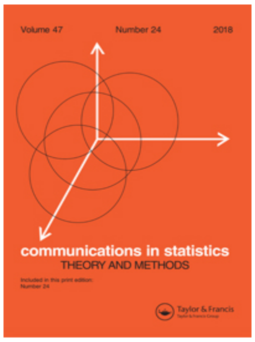


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# The geometric mean?

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**ABSTRACT**
The sample geometric mean (SGM) introduced by Cauchy in 1821, is a measure of central tendency with many applications in the natural and social sciences including environmental monitoring, scientometrics, nuclear medicine, infometrics, economics, finance, ecology, surface and groundwater hydrology, geoscience, geomechanics, machine learning, chemical engineering, poverty and human development, to name a few. Remarkably, it was not until 2013 that a theoretical definition of the population geometric mean (GM) was introduced. Analytic expressions for the GM are derived for many common probability distributions, including: lognormal, Gamma, exponential, uniform, Chi-square, F, Beta, Weibull, Power law, Pareto, generalized Pareto and Rayleigh. Many previous applications of SGM assumed lognormal data, though investigators were unaware that for that case, the GM is the median and SGM is a maximum likelihood estimator of the median. Unlike other measures of central tendency such as the mean, median, and mode, the GM lacks a clear physical interpretation and its estimator SGM exhibits considerable bias and mean square error, which depends significantly on sample size, pd, and skewness. A review of the literature reveals that there is little justification for use of the GM in many applications. Recommendations for future research and application of the GM are provided.

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Central tendency; arithmetic mean; median; log transformation; lognormal; multiplicative aggregation; effective

## 1. Introduction

For nearly two centuries we have known that the sample geometric mean,

$$SGM = \sqrt[n]{x_1 x_2 \ldots x_n} = \exp\left(\sum_{i=1}^{n} \frac{\ln(x_i)}{n}\right) \quad \text{for } x > 0 \tag{1}$$

is a measure of the central tendency of a positive random variable which is always less than its sample arithmetic mean (Cauchy 1821). In an elegant half page paper, Burk (1985) proves the following order of sample means: harmonic mean < geometric mean < arithmetic mean < root mean square (see appendix for definition of these sample means).

The SGM is now used in a very broad range of natural and social science disciplines such as: to express acceptable levels of fecal coliform counts and other contaminant

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levels in federal and state water quality criteria or standards (Landwehr 1978; Parkhurst 1998), for summarizing immunologic data (Olivier, Johnson, and Marshall 2008), for summarizing citation counts in scientometrics and infometrics (Thelwall 2016), for computing cumulative compounding rates in economics and finance (Spizman and Weinstein 2008), for maximization of investment portfolio returns (Elton and Gruber 1974), for correcting for tissue attenuation in gastric emptying studies in nuclear medicine (Ford, Kennedy, and Vogel 1992), for summarizing the suitability of ecological habitats (Hirzel and Arlettaz 2003), for summarizing ecological population growth rates (Yoshimura et al. 2009), for summarizing groundwater samples (Currens 1999), in machine learning algorithms in pattern classification and data visualization (Tao et al. 2009), for computing reaction rates in chemical engineering (Garland and Bayes 1990), for summarizing samples in pharmacokinetics (Martinez and Bartholomew 2017), for summarizing return periods or recurrence intervals (Gumbel 1961), for summarizing mammalian allometry data (Smith 1993), in seismic reliability analyses (Abyani, Asgarian, and Zarrin 2019) and for computing the Human Development Index (Human Development Report 2010). In some fields the application of the SGM is pervasive, such as for characterizing the effective permeability of heterogeneous porous media in a broad range of geoscience and geomechanics applications (Parkin and Robinson 1993; Jensen 1991; Selvadurai and Selvadurai 2014), including applications to groundwater modeling, nuclear waste characterization, earthquake hazards, geothermal energy extraction and disposal of carbon dioxide as a means of mitigating the impacts of climate change (see Selvadurai and Selvadurai 2014, for citations). In spite of numerous cogent arguments against the use of SGM in environmental monitoring (Thomas 1955; Parkhurst 1998), the SGM also has widespread use for summarizing environmental concentrations and in the implementation of water quality standards in the U.S. (U.S.E.P.A 2010). The above list is neither exhaustive, nor comprehensive, and only gives a glimpse of the broad range of applications of the *SGM* in practice.

