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Supersymmetry with scattering states

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Abstract – Supersymmetric partner potentials calculated using single-particle non-reflecting scattering states are localized complex potentials with real eigenvalues. If the initial potential is a symmetric real potential then the partner potential is PT-symmetric. In particular cases, complex conduction band edge profiles are found that yield supersymmetric partner total potentials evaluated using scattering states in the context of self-consistent Schrödinger-Poisson equations.

Introduction. – Hermitian supersymmetric quantum mechanics provides an efficient method for finding pairs of potentials with real-valued bound-state solutions to the single-particle Schrodinger equation that have the same energy eigenvalues and hence are isospectral [1–5]. While requiring Hamiltonians to be Hermitian is sufficient to ensure real eigenvalues, this constraint may be unnecessary [6], since non-Hermitian Hamiltonians, including Hamiltonians with unbroken parity-time symmetry (PT symmetry), can also have real eigenvalues [7,8]. The application of supersymmetric quantum mechanics to optical systems is an active research topic [9–12], and non-Hermitian quantum mechanics and PT symmetry have also been applied to classical optics systems [13].

Prior research on the application of supersymmetric quantum mechanics to non-Hermitian Hamiltonians has provided methods for finding complex-valued partner potentials of real-valued [8] or complex-valued [14] potentials. While this includes work on the use of scattering states with complex eigenvalues (Gamov states) to find complex-valued partner potentials [15,16], to the best of our knowledge, prior work has not addressed the use of real-valued scattering states to find complex-valued partner potentials.

In the present work, it is shown that non-reflecting scattering states can be used to find localized complex-valued partner potentials. In the case where the initial potential is symmetric, it is shown that the corresponding partner potential is PT-symmetric. Furthermore, it is shown that semi-infinite periodic partner potentials can be generated using scattering states with a non-zero reflected component. For numerical simulations, these partner potentials can be regularized using a slowly varying envelope function. The use of scattering states to evaluate partner total potentials is also examined in the context of self-consistent Schrödinger-Poisson equations.

Supersymmetric quantum mechanics. – Consider a particle of mass m moving in a real one-dimensional potential $V(x)$, and let $\psi(x)$ be an eigenstate of the Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ with eigenenergy E. The wave function $\psi(x)$ may be complex, as in the case of scattering states. Let

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi'}{\psi}, \tag{1}$$

with $\psi' = \frac{\mathrm{d}}{\mathrm{d}x}\psi$, be the superpotential, so that

$$\hat{A}^+ = \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{dx}} + W(x), \tag{2}$$

$$\hat{A}^- = -\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x), \tag{3}$$

where $\hat{A}^- = [\hat{A}^+]^\dagger$ if $W(x)$ is real, and

$$\hat{H}^{(-)} = \hat{A}^-\hat{A}^+ = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2(x) - \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W}{\mathrm{d}x}, \tag{4}$$

$$\hat{H}^{(+)} = \hat{A}^+\hat{A}^- = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2(x) + \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W}{\mathrm{d}x} \tag{5}$$

are supersymmetric partner Hamiltonians, with potentials

$$V^{(-)} = W^2(x) - \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W}{\mathrm{d}x}, \tag{6}$$

$$V^{(+)} = W^2(x) + \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W}{\mathrm{d}x}. \tag{7}$$

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The Hamiltonians $\hat{H}^{(-)}$ and $\hat{H}^{(+)}$ have the same energy eigenvalues and so are isospectral, with the possible exception that there may be no eigenstate of $\hat{H}^{(+)}$ with the same energy as the ground state of $\hat{H}^{(-)}$. Given an eigenfunction $\psi^{(-)}$ of $\hat{H}^{(-)}$ with eigenenergy $E^{(-)}$,

$$\psi^{(+)} = \frac{1}{\sqrt{E^{(-)}}}\hat{A}^{+}\psi^{(-)} \tag{8}$$

is an eigenstate of $\hat{H}^{(+)}$ with eigenenergy $E^{(+)} = E^{(-)}$.

The potentials $V^{(-)}$ and $V^{(+)}$ also exhibit similar scattering characteristics. In particular, if $V^{(-)}(x \to \pm\infty)$ and $V^{(+)}(x \to \pm\infty)$ are finite constants, so that $W(x \to \pm\infty)$ is also finite, then scattering eigenstates with real eigenenergies exist. If the transmission coefficients of $V^{(-)}$ and $V^{(+)}$ are $t^{(-)}$ and $t^{(+)}$, and the reflection coefficients are $r^{(-)}$ and $r^{(+)}$, respectively, then [5,17]

$$t^{(-)}(k) = \frac{W(+\infty) - i\frac{\hbar k'}{\sqrt{2m}}}{W(-\infty) - i\frac{\hbar k}{\sqrt{2m}}}t^{(+)}(k), \tag{9}$$

$$r^{(-)}(k) = \frac{W(-\infty) + i\frac{\hbar k}{\sqrt{2m}}}{W(-\infty) - i\frac{\hbar k}{\sqrt{2m}}}r^{(+)}(k), \tag{10}$$

where k is the wave number of the scattering state for $x \to -\infty$ while k' is the wave number of the scattering state for $x \to +\infty$.

