC does not, by itself, expose the uncertainty associated with a particular quantile estimate. For this purpose, confidence intervals should be constructed about the true quantile, to quantify the expected uncertainty associated with estimating each quantile. Another important advantage associated with the annual interpretation of FDCs is that nonparametric confidence intervals are easily constructed. To our knowledge, no procedures exist for computing confidence intervals for period-of-record FDCs. When one estimates an FDC at an ungaged site, regional regression procedures are sometimes used, in which case Fennessey and Vogel (1990) show how confidence intervals can be approximated. In this section, we describe simple nonparametric procedures for constructing confidence intervals about each quantile associated with the median annual FDC.

We define $Q_p(i)$ as an estimate of the pth quantile of streamflow based on the 365 daily streamflows in year i, using one of the estimators, $Q_{p,1}$, $Q_{p,2}$, or $Q_{p,3}$, described earlier. Given our results in Fig. 4(b), we recommend the use of $Q_{p,3}$ here; however, $Q_{p,2}$ suffices. The n years of streamflow data yield n estimates of $Q_p(i)$ for $i = 1, \ldots n$. The n values of $Q_p(i)$ are treated as a random sample from which one can estimate a $100(1 - \alpha)\%$ confidence interval about the true quantile Q_p. To obtain the upper and lower confidence interval for Q_p, one of the three estimators can be applied to the random sample $Q_p(i)$, $i = 1, \ldots n$. For example, using the estimator $Q_{p,2}$, one obtains the $100(1 - \alpha)\%$ confidence interval about Q_p as $[Q_p(L)$,

$Q_p(U)]$, where $Q_p(L)$ and $Q_p(U)$ denote the lower and upper limits of that interval computed from

$$Q_p(L) = (1 - \theta)Q_p(i) + \theta Q_p(i + 1); \quad \text{with } i = [(n + 1)\alpha/2] \text{ and } \theta = (n + 1)\alpha/2 - i \tag{10a}$$

$$Q_p(U) = (1 - \theta)Q_p(i) + \theta Q_p(i + 1); \quad \text{with } i = [(n + 1)(1 - \alpha/2)] \text{ and } \theta = (n + 1)(1 - \alpha/2) - i \tag{10b}$$

In Fig. 7(*a*) we use (10) to illustrate 90% confidence intervals for each quantile associated with the median annual FDC (solid line) for the Quabog River at West Brimfield, Massachusetts. Here, the solid line Q_p is estimated