The nearly ubiquitous application of the *SGM* in natural and social science disciplines is remarkable given that it was only recently that a theoretical definition of the geometric mean (*GM*) was introduced in this journal (Feng, Wang, and Tu 2013)

$$GM = \exp\left[E[\ln(X)]\right] = \exp\left[\int_0^\infty \ln(X) f(X) dX\right] \quad \text{for } X > 0 \quad \text{only} \tag{2}$$

where $f(X)$ denotes the probability distribution (pd) of $X$. Feng et al. (2017) introduce a more general definition of *GM* which includes the possibility that observations might equal zero, in which case $GM = 0$. Others have introduced equivalent expressions for GM in (2) with little discussion (for example, see Equation (1) in Jensen 1991; and equation (6) in Limbrunner, Vogel, and Brown 2000). Jensen (1998) introduces GM as a special case of the power mean $\mu_p = [E(X^p)]^{1/p}$ when $p = 0$, in which case the power mean reduces to GM in (2).

One might wonder why it took centuries for a theoretical definition of *GM* to appear; perhaps it is because mathematicians are reluctant to introduce a theoretical statistic which does not exist under certain conditions, as is the case in (2). Naturally, all other commonly used measures of central tendency such as the median, mode and arithmetic mean can be computed without constraints on the variable of interest and have had well developed theoretical definitions for a very long time.

It is very difficult to find other examples of a sample estimator of a statistic which has no associated theoretical definition. Imagine if we did not know that the sample median is an estimate of the value of the variable with equal exceedance and nonexceedance probabilities. Imagine if we did not know that the arithmetic mean of $X$ is an estimate of *E[X]*. Without such knowledge casinos, insurance companies, and lotteries could not earn reliable profits.

Without a theoretical definition of a statistic, it is not possible to define the bias or root mean square error (RMSE) associated with a particular sample estimator of that statistic. Until the definition of GM was advanced in (2), it was not possible to determine whether or not *SGM* provides a good approximation of the *GM*, or not. Without a theoretical definition for *GM* it was only possible to derive the expectation and variance of the *SGM*, as was done by Landwehr (1978) and others. Without the theoretical definition of *GM* in (2), it was not possible to define the bias or RMSE associated with *SGM*, for comparison with other estimators of *GM*. It is ONLY through such studies of the sampling properties of an estimator, that one can conclude which estimator is best under certain conditions.

Given the widespread usage of *SGM* combined with the lack of information concerning the theoretical properties of *GM* and the sampling properties of *SGM*, the goals of this paper are (1) to provide comparisons of the theoretical properties of *GM* with other common measures of central tendency including the arithmetic mean and the median for a very wide range of commonly used probability distributions (pds) (2) to summarize the sampling properties of *SGM* for a range of pdfs, and (3) to discuss various concerns relevant to the use of *GM* in applications.

## 2. Derivation of the geometric mean, GM, for a wide class of probability distributions

In this section, the theoretical definition of *GM* in (2) is used to derive relationships between *GM* and the parameters of various common pds. The random variable $X$ is assumed to be positive. Theorem 5 in Feng et al. (2017) proves that the expression in (2) is equivalent to

$$GM = \lim_{n\to\infty} \left[ \left( \int_0^\infty x^{1/n} f(x) dx \right)^n \right] \tag{3}$$

which is often much easier to evaluate than (2) and thus was used to derive most of the formulas for GM reported in Table 1 and illustrated in Figure 1. The GM can also be expressed in terms of the quantile function of $x$, denoted $x(p)$ so that,

$$GM = \exp\left( \int_0^1 \ln(x(p)) dp \right) \tag{4}$$

where $x(p) = F^{-1}[p]$ and p denotes nonexceedance probability given by the cumulative distribution function $p = F(x)$. Equation (4) was used to derive GM for the generalized Pareto distribution (Hosking and Wallis 1987). Arnold (2008) describes five different