Supersymmetric partners calculated from scattering states. – Typically, the eigenstate $\psi(x)$ used to calculate partner potentials is the lowest-energy *bound state*. Higher-energy bound states are avoided due to the fact that they have zeros that create singularities in the partner potentials. However, other types of states can be used to calculate partner potentials, including *scattering states*, which have no zeros. Gamov states (scattering states with complex eigenenergies) have previously been used to find non-Hermitian partner Hamiltonians of Hermitian Hamiltonians [15,16]. Scattering states with real eigenenergies, however, have generally been avoided in the literature, perhaps because they are not normalizable. In the following, it is shown that such scattering states can nevertheless be used to find complex-valued partner potentials of real-valued initial potentials.

Partner potentials calculated from non-reflecting scattering states. If a scattering state $\psi_k(x)$ with wave number k is used in eq. (1) to evaluate the superpotential W, then both W and the partner potential $V^{(+)}$ will be complex-valued, because the wave function ψ_k is complex-valued. Furthermore, because W is complex in this case, the operators $\hat{A}^{-}$ and $\hat{A}^{+}$ are not one another's adjoint operators (*i.e.*, $\hat{A}^{-} \neq [\hat{A}^{+}]^{\dagger}$). However, as the operators satisfy the required commutation relations, eqs. (1)–(7) can be used with complex superpotentials W calculated from scattering states to yield complex-valued potentials $V^{(+)}$ that are partner potentials of real-valued potentials $V^{(-)}$.

If a scattering state with a non-zero reflected component is used to calculate a superpotential as in eq. (1), then the evaluated superpotential will have a periodic part extending infinitely in the direction of reflection, so that $W(-\infty)$ is undefined. This is because the reflected component of a scattering state creates a semi-infinite standing wave. Nevertheless, with appropriate regularization, such a potential exhibits properties similar to its partner potential.

On the other hand, if the superpotential is evaluated using a non-reflecting scattering state, then the potential will be localized, and regularization is not needed. Typically, non-reflecting scattering states are resonant scattering states with unity transmission. However, certain potentials, such as the sech^2 potential [5,18], are reflectionless, so that all scattering states in such potentials are non-reflecting scattering states.

If a non-reflecting scattering state ψ_{k_r} with wave number k_r at $x \to -\infty$ and k_r' at $x \to +\infty$ is used to calculate the partner potentials, then

$$W(-\infty) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_{k_r}'(-\infty)}{\psi_{k_r}(-\infty)} = -i\frac{\hbar k_r}{\sqrt{2m}}, \tag{11}$$

$$W(+\infty) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_{k_r}'(+\infty)}{\psi_{k_r}(+\infty)} = -i\frac{\hbar k_r'}{\sqrt{2m}}, \tag{12}$$

provided that $V^{(-)}(x \to \pm\infty)$ is finite, constant, and less than the eigenenergy of the state E_{k_r}. It follows that $V^{(+)}(x \to \pm\infty)$ is also finite and constant. In this case, eqs. (9) and (10) are applicable without further adjustment.

In particular, if $V^{(-)}(x \to -\infty) = V^{(-)}(x \to +\infty)$, so that $W(-\infty) = W(+\infty)$ and $k_r = k_r'$, then it follows from eq. (9) that

$$t^{(-)}(k) = t^{(+)}(k). \tag{13}$$

Substituting $W(-\infty) = -i\frac{\hbar k_r}{\sqrt{2m}}$ into eq. (10) gives

$$r^{(+)}(k) = \frac{k + k_r}{k - k_r}r^{(-)}(k), \tag{14}$$

so that $r^{(-)}(k) \neq r^{(+)}(k)$. Note that $r^{(+)}$ appears to have an additional pole at $k = k_r$. However, since k_r is the wave number of a non-reflecting scattering state of $V^{(-)}$, then $r^{(-)}(k_r) = 0$. This zero cancels the pole, so that $r^{(+)}(k_r)$ is finite. Also, $r^{(+)}$ can have a value much greater than unity near $k = k_r$ due to the fact that $H^{(+)}$ is a non-Hermitian Hamiltonian with $V^{(+)}$ complex. While the state $\psi_{k_r}^{(-)}$ is a non-reflecting scattering state, $\psi_{k_r}^{(+)}$ has a finite reflected component.