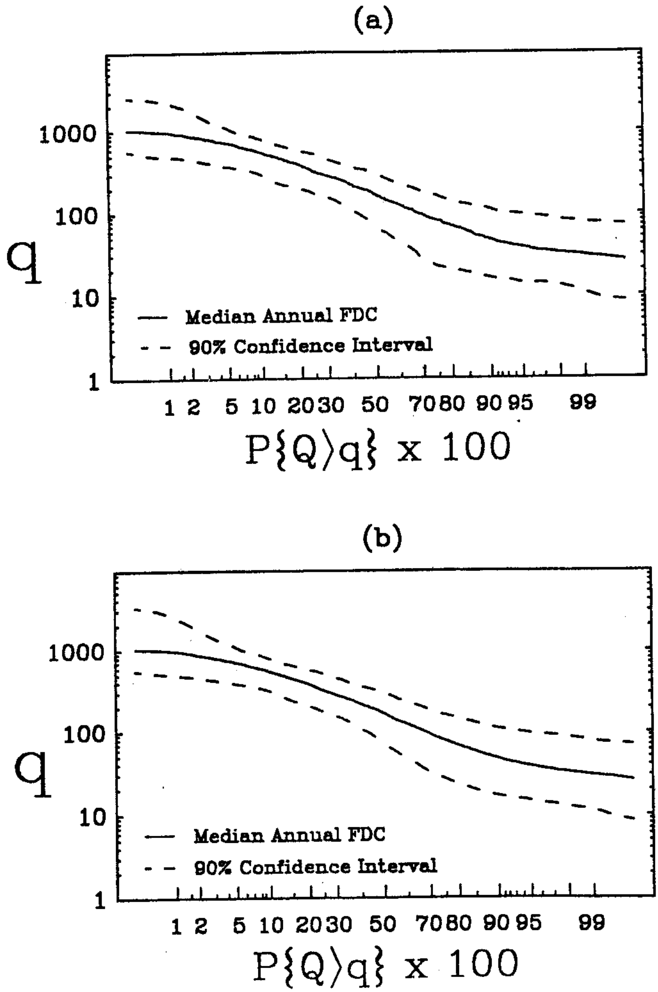

FIG. 7. 90% Confidence Intervals Associated with Quantiles of Median Annual FDC for Quabog River Using: (*a*) $Q_{p,2}$; and (*b*) $Q_{p,3}$

using the median of the sample of n values $Q_p(i)$. These confidence intervals were constructed by setting $\alpha = 0.10$ in (10) and repeating these computations 365 times to obtain values of $Q_p(U)$ and $Q_p(L)$ corresponding to $p = i/366$ for $i = 1, \ldots 365$. Smoother confidence intervals are obtained by using the estimator $Q_{p,3}$ instead of $Q_{p,2}$, which is assumed in (10). For example, Fig. 7(*b*) illustrates the 90% confidence intervals for each quantile of the median annual FDC at the Quabog River when $Q_{p,3}$ is used instead of $Q_{p,2}$.

The nonparametric confidence intervals illustrated in Fig. 7 and described by (10) have a precise interpretation that corresponds to each individual quantile Q_p associated with the median annual FDC. For a particular quantile Q_p, the confidence intervals $Q_p(L)$ and $Q_p(U)$ represent the random interval within which one would expect the true annual median quantile Q_p to fall $100(1 - \alpha)\%$ of the time.

GENERALIZED NONPARAMETRIC HYDROLOGIC FREQUENCY ANALYSIS

Given the annual interpretation of FDCs illustrated in Figs. 5–7, one may now think of each individual annual FDC in much the same way one thinks of each annual maximum flood flow or each annual minimum low flow drawn from a streamflow record of length n years. One may envision each annual FDC as a continuum bounded by two end points that are the annual maximum and annual minimum daily streamflow for that particular year. Then, similar to our assumption of independence among the n annual minimum and n annual maximum streamflows, one envisions n independent annual FDCs.

Another advantage of defining and estimating annual-based FDCs is that it allows us to define an annual probability of exceedance, which we term ε, (or probability of nonexceedance, which we term ν) associated with each annual-based FDC or quantile function Q_p. The definition of an annual probability of exceedance or nonexceedance associated with each annual FDC allows one to define an annual FDC with a specified average recurrence interval T, where $T = 1/\varepsilon$ for high flows, and $T = 1/\nu$ for low flows. For example, one could easily define and estimate the annual FDC with an exceedance probability of $\varepsilon = 5\%$, which would be the hypothetical annual FDC that is exceeded on average, once every $T = 1/0.05 = 20$ years, assuming independence among years. Unfortunately, one would never actually observe a T-year FDC. Nevertheless, if one wishes to understand the frequency of daily streamflow during an unusual but hypothetical year, a T-year FDC provides such information. This concept is analogous to the use of T-year design hydrographs and hyetographs used in flood studies. One could never observe a T-year hydrograph or a T-year hyetograph.

To formalize this concept, we first recall that $Q_p(i)$ is the estimate of the pth quantile of streamflow based on the 365 daily streamflows in year i, using one of the estimators $Q_{p,1}$, $Q_{p,2}$, or $Q_{p,3}$, described earlier. The n years of streamflow data yield n estimates of $Q_p(i)$ for $i = 1, \ldots n$. Now suppose we wish to estimate the annual FDC with a prespecified annual probability of exceedance ε, or probability of nonexceedance ν. The three nonparametric quantile estimators can be used again to define the εth and νth quantiles of $Q_p(i)$, which we term $Q_{p,\varepsilon}$ and $Q_{p,\nu}$, respectively. We recommend the use of $Q_{p,3}$ for this purpose, since there are only n values of $Q_p(i)$ for each value of p, and $Q_{p,3}$ leads to much smoother estimates than any of the alternative quantile function estimators.