**Table 1.** Mean, median, geometric mean, and skewness derived for numerous common distributions.

| Name of PD | Probability Density Function, f(x) | Mean | Median | Geometric Mean *GM* | Coefficient of Skew $\gamma$ |
|---|---|---|---|---|---|
| Gamma | $\frac{1}{a\Gamma(b)}\left(\frac{x}{a}\right)^{b-1}\exp\left(\frac{-x}{a}\right)$ <br> $0 \le x < \infty \quad a,b>0$ | $ab$ | $*$ | $a\exp\left(\psi(b)\right)^{**}$ | $\frac{2}{\sqrt{b}}$ |
| Lognormal | $\frac{1}{bx\sqrt{2\pi}}\exp\left(\frac{-1}{2}\left(\frac{\ln(x)-a}{b}\right)^2\right)$ <br> $0 \le x < \infty \quad b>0$ | $\exp\left(a+\frac{b^2}{2}\right)$ | $\exp(a)$ | $\exp(a)$ | $(w+2)\sqrt{w-1}$ <br> $w = \exp(b^2)$ |
| Weibull | $\frac{b}{a}\left(\frac{x}{a}\right)^{b-1}\exp\left(-\left(\frac{x}{a}\right)^b\right)$ <br> $0 \le x < \infty \quad a,b>0$ | $a\Gamma\left(1+\frac{1}{b}\right)$ | $a(\ln(2))^{1/b}$ | $a\exp\left(\frac{-\gamma}{b}\right)^{***}$ | $\frac{w_3-3w_2w_1+2w_1^3}{(w_2-w_1^2)^{3/2}}$ <br> $w_i=\Gamma\left(1+\frac{i}{b}\right)$ |
| Pareto | $\frac{ba^b}{x^{b+1}}$ <br> $a \le x < \infty \quad a,b>0$ | $\frac{ab}{b-1}$ | $a2^{1/b}$ | $a\exp\left(\frac{1}{b}\right)$ | $\frac{2(1+b)}{b-3}\sqrt{\frac{b-2}{b}}$ <br> $b>3$ |
| Generalized Pareto | $\frac{1}{a}\left[1-b\left(\frac{x}{a}\right)\right]^{\frac{1}{b}-1} \quad for \quad b\neq 0$ | $\frac{a}{1+b}$ <br> $b>-1$ | $\frac{a}{b}\left[1-0.5^b\right]$ | $\frac{a}{b}\exp\left(\psi(1)-\psi\left(1+\frac{1}{b}\right)\right) \quad b>0$ <br> $a\exp\left(\psi(1)\right) \quad b=0$ <br> $\frac{a}{-b}\exp\left(\psi(1)-\psi\left(1-\frac{1}{b}\right)-b\right) \quad b<0$ | $\frac{2(1-b)\sqrt{1+2b}}{1+3b}$ <br> $b>-\frac{1}{3}$ |
| Power Function | $\frac{bx^{b-1}}{a^b}$ <br> $0 \le x < a \quad a,b>0$ | $\frac{ab}{b+1}$ | $\frac{a}{2^{1/b}}$ | $a\exp\left(\frac{-1}{b}\right)$ | $\frac{2(1-b)\sqrt{2+b}}{(3+b)\sqrt{b}}$ |