This lack of a corresponding non-reflecting scattering state can be understood as being analogous to the lack of a bound state $\psi_0^{(+)}$ when a bound state $\psi_0^{(-)}$ is used to calculate partner potentials. In that case, $\psi_0^{(+)} \propto \hat{A}^{+}\psi_0^{(-)} = 0$. Similarly, if a non-reflecting scattering state $\psi_{k_r}^{(-)}$ is used to evaluate partner potentials, then $\hat{A}^{+}\psi_{k_r}^{(-)} = 0$. However, partner potentials evaluated using scattering states

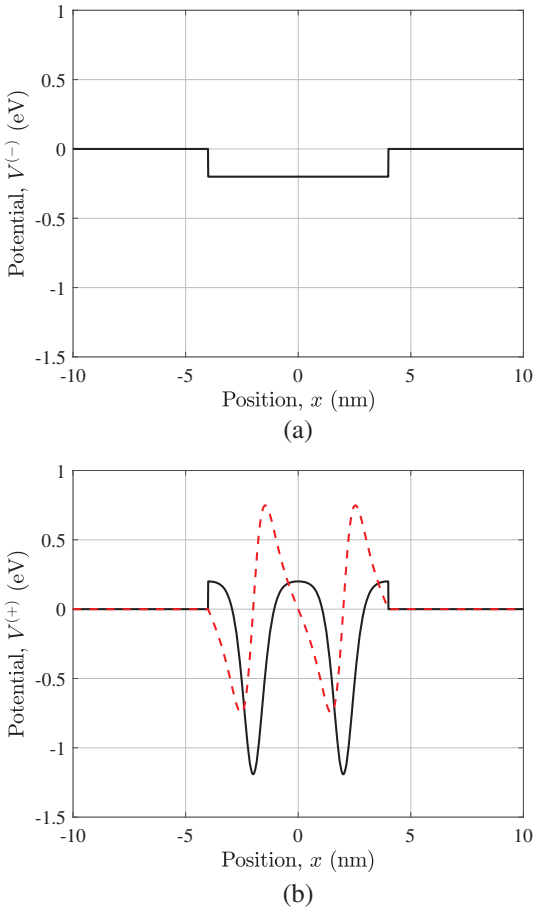

Fig. 1: (a) Real-valued rectangular potential well $V^{(-)}(x)$ and (b) its complex partner potential $V^{(+)}(x)$. The partner potential $V^{(+)}$ is calculated using the lowest-energy *resonant* scattering eigenstate of $\hat{H}^{(-)}$. The solid black curves indicate the real part of the potential, and the dashed red curves indicate the imaginary part of the potential. The rectangular potential well has thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$, and the particle has effective electron mass $m_e^* = 0.07 \times m_0$ where m_0 is the bare electron mass.

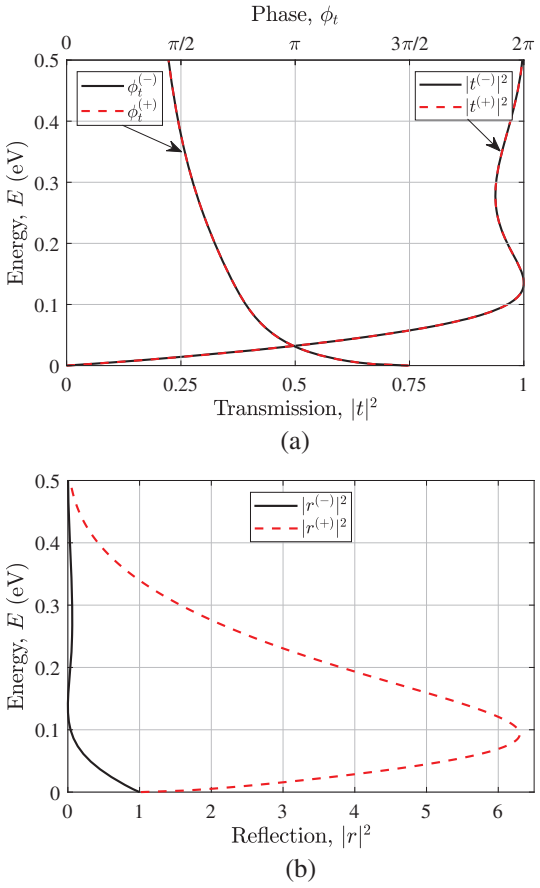

Fig. 2: (a) The magnitude squared $|t|^2$ and phase ϕ_t of the transmission amplitude t of a rectangular potential well $V^{(-)}$ (solid black curves) and its partner potential $V^{(+)}$ (dashed red curves), where $V^{(+)}$ is calculated using the lowest-energy *resonant* scattering eigenstate of $\hat{H}^{(-)}$. (b) The magnitude squared $|r|^2$ of the reflection amplitude r of a rectangular potential well $V^{(-)}$ (solid black curve) and its partner potential $V^{(+)}$ (dashed red curve), where $V^{(+)}$ is calculated using the lowest-energy resonant scattering eigenstate of $\hat{H}^{(-)}$. The rectangular potential well has thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$, and the particle has effective electron mass $m_e^* = 0.07 \times m_0$.

still admit a scattering state at $E = E_{k_r}$, though it is not a non-reflecting state.

