NMR Sensitivity
the Reciprocity Theorem from a Power Viewpoint

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• A version emphasizing power (not volts & amps)
• Dispel some common misconceptions
  ❖ A tutorial, nothing new
Reciprocity Theorem – standard view

\[ V_{\text{NMR}} = \omega m \left( \frac{B_1}{I} \right) \]

- induced voltage in rcvr coil.

\[ \omega = \gamma B_0 \quad m = \text{total sample magnetic moment} \]

\[ \frac{B_1}{I} = \text{field per ampere of rcvg coil, used as transmit coil.} \]

Roughly, “the best transmit coil is the best receive coil.”

Integral generalization:

\[ V_{\text{NMR}} = \omega \int_{\text{volume}} \frac{\vec{B}_1(r)}{I} \cdot \vec{M}(r) \, d(\text{volume}) \]
1) Replace spins by a current loop (call it #2).

\[ m_2 = I_2 A_2 = m_{\text{spins}} \]

2) Recall mutual inductances are equal:

\[ M_{12} = M_{21} \quad (\text{Neumann’s formula}) \]

3) \( \Phi_1 = M_{12} I_2 \) so \( V_{\text{NMR}} = \frac{d\Phi_1}{dt} \)

\[ V_{\text{NMR}} = \omega M_{12} I_2 = \omega M_{21} \left( \frac{m_2}{A_2} \right) \]

\[ \text{but} \quad M_{21} \equiv \frac{\Phi_2}{I_1} = \frac{A_2 B_{\text{at}_2}}{I_1} \]

\[ \text{so} \quad V_{\text{NMR}} = \frac{B_{\text{at}_2} A_2}{I_1} \omega \frac{m_2}{A_2} = \left( \frac{B_{\text{at} \text{spins}}}{I} \right) \omega m_{\text{spins}} \]

QED
First Possible Misconception

\[ V_{\text{NMR}} = \left( \frac{B_1}{I} \right) \omega m \]

“So we want coil with largest \(B_1\)-per-amp, meaning huge number of turns of very fine wire.” e.g., solenoid:

\[ B_1 = \mu_0 \frac{NI}{\ell} \]

True theorem, but received voltage is not the goal. After all, could use transformer to step-up voltage.

We want to deal in power delivered to rcvg preamp; power is conserved and not trivially transformed.

Alternate view: huge number turns increases \(V_{\text{NMR}}\) but also increases impedance. In end, power delivered is the same.
Power in Reciprocity

During transmit

\[ P_{\text{trans}} = I^2 r \]

We had \[ V_{\text{NMR}} = \frac{B_1}{I} \omega m \]. Square and divide by \( r \),

\[ \frac{V_{\text{NMR}}^2}{r} = \left( \frac{B_1}{I^2 r} \right) \omega^2 m^2 = \left( \frac{B_1^2}{P_{\text{trans}}} \right) \omega^2 m^2 \]

“\( B_1^2 \)-per-watt”

During receive

\[ P_{\text{rcvr}} = \frac{V_{\text{NMR}}^2}{4r} = \left( \frac{B_1^2}{P_{\text{trans}}} \right) \frac{\omega^2 m^2}{4} \]

\[ \frac{S}{N} \propto \sqrt{P_{\text{rcvr}}} = \frac{1}{2} \omega m \left( \frac{B_1}{\sqrt{P_{\text{trans}}}} \right) \]

B\(_1\)-per-root-watt
Our Task

\[ P_{rcvr} = \frac{1}{4} \omega^2 m^2 \left( \frac{B_1^2}{P_{trans}} \right) \]

Find/design coil with best \( B_1^2 \)-per-watt. We see there is no advantage to "many turns of fine wire."

In ESR, cavities have been used. Notions of voltage and current become ambiguous, while power remains well-defined.

As \( B_0 \) fields get larger, NMR coils look more like resonators or cavities.
RF Coil: Turns in Series vs Parallel

Keep current density \( J \) same:

- Same \( B_1 \)
  \[ \overrightarrow{B_1} = \frac{\mu_0}{4\pi} \int_{\text{vol}} \mathbf{J} \times \frac{\mathbf{R}}{R^2} \, d(\text{vol}) \]

- Same power (resistive heating)
  \[ P = \frac{1}{\sigma} \int_{\text{vol}} J^2 \, d(\text{vol}) \]

Result: turns in series vs parallel give same \( B_1^2 \)-per-watt, so same receiving S/N.
S/N: Compare NMR to Coil-Noise

\[ V_{NMR}^2 = \left( \frac{B_1}{I} \right)^2 \omega^2 m^2 \]

\[ V_{noise}^2 = 4k_B Tr(\Delta f) \]

Nyquist

\[
\left( \frac{S}{N} \right)^2 = \frac{V_{NMR}^2}{V_{noise}^2} = \left( \frac{B_1^2}{I^2 r} \right) \omega^2 m^2 \frac{1}{4k_B T} \frac{1}{\Delta f}
\]

So \( \left( \frac{S}{N} \right)^2 \) is again prop to \( \frac{B_1^2}{P_{\text{trans}}} \), \( B_1^2 \)-per-watt

Note: \( \left( \frac{S}{N} \right)^2 \propto \frac{1}{\Delta f} \), \( \frac{1}{T_{\text{coil}}} \) Cryocoil?
Where Does $P_{\text{rcvr}}$ Come From?