| | | | | | |
|---|---|---|---|---|---|
| Rayleigh | $\frac{x}{b^2}\exp\left(\frac{-x^2}{2b^2}\right)$ <br> $0 \le x < \infty \quad b > 0$ | $b\sqrt{\frac{\pi}{2}}$ | $b\sqrt{\ln(4)}$ | $b\sqrt{2}\exp\left(\frac{-\gamma}{2}\right)^{***}$ | $\frac{2(\pi-3)\sqrt{\pi}}{(4-\pi)^{3/2}} \approx 0.63$ |
| Exponential | $\frac{1}{b}\exp\left(-\frac{x}{b}\right)$ <br> $0 \le x < \infty \quad b > 0$ | $b$ | $b\ln(2)$ | $b\exp(-\gamma)^{***}$ | $2$ |
| Chi-Square | $\frac{x^{\frac{b-2}{2}}\exp\left(-\frac{x}{2}\right)}{2^{\frac{b}{2}}\Gamma\left(\frac{b}{2}\right)}$ <br> $0 \le x < \infty \quad b > 0$ | $b$ | * | $2\exp\left(\psi\left(\frac{b}{2}\right)\right)$ | $2^{3}/\frac{2}{\sqrt{b}}$ |
| F | $\frac{\Gamma\left(\frac{a+b}{2}\right)\left(\frac{a}{b}\right)^{a/2}x^{(a-2)/2}}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\left[1+\frac{a}{b}x\right]^{(a+b)}/_2}$ <br> $0 \le x < \infty \quad a,b > 0$ | $\frac{b}{b-2}$ <br> $b > 2$ | * | $\left(\frac{b}{a}\right)\exp\left(\psi\left(\frac{a}{2}\right)-\psi\left(\frac{b}{2}\right)\right)^{**}$ | $\frac{(2a+b-2)\sqrt{8(b-4)}}{(b-6)\sqrt{a(a+b-2)}}$ <br> $b > 6$ |
| Beta | $\frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)^{****}}\ 0 \le x < 1$ <br> $a,b > 0$ | $\frac{a}{a+b}$ | * | $\exp\left(\psi(a)-\psi(a+b)\right)$ | $\frac{2(b-a)\sqrt{a+b+1}}{(a+b+2)\sqrt{ab}}$ |
| Uniform | $\frac{1}{b-a} \quad a \le x < b$ | $\frac{a+b}{2}$ | $\frac{a+b}{2}$ | $\frac{1}{e}\left(\frac{a}{b}\right)^{\left(a/_{(a-b)}\right)} \quad a > 0$ | $0$ |

*No closed form expression exists, numerical integration employed in Figures
**$\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function
***$\gamma = -\psi(1) = 0.5772$ is the Euler number
****$\beta(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$

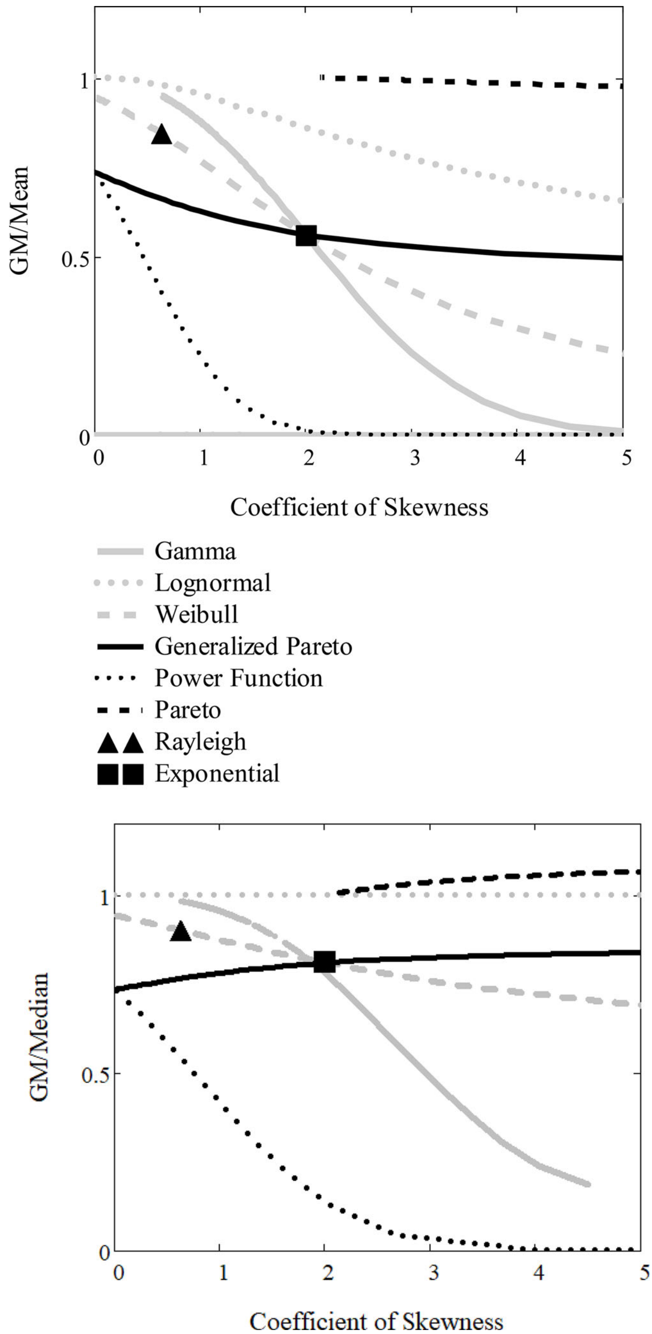


**Figure 1.** The ratios GM/Median and GM/Mean, for the Gamma, lognormal, Weibull, generalized Pareto, Power Function, Pareto, Rayleigh, and Exponential probability distributions.