If $V^{(-)}$ and $V^{(+)}$ are partner potentials calculated using a scattering state of $\hat{H}^{(-)}$, then, as in the case of bound states, if $\psi^{(-)}$ is an eigenstate of $\hat{H}^{(-)}$ then $\psi^{(+)} = \hat{A}^+\psi^{(-)}$ is an eigenstate of $\hat{H}^{(+)}$. This applies regardless of whether $\psi^{(-)}$ is a scattering state or a bound state. Unlike in the case of partner potentials calculated using bound states, there is no bound state of $\hat{H}^{(-)}$ for which there is no corresponding bound state of $\hat{H}^{(+)}$ if a scattering state is used to evaluate the partner potentials. However, due to the non-hermiticity of $\hat{H}^{(+)}$, its bound states may be non-orthogonal.

Figure 1 shows a real-valued rectangular potential well $V^{(-)}$ of thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$, and its complex partner potential $V^{(+)}$ calculated using the lowest-energy resonant scattering state $\psi^{(-)}_{k_{r_1}}$ of $\hat{H}^{(-)}$, with energy $E_{k_{r_1}} \approx 0.135\,\text{eV}$. Note the PT symmetry with symmetric real part and anti-symmetric imaginary part of $V^{(+)}$.

Figure 2(a) shows the magnitude squared of the transmission amplitudes $|t^{(-)}|^2$ (solid black line) and $|t^{(+)}|^2$ (dashed red line), as well as the phases of the transmission amplitudes $\phi_t^{(-)}$ (solid black line) and $\phi_t^{(+)}$ (dashed red line). Both the magnitudes and phases of the transmission amplitudes are plotted as a function of the energy $E(k)$. Note that the $t^{(-)}$ and $t^{(+)}$ are equal in both magnitude and phase, as predicted by eq. (9) when $W(-\infty) = W(+\infty)$. In contrast, fig. 2(b) shows the magnitude squared of the reflection amplitudes $|r^{(-)}|^2$ and $|r^{(+)}|^2$ as a function of the energy $E(k)$. The magnitudes of the reflection amplitudes are not equal, as

indicated by eq. (14). Furthermore, while $|r^{(-)}(k_{r_1})|^2 = 0$, $|r^{(+)}(k_{r_1})|^2 \approx 5.68 > 1$, indicating gain due to the non-Hermiticity of the Hamiltonian $H^{(+)}$.

As expected in the case of partner potentials calculated from scattering states, both bound states of $\hat{H}^{(-)}$ have corresponding bound states of $\hat{H}^{(+)}$ with the same eigenenergies. However, the bound states of $\hat{H}^{(+)}$ can be similar because they are non-orthogonal due to the non-Hermiticity of $\hat{H}^{(+)}$.

Symmetric potentials have PT-symmetric partner potentials. While the potential $V^{(-)}$ shown in fig. 1 is symmetric, the corresponding partner potential $V^{(+)}$ is PT-symmetric. In general, the partner potential of a symmetric potential is PT-symmetric if it is calculated from a non-reflecting scattering state.

Consider a real-valued potential $V(x)$ that is symmetric, so that $V(x) = V(-x)$. Suppose furthermore that $V(x \to \pm\infty)$ is finite, so that scattering states exist. For an electron incident upon such a potential barrier, there are energies E_{k_r} at which the magnitude of the reflection coefficient is zero ($|r| = 0$). If $\psi_{k_r}(x)$ is the wave function of a non-reflecting scattering state with energy E_{k_r} in the symmetric potential $V(x)$, then it can be written as

$$\psi_{k_r}(x) = e^{i\phi}\psi^{(0)}_{k_r}(x), \tag{15}$$

where $\psi^{(0)}_{k_r}(x)$ is a PT-symmetric function.