Energy of spins = $-\mathbf{m} \cdot \mathbf{B}_0 = -B_0 m_z$

As spins precess, the induced current in rcvr coil nutates spins back to low energy ($\mathbf{m}$ parallel to $\mathbf{B}_0$)

This is source of “radiation damping.”

As spins are nutated closer to low energy state, decrease in energy explains power delivered to rcvg preamp. (And $I^2 r$ of rcvg coil.)
Gil Clark’s Formula: Stored Energy

\[ P_{\text{trans}} = \frac{\text{stored energy}}{\text{time for energy to decay}} = \frac{1}{\mu_0} \int_{\text{vol}} B_1^2 \, d(\text{vol}) \approx \frac{1}{\mu_0} \frac{B_1^2 V_C}{Q/\omega} \]

- \[ \frac{B_1^2}{P_{\text{trans}}} = \frac{\mu_0 Q}{\omega V_C} \], Clark’s result.

- \[ \frac{S}{N} \propto \sqrt{P_{\text{rcvr}}} = \frac{1}{2} \omega m \sqrt{\frac{B_1^2}{P_{\text{trans}}}} \]

- \[ \frac{S}{N} \propto \frac{1}{2} \sqrt{\mu_0 m} \sqrt{\frac{Q \omega}{V_C}} \]

- \[ \frac{S}{N} \propto m \] [more spins in coil, more \( \frac{S}{N} \)]

- \[ \frac{S}{N} \propto \omega^{1/2} \], at fixed \( m \).
  
  So if \( m \propto \omega \) (Boltzmann eq.), \( \frac{S}{N} \propto \omega^{3/2} \)

- \[ \frac{S}{N} \propto Q^{1/2} \], all else constant.

- \[ \frac{S}{N} \propto \frac{1}{\sqrt{V_C}} \], so keep \( V_C \) only big enough to hold sample.
Filling Factor, $\eta_{FF} \equiv \frac{V_{\text{sample}}}{V_{\text{coil}}}$

\[
\frac{S}{N} \propto \sqrt{P_{\text{rcvr}}} = \frac{\sqrt{\mu_0}}{2} \frac{m\sqrt{Q\omega}}{\sqrt{V_C}}
\]

But $m$ is proportional to number of spins = density $x V_{\text{sample}}$

\[
\frac{V_{\text{sample}}}{\sqrt{V_{\text{coil}}}} = \left(\frac{V_{\text{sample}}}{V_{\text{coil}}}\right)\sqrt{V_{\text{coil}}} = \eta_{FF} \sqrt{V_{\text{coil}}}
\]

\[
\frac{S}{N} \propto \sqrt{P_{\text{rcvr}}} = \frac{\sqrt{\mu_0}}{2} (\text{spin-density}) \eta_{FF} \sqrt{Q\omega V_{\text{coil}}}
\]

- At const fill factor, $\frac{S}{N} \propto \sqrt{V_{\text{coil}}}$

- At fixed $V_{\text{coil}}$, $\frac{S}{N} \propto \text{fill-factor} \propto V_{\text{sample}}$ provided all sample is in coil.
Second Misconception

\[ \frac{S}{N} \propto \eta_{FF} = \frac{V_{\text{sample}}}{V_{\text{coil}}} \]

We have thin, flat sample. “Let’s wind flat coil (“squashed”) so \( V_{\text{coil}} \) is small, to get better S/N.”

Examples: spins are on a glass slide. Or spins are absorbed onto thin substrate.

End views:

\[ \frac{V_{\text{flat}}}{V_{\text{round}}} \approx \frac{t}{2r} \ll 1 \]
But: \[ \frac{S}{N} \propto \sqrt{P_{rcvr}} = \frac{1}{2} \omega m \sqrt{\frac{B_1^2}{P_{trans}}} = \frac{1}{2} \omega m \left( \frac{B_1}{I} \right) \frac{1}{\sqrt{r}} \]

Both coils have same \( \frac{B_1}{I} \). Resistance \( r \) and length of wire is about 1.6 x larger for round \( \Rightarrow \frac{S}{N} \) of flat better by 1.25 than round. Only 1.25!

But we had \[ \frac{S}{N} \propto \sqrt{P_{rcvr}} = \frac{\sqrt{\mu_0}}{2} m \sqrt{\frac{Q}{V_C}} \sqrt{\omega} \]

Isn’t \( V_C \) much smaller for flat coil? Yes, but \( Q \) is much smaller too. Note that \( Q = \frac{\omega L}{r} \) and \( L \) is much smaller for flat coil because of the anti-parallel currents, so \( \frac{S}{N} \) is essentially same for round as flat.
Reciprocity Theorems in MR

\[ V_{\text{NMR}} = \omega m \left( \frac{B_1}{I} \right) \]
\[ \sqrt{P_{\text{rcvr}}} = \frac{\omega m}{2} \left( \frac{B_1}{\sqrt{P_{\text{trans}}}} \right) \]

Volts, amps, ohms: easily transformed
Watts and stored energy: permanent quantities

Misconception 1:
Don’t be fooled into using a huge number of turns

Misconception 2: flat coil vs round coil, for flat sample.
Don’t focus on only one of the many factors.