The annual FDCs described by $Q_{p,\varepsilon}$ and $Q_{p,\nu}$ are identical when $\varepsilon = \nu = 0.5$ or, equivalently, $T = 1/\varepsilon = 1/\nu = 2$ years. In this case both FDCs correspond to the median annual FDC, which we term $Q_{p,0.5}$. In general $Q_{p,\varepsilon}$ lies above $Q_{p,0.5}$ and $Q_{p,\nu}$ lies below $Q_{p,0.5}$.

In Fig. 8 we apply the generalized annual FDCs described to the Quabog River. Using the 77-year record of available daily streamflows, we first estimated 77 individual annual quantile functions $Q_p(i)$, $i = 1, \ldots 77$, using the estimator $Q_{p,2}$. Next, in Figs. 8(*a*) and 8(*b*), the estimators $Q_{p,2}$ and $Q_{p,3}$ are used, respectively, to estimate the median annual quantile function $Q_{p,0.5}$ (depicted using solid lines) along with the generalized quantile functions $Q_{p,\varepsilon}$ for $T = 1/\varepsilon = 20$ and 50 years, and $Q_{p,\nu}$ for $T = 1/\nu = 20$ and 50 years (depicted using dashed and dotted lines). For example, the dashed and dotted lines above each solid line represent the annual FDC that is exceeded on average once every $T = 20$ and 50 years, respectively.

We illustrate the generalized annual FDCs in Fig. 8 using lognormal probability paper, although we make no distributional assumptions. The generalized FDCs depicted in Fig. 8(*b*) are slightly smoother than the cor-

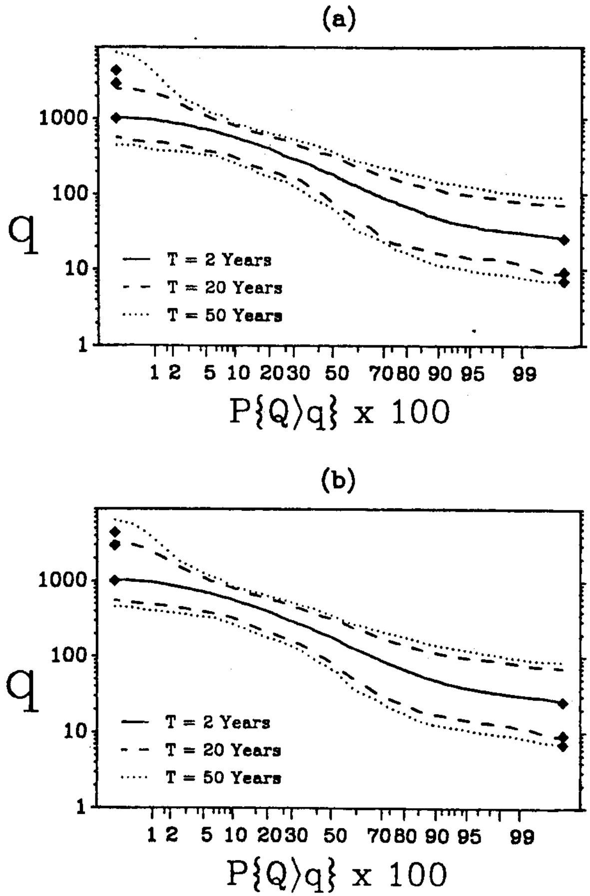

FIG. 8. Generalized Annual FDCs for Quabog River Using: (*a*) $Q_{p,2}$; and (*b*) $Q_{p,3}$

responding curves depicted in Fig. 8(*a*). This is because the estimator $Q_{p,3}$ provides a smoother estimate of the generalized quantile functions than the estimator $Q_{p,2}$. This effect is particularly apparent in the tails of the generalized quantile functions. In general, we recommend the use of the estimator $Q_{p,3}$ for constructing generalized quantile functions, as shown in Fig. 8(*b*).