The wave function ψ_{k_r} can be substituted into eqs. (1), (6), and (7) to evaluate the supersymmetric partner potentials

$$V^{(-)} = V(x) - E_{k_r}, \tag{16}$$

$$V^{(+)} = -V^{(-)} + \frac{\hbar^2}{m}\left[\frac{\psi'_{k_r}}{\psi_{k_r}}\right]^2. \tag{17}$$

Since $\psi^{(0)}_{k_r}(x)$ is PT-symmetric, its first derivative $\psi^{(0)'}_{k_r}(x)$ is anti-PT-symmetric, so that

$$\frac{\psi'_{k_r}}{\psi_{k_r}} = \frac{\psi^{(0)'}_{k_r}}{\psi^{(0)}_{k_r}} \tag{18}$$

is also anti-PT-symmetric. It follows that $[\frac{\psi'_{k_r}}{\psi_{k_r}}]^2$ is PT-symmetric, and

$$V^{(+)} = -V^{(-)} + \frac{\hbar^2}{m}\left[\frac{\psi'_{k_r}}{\psi_{k_r}}\right]^2 \tag{19}$$

is PT-symmetric, since $V^{(-)}$ is a symmetric real potential. Thus, partner potentials of symmetric potentials are PT-symmetric if they are evaluated using non-reflecting scattering states.

Partner potentials calculated from states with nonzero reflection. Due to standing waves created by the reflected component of states with non-zero reflection, partner potentials calculated using such states contain a periodic part extending infinitely in the direction of reflection. In addition to this potential being undefined at infinity, the equations in eqs. (9) and (10) are not immediately applicable to such a potential due to $W(\pm\infty)$ being undefined. For example, fig. 1(a) shows a real-valued rectangular potential well $V^{(-)}$ of thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$ and fig. 3 shows its complex oscillatory partner potential $V^{(+)}$ calculated using the *off-resonance* eigenstate $\psi_{k_{m_1}}$ with eigenenergy $E_{k_{m_1}} \approx 0.280\,\text{eV}$. This energy corresponds to the lowest-energy transmission local minimum.

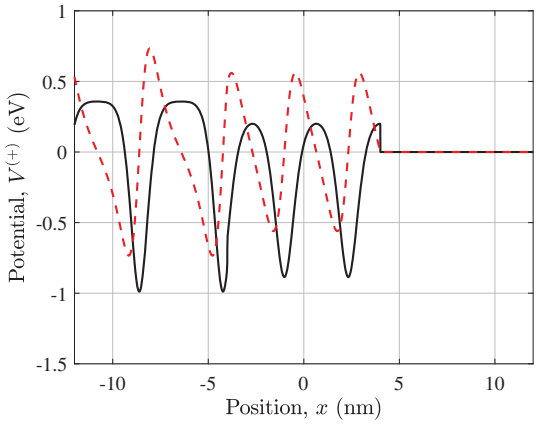

Fig. 3: A partner potential $V^{(+)}(x)$ of the rectangular potential well $V^{(-)}(x)$ shown in fig. 1(a). The partner potential $V^{(+)}(x)$ is calculated using an *off-resonance* scattering eigenstate of $V^{(-)}(x)$ with energy eigenvalue $E_{k_{m_1}} \approx 0.280\,\text{eV}$. The solid black curves indicate the real part of the potential, and the dashed red curves indicate the imaginary part of the potential. The rectangular potential well has thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$, and the particle has effective electron mass $m^*_e = 0.07 \times m_0$.

The scattering characteristics of such a potential may be determined by applying a real envelope function to $V^{(+)}$ that varies slowly with respect to the period of the potential, is unity at the location of the potential barrier $V^{(-)}$, and goes to zero as $x \to -\infty$ or $+\infty$, depending on which side the periodic part of the potential appears. Rather than determine the envelope function by formal optimization methods, a raised cosine of the form

$$f(x) = \begin{cases} \dfrac{1+\cos(2\pi x/d)}{2}, & x \in [-d/2, d/2]\,, \\ 0, & x \notin [-d/2, d/2] \end{cases} \tag{20}$$

was used with $d = 195\,\text{nm}$ to regularize the potential shown in fig. 3. This value was found by trial and error, taking into account the fact that larger values of d yield lower error, but are less computationally efficient. Balancing these two competing considerations is an optimization problem.

Figure 4(a) shows the corresponding magnitude squared of the transmission amplitude $|t|^2$ and the phase of the transmission amplitude ϕ_t for both $V^{(-)}$ (solid black line) and $V^{(+)}$ (dashed red line). The transmission amplitudes