Pareto distributions. Here we consider four of those distributions including the three Pareto models which form the basis of the generalized Pareto distribution (Pickands 1975) as well as the classical Pareto model.

Table 1 reports the probability density function *f(x)*, and its associated mean, median, GM and coefficient of skewness for the pds: lognormal, Gamma, exponential, uniform, Chi-square, F, Beta, Weibull, Power law, Pareto, generalized Pareto, and Rayleigh. Figure 1 illustrates the ratios *GM*/Mean and *GM*/Median for the lognormal, gamma, exponential, Weibull, power law, Pareto, generalized Pareto and Rayleigh distributions.

Figure 1 illustrates that among all the pds considered, GM is always less than the arithmetic mean and with the exception of the Pareto and lognormal pds, GM is always less than the median. The relationship between GM, mean and median is highly dependent upon both the pd and its coefficient of variation or skewness. Importantly, among all the distributions considered, it is only for the lognormal case that GM is equivalent to the median, yet for data from other positively skewed distributions, with the exception of the Pareto pd, GM is generally not very close to the median.

To my knowledge, Table 1 and Figure 1 are the first examples which compare the theoretical properties of GM for a variety of pds. Such comparisons are necessary for a complete understanding of the behavior of one measure of central tendency versus another and are commonplace for other measures of central tendency and for a wide class of other statistics such for moment ratios and L-moment ratios (Vogel and Fennessey 1993). If GM is to find further use in scientific investigations and applications, further developments and comparisons analogous to Figure 1 are needed.

## 3. Geometric mean applications and the lognormal distribution

Many previous applications of the *SGM* assume the variable of interest follows a lognormal distribution. For example, a lognormal pd was assumed in their analysis of the properties of *SGM* for: environmental concentration data (Parkhurst 1998; also see Ott 1990), immunologic data (Olivier, Johnson, and Marshall 2008), citation data (Thelwall 2016), mammalian allometry data (Smith 1993), investment and portfolio return data (Elton and Gruber 1974), ecological population growth data (Yoshimura et al. 2009), and for permeability data in geoscience and geomechanics applications (Selvadurai and Selvadurai 2014). For example, within the context of the widespread use of *SGM* for characterizing the effective permeability of heterogeneous subsurface materials, Jensen (1991) concluded on the basis of numerous previous studies that "Their results indicate that the geometric mean applies when permeabilities are log-normally distributed". Similarly, within the context of developing water quality regulatory guidelines for the EPA, Stephen et al. (1985) suggest that "Geometric means, rather than arithmetic means, are used here because the distributions of sensitivities of individual organisms in toxicity tests on most materials and the distributions of sensitivities of species within a genus are more likely to be lognormal than normal. Similarly, geometric means are used for acute-chronic ratios and bioconcentration factors because quotients are likely to be closer to lognormal than normal distributions."

Surely the choice of a suitable summary statistic of central tendency of a random variable should consider first, the application, interpretation and meaning of that statistic instead of its probability distribution. For example, if one's wish were to identify the most typical value of a random variable, one might choose a median or mode, regardless of the pd of the observations. Similarly, a casino, insurance company or lottery should focus on estimation of the expectation of their revenues if they are hoping to realize a profit, over the long term, regardless of the pd of those revenues, thus they may wish to compute an arithmetic mean because it would be an estimate of the long term expectation of those revenues, and other measures of central tendency would not be appropriate for that application, regardless of the pd of the revenues.

## 4. Sampling properties of sample geometric mean, SGM

Galton (1879) and McAlister (1879) were the first to investigate the sampling distribution of the *SGM*, which later led to both a large sample theory (Aitchison and Brown 1957) and sampling distributions of SGM for small samples from particular parent distributions (Camp 1938). Feng, Wang, and Tu (2013) document that although SGM is generally biased regardless of its pd, that bias disappears as n increases. Some investigators have developed general relationships between the moments of various parent distributions and the moments of the SGM including Norris (1940), Landwehr (1978), and Wilson and Martin (2006). For example, Landwehr (1978) derived the expectation and variance of *SGM* for lognormal, Gamma, Weibull and uniform samples.