Since there are only $n = 77$ annual values of $Q_p(i)$ available for estimating the annual FDCs $Q_{p,\varepsilon}$ and $Q_{p,\nu}$, the selection of an appropriate quantile estimator is extremely important. In most hydrologic applications for which these procedures are appropriate, n is in the range $10 \le n \le 100$. The nonparametric quantile estimators presented are limited to applications in which both ε and ν are greater than $1/n$, or, equivalently, T is less than n.

FLOW-DURATION CURVES IN FLOOD FLOW AND LOW-FLOW FREQUENCY ANALYSIS

The nonparametric quantile-estimation procedures provide a generalized alternative for estimating the magnitude and frequency of the complete continuum of daily streamflow ranging from the T-year annual minimum low flow to the T-year annual maximum flood flow. In this section, we describe how an annual FDC can be used to estimate extreme design events. Beard (1943) first suggested the use of FDCs in flood frequency analysis.

In a given year, an unbiased estimate of the expected probability of exceedance associated with the largest observed average daily streamflow $q_{(1)}$, is $p = 1/(n + 1) = 1/366 = 0.002732$. Similarly, an unbiased estimate of the expected probability of exceedance associated with the smallest observed average daily streamflow $q_{(n)}$ is $p = n/(n + 1) = 365/366 = 0.99726$. Hence, the distribution of annual maximum average daily flood flows is given by $Q_{p,\varepsilon}$ with $p = 0.002732$, and the distribution of annual minimum average daily low flows is given by $Q_{p,\nu}$ with $p = 0.99726$. In Fig. 8, we plot, using filled diamonds, the log Pearson type 3 estimators of the T = 2-year, 20-year, and 50-year annual maximum flood flows. Similarly, we plot using filled diamonds, the log Pearson type 3 (LP3) estimators of the T = 2-year, 20-year, and 50-year annual minimum low flows. We use the standard method-of-moment estimators in log space for estimating quantiles of the LP3 distribution as described in "Guidelines" (1982) and in most hydrology textbooks.

Fig. 9 uses lognormal probability paper to illustrate the goodness of fit of a LP3 distribution to the annual maximum flood flows and annual minimum low flows of the Quabog River. The LP3 distribution provides a good approximation to the distribution of annual minimum low flows and only a fair approximation to the distribution of annual maximum flood flows at this site. This is why there is good agreement in Fig. 8 between the nonparametric and parametric LP3 procedures for the annual minimum low flows, and poor agreement between the nonparametric and parametric LP3 procedures for the annual maximum flood flows. Since Vogel and Kroll (1989) showed that the LP3 distribution provides an excellent approximation to the distribution of annual minimum 7-day low flows at 23 sites in Massachusetts, the agreement between parametric and nonparametric procedures for the annual minimum low flows is not surprising. The smoother generalized nonparametric quantile functions based on the estimator $Q_{p,3}$ in Fig. 8(*b*) provide better agreement with the parametric LP3 procedures than the generalized quantile functions based on the estimator $Q_{p,2}$ in Fig. 8(*a*). This is to be expected since the estimator $Q_{p,3}$ uses all the observations

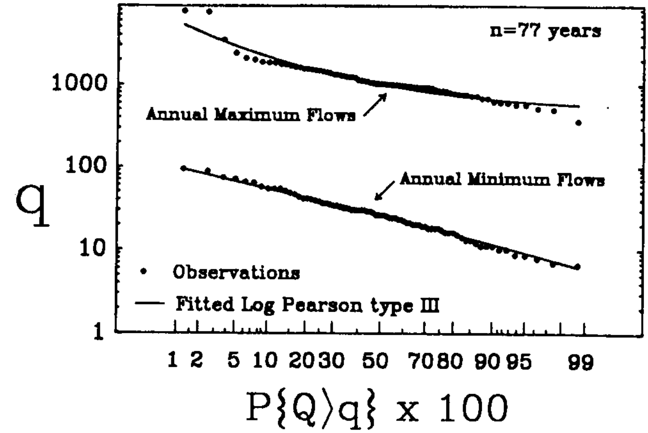

FIG. 9. LP3 Distribution Fit to Annual Maximum and Annual Minimum Streamflows at Quabog River

to obtain each value of $Q_{p,\varepsilon}$ and $Q_{p,\nu}$, unlike $Q_{p,2}$, which only uses two adjacent observations.