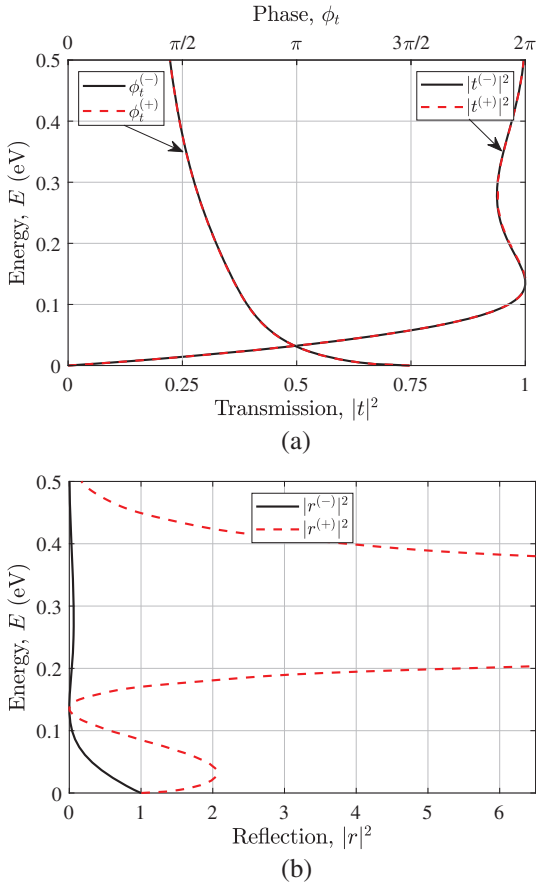

Fig. 4: (a) The magnitude squared $|t|^2$ and phase ϕ_t of the transmission amplitude t of a rectangular potential well $V^{(-)}$ (solid black curve) and its partner potential $V^{(+)}$ (dashed red curve), where $V^{(-)}(x)$ is calculated using an off-resonance scattering eigenstate of $V^{(-)}(x)$. (b) The magnitude squared $|r|^2$ of the reflection amplitude r of a rectangular potential well $V^{(-)}$ (solid black curve) and its partner potential $V^{(+)}$ (dashed red curve), where $V^{(+)}(x)$ is calculated using an off-resonance scattering eigenstate of $V^{(-)}(x)$. The rectangular potential well has thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$, and the particle has effective electron mass $m_e^* = 0.07 \times m_0$.

are equal in both magnitude and phase. Similar to the non-reflecting case discussed previously, the reflection amplitudes of states scattering from $V^{(+)}$ and $V^{(-)}$ are related by the equation

$$r^{(+)} = \frac{k + k_0}{k - k_0} r^{(-)}, \tag{21}$$

where k_0 is the wave number of the state used to calculate $V^{(+)}$ and $V^{(-)}$. Consequently, $|r^{(+)}|^2$ can exceed unity, as seen in fig. 4(b). However, unlike the non-reflecting case, the pole at $k = k_0$ is not cancelled by a zero in $r^{(-)}$, so $|r^{(+)}|^2 \to \infty$ as $k \to k_0$. If $V^{(+)}$ is regularized using an envelope function, then $|r^{(+)}|^2$ will be finite at $k = k_0$, but will approach infinity as the width of the envelope function approaches infinity.

Self-consistent Schrödinger-Poisson equations with partner total potentials evaluated from scattering states. – It has previously been shown that bound-state solutions in doped semiconductors can be used to find conduction band edge profiles $V_{\text{be}}^{(+)}$ and $V_{\text{be}}^{(-)}$ that yield supersymmetric partner total potentials $V_{\text{tot}}^{(+)}$ and $V_{\text{tot}}^{(-)}$ when solved self-consistently using the Schrödinger and Poisson equations [19]. This is done by finding the total potential $V_{\text{tot}}^{(-)}$ corresponding to a given conduction band edge profile $V_{\text{be}}^{(-)}$, and using the ground state wave function $\psi_0^{(-)}$ to evaluate its partner total potential $V_{\text{tot}}^{(+)}$ by the usual methods of supersymmetric quantum mechanics. The bound states $\psi_i^{(+)}$ are then used to evaluate the electrostatic potential energy $V_{\text{c}}^{(+)}$ generated by the charges, from which the conduction band edge profile $V_{\text{be}}^{(+)} = V_{\text{tot}}^{(+)} - V_{\text{c}}^{(+)}$ is evaluated.

If instead the partner total potential $V_{\text{tot}}^{(+)}$ is evaluated using a scattering state ψ_k, then $V_{\text{tot}}^{(+)}$ will be complex-valued. In general, complex potentials do not necessarily have real eigenvalues, but since $V_{\text{tot}}^{(+)}$ is the partner of the real-valued potential $V_{\text{tot}}^{(-)}$, its eigenvalues are real. However, it is not presently known under what circumstances eigenvalues can be guaranteed to be real when iterating to find $V_{\text{tot}}^{(+)}$ from $V_{\text{be}}^{(+)}$.

