There seems to be a little bit of confusion about this, so I thought a brief note might help. Let Ψ_t be the time-t flow of some vector field v on a manifold M. Let α be a differential form on M.

In symplectic geometry, we are often interested in the expression

$$\frac{d}{dt}(\Psi_t^* \alpha).$$

One nice way to evaluate this is via "Cartan's magic formula". We first define the *Lie Derivative*

$$\mathcal{L}_v \alpha = \frac{d}{dt}(\Psi_t^* \alpha)|_{t=0}. \tag{1}$$

We show that this satisfies the relation

$$\mathcal{L}_v \alpha = d\iota_v \alpha + \iota_v d\alpha, \tag{2}$$

and we show that

$$\frac{d}{dt}(\Psi_t^* \alpha)|_{t=t_0} = (\Psi_{t_0})^* \mathcal{L}_v \alpha. \tag{3}$$

This works very well when v is independent of time, but when v_t is dependent on time there is a minor subtlety that needs to be addressed. There are several reasonable ways to address it, but one should be aware of the subtlety.

The issue. First, let us say what does not work. The *word-for-word* repeat of the above story, with v_t replacing v, does not quite work. The issue is, if we let Ψ_t be the time-t-flow of the time-varying vector field v_t, then the expression $\frac{d}{dt}(\Psi_t^* \alpha)|_{t=0}$ is not really what we want. Indeed, it has no real way to access v_{t_0} for general t_0 at all, and so we definitely should *not* denote it $\mathcal{L}_{v_t}$ and assert that it satisfies the Cartan equation (2), with v_t substituted for v.

Along these lines, strictly speaking the exercise in Canas da Silva (see homework assignment) is not quite correct, in the case where v depends on time. Actually, the Cartan magic formula is only stated for vector fields that do not depend on time there; however, with the definition of the Lie Derivative there, the equation labeled (*) there is neither correct, nor what we would anyways want to use in our arguments, because it would lead to the wrong vector field being present in the Cartan magic formula.

Fixes. What we really want is for the formula

$$\frac{d}{dt}(\Psi_t^* \alpha) = (\Psi_t)^* [d\iota_{v_t} \alpha + \iota_{v_t} d\alpha], \tag{4}$$

where Ψ_t denotes the time-t flow of v_t, to hold. This is a correct formula, but we'd like to see why it is true.

There are now several ways to think about this.

We could still interpret the bracketed expression in the right hand side of the above equation as a Lie derivative, but we should think of it as the Lie derivative for the *time-independent vector field* v_t. That is, let Ψ'_s denote the time s-flow of v_t, regarded now as a time-independent vector field, and define $\mathcal{L}_{v_t}\alpha$ by (1), with Ψ' instead of Ψ and s instead of t. Then, everything will work just as we want, and we can derive (4) from this point of view. This is essentially the approach taken in McDuff-Salamon, see for example the first line in the proof of Proposition 3.1.5. One just has to be aware that at each t, $\mathcal{L}_{v_t}$ is computed with respect to a different isotopy, namely the one, denoted Ψ' above, which is the flow of the vector field v_t, regarded as a time independent vector field.

This is a perfectly fine approach, in a way probably the best (it can be proved by essentially the same argument as in the time-independent case). From an expository point of view, one just has to be careful when reading it to keep track of what is going on: there is a lot that is depending on t, and we introduced the parameter s to help with this.

Alternatively, if one wanted to stick with a fixed vector field on a manifold, rather than a time-varying one, so as to be able to quote the Canas da Silva exercise directly and not worry about any subtleties, one could regard v_t as a vector field on the manifold $\mathbb{R} \times M$, consider the flow of $X := \partial_t + v_t$, and consider α as a form on $\mathbb{R} \times M$ that annihilates the $\mathbb{R}$ direction. In this way, one can apply the equations from the time-independent case directly and derive (4).

Both ways are completely fine. I just wanted to clarify the relevant mathematics, which are slightly subtle.

Time-varying differential forms. In applications (e.g. Moser's trick), one sometimes also wants to understand expressions of the form

$$\frac{d}{dt}(\Psi_t^* \alpha_t),$$

in other words where the forms α themselves depend on time. In this case, there is another term in the analogue of (4), coming from the variation in α itself. That is, we have

$$\frac{d}{dt}(\Psi_t^* \alpha) = (\Psi_t)^*[\frac{d}{dt}\alpha_t + d\iota_{v_t}\alpha_t + \iota_{v_t} d\alpha_t]. \tag{5}$$

This is stated, for example, in the equation between (3.2.3) and (3.2.4) in the McDuff-Salamon book.

To derive this, one can apply the multivariate chain rule in combination with (4). Alternatively, in the spirit of what was written above, one could regard α_t as a single differential form on $\mathbb{R} \times M$ (annihilating the $\mathbb{R}$ direction), and restrict $\frac{d}{dt}(\Psi_t^* \alpha_t)$ to the M direction.

Derivation of (5)**.** Here is one derivation of the desired equation. Let us now assume that we have verified the Cartan formulas for autonomous (i.e. time-independent) vector fields, and derive (5).

As suggested above, we consider the flow ψ_t of the vector field $X = \partial_t + v_t$, which we regard as an autonomous vector field on $\mathbb{R} \times M$, and we regard α_t as a form on $\mathbb{R} \times M$, which we denote by β. Then, we have $\psi_t(t_0, x) = (t_0 + t, \psi'_{t+t_0}(x))$, where ψ'_t is the flow along the time-varying vector field v_t. In particular, the map $M \to M$ sending $x \to \pi(\psi_t(0, x))$, where π denotes the canonical projection, is exactly the flow ψ'_t.

On the other hand, by the Cartan formula, we have

$$\mathcal{L}_X \beta = d\iota_X \beta + \iota_X d\beta = d\iota_{v_t}\alpha_t + \iota_X(\frac{d}{dt}\alpha_t)dt + \iota_{v_t} d\alpha_t.$$

If we restrict to tangent vectors along M, then $\iota_X(\frac{d}{dt}\alpha_t)dt = \frac{d}{dt}\alpha_t$, hence (5) in view of the expression for ψ' in terms of ψ above.