The comparison in Fig. 8 does not validate either procedure; however, it does document the correspondence between the classical parametric annual maximum flood-flow and annual minimum low-flow computations, and the generalized nonparametric procedures recommended. Future research is required to ascertain the efficiency of the nonparametric procedures relative to the parametric alternatives. Even if the nonparametric procedures are less efficient, which they are likely to be, the nonparametric procedures provide a wealth of information regarding the frequency and magnitude of streamflow, in excess of the traditional parametric low-flow and flood-flow procedures.

CONCLUSIONS

The present study introduces a variety of nonparametric quantile-estimation procedures useful for estimating and interpreting FDCs. The traditional period-of-record FDC can be interpreted as representing the magnitude and frequency of daily streamflow during the period of record or, in the limit, over some long period of time. Experiments indicate that the lower tail of such FDCs are highly sensitive to the particular period of record used. This fact led us to consider the alternative of representing and interpreting FDCs on an annual basis. We introduced the median annual FDC that represents the exceedance probability of daily streamflow in a median, or typical, but hypothetical year. The median annual FDC is not influenced by the occurrence of extreme low-flow periods or extreme floods over the period of record, yet it still captures the frequency and magnitude of daily streamflow in a typical year. The use of annual FDCs also allowed us to introduce simple nonparametric procedures for computing confidence intervals and for estimating a hypothetical T-year FDC.

Annual FDCs provide an alternative to the traditional approach of estimating a period-of-record FDC. A period-of-record FDC represents the exceedance probability of streamflow over a long period of time. This interpretation can be quite useful, as long as the period of record used to construct the FDC is long enough to provide the "limiting" distribution of streamflow or if the period of record corresponds to a particular planning period or design life. As an alternative, the median annual FDC represents the exceedance probability of streamflow in a typical year.

Engineers often wish to estimate quantiles of daily streamflow for use in hydrologic design and planning. Such studies typically define a design event using the concept of average recurrence intervals. For example, storm sewers may be designed for the 50-year peak annual flood flow and waste-load allocations may be based upon the 7-day 10-year low-flow event. The present study shows how FDCs can be constructed so as to provide a generalized description of hydrologic frequency analysis using average recurrence intervals. In Fig. 8, generalized FDCs were constructed to illustrate how one could estimate a hypothetical FDC with a specified annual probability of exceedance ε or nonexceedance ν (or corresponding average return period $T = 1/\varepsilon$ or $T = 1/\nu$). Such FDCs provide a description of the frequency and magnitude of the entire continuum of daily streamflows ranging from the T-year annual maximum flood flow to the T-year annual minimum low flow.

The annual FDC procedures introduced are appealing because they generalize both flood-flow and low-flow events in addition to all events in between. Yet FDCs are characterized by complex shapes requiring much more complex probability distributions than for the usual flood-flow and low-flow applications. The procedures developed are entirely nonparametric, unlike the parametric procedures usually recommended in flood-flow and low-flow frequency analysis. Nonparametric quantile-estimation procedures are chosen to simplify the analysis and to alleviate the need for "goodness-of-fit" procedures that typically have very low power [see Vogel and McMartin (1991)]. Nevertheless, studies should be undertaken to determine the efficiency of the proposed nonparametric quantile-estimation procedures relative to the usually recommended parametric procedures, such as the LP3 distribution, recommended by the Interagency Advisory Committee on Water Data ("Guidelines" 1982).

ACKNOWLEDGMENTS

The writers gratefully acknowledge the financial support of the U.S. Geological Survey who sponsored this research under the terms of contract number 14-08-0001-G2071 titled "The Application of Streamflow Duration Curves in Water Resources Engineering." In addition, the writers acknowledge the partial financial support of the U.S. Environmental Protection Agency Assistance Agreement CR820301 to the Tufts University Center for Environmental Management. The writers are also indebted to Wilbert O. Thomas, Gary Tasker, William Kirby and Thomas A. McMahon for their comments on an early version of this manuscript.

The views and conclusions contained in this document are those of the writers and should not be interpreted as necessarily representing the official policies, either expressed or implied of the U.S. Government.

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