Figure 5(a) shows a rectangular potential well conduction band edge profile $V_{\text{be}}^{(-)}$ of thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$, while fig. 5(b) shows the corresponding total potential $V_{\text{tot}}^{(-)}$. The ionized donor density is $N_{\text{D}} = 1.5 \times 10^{18}\,\text{cm}^{-3}$ for $-10\,\text{nm} < x < 10\,\text{nm}$, and zero elsewhere. The effective electron mass is $m_e^* = 0.07 \times m_0$, the permittivity is $\epsilon_{\text{r0}} = 13.2$, and the temperature is $T = 0\,\text{K}$. For these parameters, only the two bound states are occupied and contribute to the charge distribution in the system (the scattering states are unoccupied). Nevertheless, the lowest-energy resonant scattering state $\psi_{k_{r_1}}^{(-)}$ of $\hat{H}^{(-)}$ with energy $E_{k_{r_1}} \approx 0.191\,\text{eV}$ was used to evaluate $V_{\text{tot}}^{(+)}$ (fig. 5(d)), a supersymmetric partner total potential of $V_{\text{tot}}^{(-)}$. Figure 5(c) shows the conduction band edge profile $V_{\text{be}}^{(+)}$ that, when substituted into the Schrödinger-Poisson equations, yields the total potential $V_{\text{tot}}^{(+)}$. Remarkably, in this case, even though $V_{\text{be}}^{(+)}$ is complex-valued, the eigenvalues evaluated while iterating to find $V_{\text{tot}}^{(+)}$ are real, ensuring that the total potential is static and converges. The partner potentials $V_{\text{tot}}^{(-)}$ and $V_{\text{tot}}^{(+)}$ result in isospectral systems with equal transmission spectra (eq. (13)) and with reflection spectra related by eq. (14).

Applications. – Supersymmetry with scattering states in the context of self-consistent Schrödinger-Poisson equations is relevant to condensed matter systems. The application of Schrödinger-Poisson equations to systems exhibiting supersymmetry and PT-symmetry builds on