Landwehr (1978) and others have shown that the expectation of *SGM* is a function of sample size and is very sensitive to the skewness and the shape of the probability distribution (pd) from which the samples arise. It is difficult to find other estimators of central tendency which exhibit bias that depends on both the pd and sample size. For example, common estimators of central tendency such as the arithmetic mean and median, have expectations which typically do not depend on sample size. Bias associated with *SGM* can be particularly relevant when considering its use in evaluation of compliance with environmental regulations as is now commonplace (Stephen et al. 1985). One environmental monitoring location may be considered compliant, whereas another is not, simply due to differences in sample size, rather than differences in environmental pollution.

Norris (1940) derived the variance of the sample geometric mean of a random variable x

$$\sigma^2_{SGM} = GM \cdot \left[\frac{\sigma^2_{\ln(x)}}{n}\right] \tag{5}$$

where $\sigma^2_{SGM}$ and $\sigma^2_{SGM}$ are the variance of SGM and ln(x), respectively.

### 4.1. Lognormal sampling

In consideration of the sampling properties of *SGM*, a unique and important case involves lognormal data, because as described below, most previous applications of *SGM* have assumed the variable of interest is well approximated by a lognormal pd. For the lognormal case, *GM* is equivalent to the median (see Table 1), in which case the *SGM* is a maximum likelihood estimator (MLE) of both *GM* and the median. Gilbert (1987) provides analytic expressions for comparing the sampling properties of *SGM* under lognormal sampling. For lognormal samples, Aitchison and Brown (1957) and Landwehr (1978) derived the mean and variance of *SGM* as

$$E[SGM] = \exp\left(a + \frac{b^2}{2n}\right) = GM \cdot \exp\left(\frac{b^2}{2n}\right) \tag{6a}$$

$$Var[SGM] = \exp(2a)\left[\exp\left(\frac{2b^2}{n}\right) - \exp\left(\frac{b^2}{n}\right)\right] \tag{6b}$$

where a and b are the mean and standard deviation of the logarithms of x (see Table 1). Stedinger (1983, Equation (4)) derives exact confidence intervals for quantiles of a lognormal variable, based on the noncentral t-distribution, which can be used to construct an exact confidence interval for both *GM* and the median, because *GM* and the median are both the 50th percentile of a lognormal variate.

In contrast with estimators of the mean of a lognormal variate, relatively little attention has been given to a comparison of the sampling properties (bias and MSE) of alternative estimators of *GM*. Shih and Binkowitz (1987) and Parkin and Robinson (1993) compared the sampling properties (bias and mean square error) of *SGM* with the widely used nonparametric rank based estimator of the median, M. As expected, for lognormal samples, the *SGM* was preferred to the sample median M, for nearly all lognormal samples considered, however importantly, in a robustness evaluation considering contaminated lognormal samples, Shih and Binkowitz (1987) found the nonparametric rank-based estimator of the median M, to have much lower *MSE* than *SGM*. The bias associated with *SGM* described in (6a), is always positive and can be quite considerable for small samples which led Finney (1941) and Bradu and Mundlak (1970) to derive minimum variance unbiased estimators of *SGM* (see summary in section 13.3.2 in Gilbert 1987). Zellner (1971) used a Bayesian approach to derive a minimum MSE (biased) estimator of *GM* and showed that it is much more efficient than Finney's (1941) minimum variance unbiased estimator of *GM*, especially for large values of the coefficient of variation of the observations. Given the wealth of parametric, nonparametric, biased and unbiased estimators available for estimation of the median and *GM* for the lognormal case, a need exists for more rigorous Monte-Carlo experiments which compare and contrast their sampling properties, with special attention given to their robustness. Such experiments may also consider improvements in estimation of GM under lognormal sampling resulting from fitting a three parameter lognormal pd using the attractive lower bound estimator given in equation (20) of Stedinger (1980) which has found widespread use within the field of hydrology.