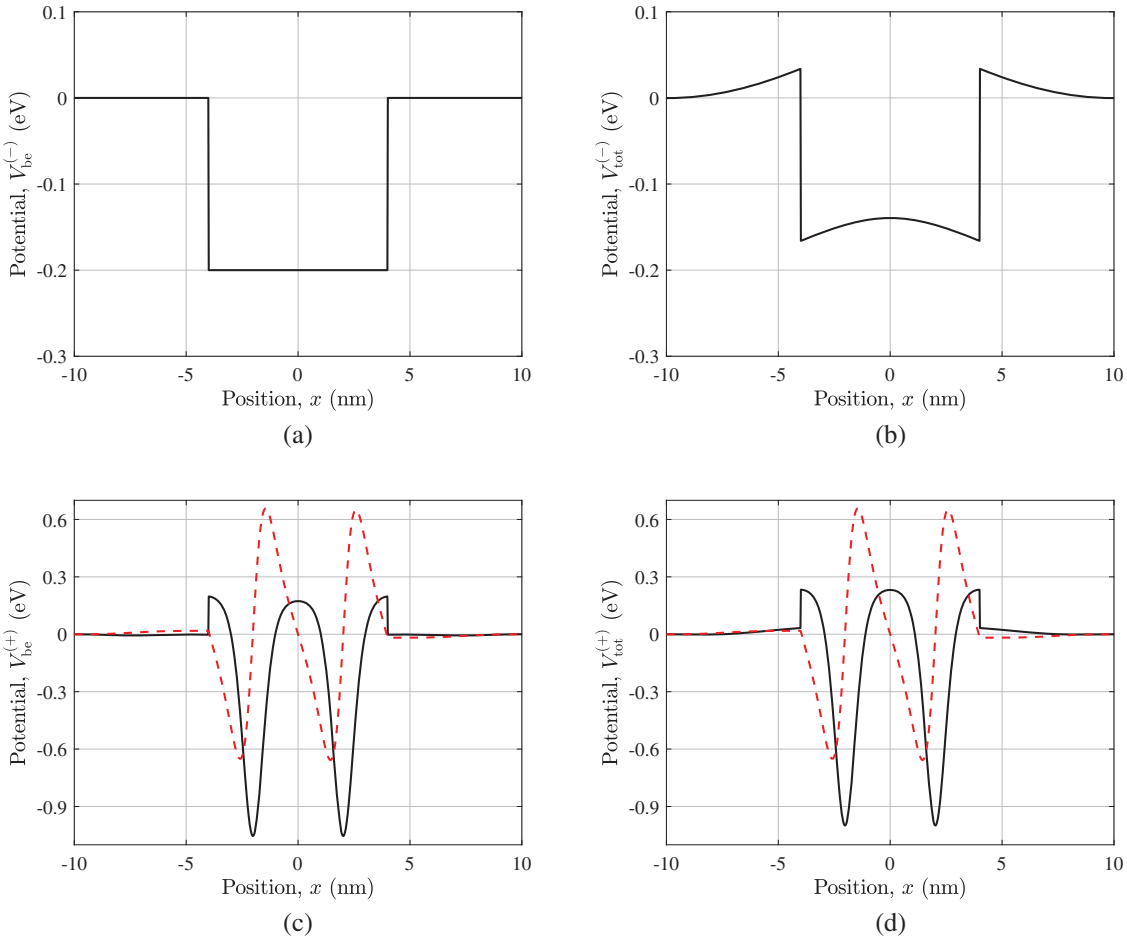

Fig. 5: (a) A rectangular potential well conduction band edge profile $V_{\rm be}^{(-)}$ and (b) the corresponding total potential $V_{\rm tot}^{(-)}$. (c) A conduction band edge profile $V_{\rm be}^{(+)}$ and (d) the corresponding total potential $V_{\rm tot}^{(+)}$ that is a supersymmetric partner potential of $V_{\rm tot}^{(-)}$. The total potential $V_{\rm tot}^{(+)}$ is evaluated from the lowest-energy resonant scattering state of $V_{\rm tot}^{(-)}$. The solid black curves indicate the real part of the potential, and the dashed red curves indicate the imaginary part. The rectangular potential well has thickness $w = 8\,\text{nm}$ and potential $V_0 = -0.2\,\text{eV}$. The effective electron mass is $m_{\rm e}^* = 0.07 \times m_0$, permittivity is $\epsilon_{\rm r0} = 13.2$, temperature is $T = 0\,\text{K}$, and ionized donor density is $N_{\rm D} = 1.5 \times 10^{18}\,\text{cm}^{-3}$ for $-10\,\text{nm} < x < 10\,\text{nm}$, and zero elsewhere.

prior work introducing nonlocal exchange effects to Schrödinger-Poisson equations [20] or exploring the existence of multiple solutions in some circumstances [21].

The notion of supersymmetry with scattering states more generally applies naturally to classical optical systems. Supersymmetry of classical optical systems can be understood using an analogy between the Schrödinger equation and the Helmholtz equation [22,23]. Dividing the usual form of the one-dimensional time-independent Schrödinger equation of a system with potential $V(x)$ by $\frac{-\hbar^2}{2m}$ gives

$$\left[\frac{\mathrm{d}}{\mathrm{d}x^2} - \frac{2m}{\hbar^2}V(x)\right]\psi(x) = -\frac{2m}{\hbar^2}E\psi(x). \tag{22}$$

Substituting $-\frac{2m}{\hbar^2}V(x) \mapsto \omega^2\mu(x)\epsilon(x)$, $-\frac{2m}{\hbar^2}E \mapsto \beta^2$ and $\psi(x) \mapsto \mathbf{E}(x)$ yields

$$\left[\frac{\mathrm{d}}{\mathrm{d}x^2} + \omega^2\mu(x)\epsilon(x)\right]\mathbf{E}(x) = \beta^2\mathbf{E}(x), \tag{23}$$

which is the Helmholtz equation for a waveguide with transverse permittivity and permeability profiles $\epsilon(x)$ and $\mu(x)$, where β is the phase constant in the waveguide. If the permeability $\mu(x) = \mu$ is a real constant, then the permittivity $\epsilon(x)$ becomes analogous to the potential $V(x)$ in supersymmetric quantum mechanics, and partner permittivity profiles can be evaluated using equations analogous to eqs. (6) and (7).

The application of supersymmetry to classical optical systems is an active field of both theoretical [9,10] and experimental [11,12] research. There is also an active effort to apply results from non-Hermitian and PT-symmetric quantum mechanics to classical optics [13], including the application of supersymmetry to non-Hermitian or PT-symmetric optical structures [24]. Such research is facilitated by the fact that complex-valued permittivities are a feature of classical optical systems with gain or loss. The present results contribute to this research by revealing a link between PT-symmetric structures and

real-valued symmetric structures through the use of non-reflecting scattering states to evaluate supersymmetric partners. This provides a new method for designing PT-symmetric optical structures that are isospectral to real-valued symmetric structures and possess equal amplitude and phase on transmission.

Conclusion. – It has been shown that supersymmetric partner potentials evaluated from quantum mechanical single-particle non-reflecting scattering states are localized in space. This provides a method for finding complex-valued potentials with real eigenvalues that are isospectral to known real-valued potentials, and have similar scattering characteristics. For the special case when the initial potential is spatially symmetric, the partner potential is PT-symmetric. Partner potentials evaluated from reflecting scattering states can be numerically simulated after regularization using an appropriate envelope function.

Scattering states can also be used to find supersymmetric partner total potentials in the context of self-consistent Schrödinger-Poisson equations, and semiconductor conduction band edge profiles that yield these supersymmetric partner total potentials can be evaluated. Although the corresponding conduction band edge profiles are complex-valued, it has been shown that the total potential nevertheless converges as expected in particular cases. Further research could examine under what conditions such complex-valued conduction band edge profiles converge to the target total potential.

Data availability statement: The data that support the findings of this study are available upon reasonable request from the authors.

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