## 5. Summary and recommendations

In spite of the widespread application of the *SGM* estimator across many disciplines partially summarized here, it is only very recently that a mathematical definition of the *GM* has been suggested (Feng, Wang, and Tu 2013; Feng et al. 2017). Using that definition, theoretical expressions for GM were derived for a wide range of common pds, including the lognormal, Gamma, exponential, uniform, Chi-square, F, Beta, Weibull, Power law, Pareto, generalized Pareto and Rayleigh distributions. Table 1 and Figure 1 summarize the behavior of the *GM* relative to the median and mean for those pds. Those relationships indicate that the value of the *GM* is extremely sensitive to both the pd and its skewness. A summary of the sampling properties of *SGM* also reveals that, unlike most other estimators of central tendency, *SGM* exhibits considerable bias for small samples which depends upon the underlying pd as well as sample size, a fact which should be considered carefully before its widespread application as a measure of central tendency.

Many previous applications of the *GM* reviewed here, have assumed the variable of interest is well approximated by the lognormal distribution, in which case, we now know that the *GM* is simply the median of that variable and thus *SGM* is an MLE of both *GM* and the median, for that case. Unlike the *GM*, the median is a statistic with a very clear interpretation, with half the observations falling above and below the median. Thus, in situations in which the variable of interest follows a lognormal distribution, it may make more sense to refer to the interpretation of the median, when computing either the *GM* or the median, than to refer to the term geometric mean, because unlike the median, the *GM* lacks a clear and concise physical interpretation. In the case of lognormal observations, there is also the question of whether or not to employ an unbiased estimator of *GM* (and median) such as one of those introduced by Finney (1941) and Bradu and Mundlak (1970) or perhaps the minimum MSE Bayesian estimator of the median and GM of a lognormal variable introduced by Zellner (1971). Shih and Binkowitz (1987) provide evidence that for small departures from lognormal, the nonparametric rank based median estimator has considerably lower MSE than *SGM*. A rigorous Monte-Carlo study is needed to compare the sampling properties and robustness of the various estimators of *GM* and median for the lognormal case. In the case of lognormal observations, given the findings of Shih and Binkowitz (1987), it remains an open question as to which estimator of *GM* and median is most efficient and robust. Under lognormal sampling, investigators are encouraged to report both *SGM* and the nonparametric rank based median, along with their associated confidence intervals given by Stedinger (1983) and Helsel and Hirsch (2002, section 3.3).

There are many other situations, apart from lognormal sampling, in which the *GM* may be a useful and sensible choice as a summary measure. For example, Fleming and Wallace (1986) show that the *GM* is useful and appropriate for summarizing normalized results, and they show why the arithmetic mean, when used in this context, can lead to grossly incorrect conclusions. The guidance of Fleming and Wallace (1986) is valuable because it extends to the summary of a wide range of normalized results including, but not limited to: relative errors, relative efficiencies, effectiveness, averaging ratios, rate constants, rates of return, ratio indices, normalized counts, normalized indicators, elasticities, and relative scores. Parkhurst (1998) suggests that "Geometric means may be useful for representing the average of a series of values that are always multiplied. For example, the average efficiency for a sequence of five processes involved in transforming the energy stored in underground oil to electrical energy in the home can be calculated as the one-fifth root of the product of the five process efficiencies. If that value were multiplied by itself five times, it would yield the overall efficiency." Mahajan (2019) describes numerous practical examples in which the SGM provides useful insights.

Table 1 and Figure 1 are the first example of a comparison of the theoretical properties of GM for a variety of pds. Such comparisons are necessary for a complete understanding of the behavior of one measure of central tendency versus another and are commonplace for other measures of central tendency and for a wide class of other statistics including moment and Lmoment ratios (Vogel and Fennessey 1993). If *GM* and *SGM* are to find further use in scientific investigations, further comparisons analogous to Table 1 and Figure 1 are needed, along with the development and evaluation of alternative parametric and nonparametric estimators, confidence intervals, and hypothesis tests, all of which do not presently exist for the GM, except to some extent, for the lognormal case.

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## Appendix

Burk (1985) proves the following order of sample means:

harmonic mean < geometric mean < arithmetic mean < root mean square

For a sample of size n, those sample means are defined, in order:

$$\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} < \sqrt{\prod_{i=1}^{n} x_i} < \frac{1}{n}\sum_{i=1}^{n} x_i < \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$