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Strong Axioms of Infinity and the search for V

Abstract.

The axioms ZFC do not provide a concise conception of the Universe of Sets. This claim has been well documented in the 50 years since Paul Cohen established that the problem of the Continuum Hypothesis cannot be solved on the basis of these axioms.

Gödel's Axiom of Constructibility, $V = L$, provides a conception of the Universe of Sets which is perfectly concise modulo only large cardinal axioms which are strong axioms of infinity. However the axiom $V = L$ limits the large cardinal axioms which can hold and so the axiom is *false*. The Inner Model Program which seeks generalizations which are compatible with large cardinal axioms has been extremely successful, but incremental, and therefore by its very nature unable to yield an ultimate enlargement of L. The situation has now changed dramatically and there is, for the first time, a genuine prospect for the construction of an ultimate enlargement of L.

1. Introduction

Paul Cohen showed in 1963 that Cantor's problem of the *Continuum Hypothesis* cannot be (formally) solved on the basis of the ZFC axioms for Set Theory. This result and its mathematical descendents have severely challenged any hope for a concise conception of the Universe of Sets.

Gödel's *Axiom of Constructibility*, this is the axiom $V = L$ defined in Section 6, does provide a clear conception of the Universe of Sets–a view that is arguably absolutely concise modulo only *large cardinal axioms* which are strong versions of the *Axiom of Infinity*. A conception of sets, perfectly concise modulo only large cardinal axioms, is clearly the idealized goal of Set Theory. But the axiom $V = L$ limits the large cardinal axioms which can hold and so the axiom is *false*.

The obvious remedy is to seek generalizations of the axiom $V = L$ which are *compatible* with large cardinal axioms. This program has been very successful, producing some of the most fundamental insights we currently have into the Universe of Sets. But at the same time the incremental nature of the program has seemed to be an absolutely fundamental aspect of the program: each new construction of an enlargement of L meeting the challenge of a specific large cardinal axiom comes with a theorem that no *stronger* large cardinal axiom can hold in that enlargement. Since it seems very unlikely that there could ever be a *strongest* large cardinal axiom, this methodology seems unable by its very nature to ever succeed in providing the requisite axiom for clarifying the conception of the Universe of Sets.

The situation has now changed dramatically and there is for the first time a genuine prospect for the construction of an *ultimate* enlargement of L. This arises *not* from the identification of a strongest large cardinal axiom but from the unexpected discovery that at a specific critical stage in the hierarchy of large cardinal axioms, the construction of an enlargement of L compatible with this large cardinal axiom must yield the ultimate enlargement of L. More precisely this construction must yield an enlargement which is compatible with all stronger large cardinal axioms.

In this paper I shall begin with an example which illustrates how large cardinal axioms have been successful in solving questions some of which date back to the early 1900's and which were conjectured at the time to be *absolutely* unsolvable. This success raises a fundamental issue. Can the basic methodology be extended to solve a much wider class of questions such as that of the Continuum Hypothesis?

I shall briefly review the construction of L, the basic template for large cardinal axioms, and describe the program which seeks enlargements of L compatible with large cardinal axioms.

Finally I will introduce Ω-logic, explain how on the basis of the Ω Conjecture a multiverse conception of V is untenable, and review the recent developments on the prospects for an ultimate version of L. I will end by stating an axiom which I conjecture is the axiom that V is this ultimate L even though the definition of this ultimate L is *not yet known*.

This account follows a thread over nearly 100 years but neither it nor the list of references is intended to be comprehensive, see [10] and [18] for far more elegant and thorough accounts.

2. The projective sets and two questions of Luzin

The projective sets are those sets of real numbers $A \subseteq \mathbb{R}$ which can be generated from the open subsets of $\mathbb{R}$ in finitely many steps of taking complements and images by continuous functions,

$$f : \mathbb{R} \to \mathbb{R}.$$

Similarly one defines the projective sets $A \subseteq \mathbb{R}^n$ or one can simply use a borel bijection,

$$\pi : \mathbb{R} \to \mathbb{R}^n$$

and define $A \subseteq \mathbb{R}^n$ to be projective if the preimage of A by π is projective.

From perspective of set theoretic complexity, projective sets are quite simple and one might expect that their basic properties can be established directly on the basis of the axioms ZFC.

The projective sets were defined by Luzin who posed two basic questions, [13] and [14]. A definition is required. Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$. A function f *uniformizes* A if for all $x \in \mathbb{R}$, if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$. The Axiom of Choice implies that for every set $A \subseteq \mathbb{R} \times \mathbb{R}$ there exists a function which uniformizes A. But if A is projective the Axiom of Choice seems to offer little insight into whether there is a function f which uniformizes the set A and which is *also* projective (in the sense that the graph of f is a projective subset of $\mathbb{R} \times \mathbb{R}$).

The two questions of Luzin are the following but I have expanded the scope of the second question–this is the measure question–to include the property of Baire.

1. Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is projective. Can A be uniformized by a projective function?
2. Suppose $A \subseteq \mathbb{R}$ is projective. Is A Lebesgue measurable and does A have the property of Baire?

Luzin conjectured that "we will never know the answer to the measure question for the projective sets". Luzin's reason for such a bold conjecture is the obvious fact that Lebesgue measurability is not preserved under continuous images since any set $A \subseteq \mathbb{R}$ is the continuous image of a Lebesgue null set.

The exact mathematical constructions of Gödel [6], [7] and Cohen [1],[2] which were used to show that the Continuum Hypothesis can neither be proved nor refuted on the basis of the ZFC axioms, show that the uniformization question for the projective sets can also neither be proved or refuted from the axioms ZFC.

The measure question is more subtle but the construction of Gödel and a refinement of Cohen's construction due to Solovay [25] show the same is true for the measure question. A curious wrinkle is that for Solovay's construction a modest large cardinal hypothesis is necessary.

The structure of the projective sets is of fundamental mathematical interest since it is simply the structure of the standard model of Second Order Number Theory:

$$\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle.$$

3. Logical definability from parameters, elementary embeddings, and ultrapowers

The answers to Luzin's questions involve aspects of Mathematical Logic and Set Theory which were unknown and unimagined at the time and I briefly review the basic context.

A set X is *transitive* if every element of X is a subset of X. Naively transitive sets can be viewed as *initial segments* of the Universe of Sets. The *ordinals* are the transitive sets α which are totally ordered by the set membership relation. There is a least infinite ordinal, denoted ω, and ω is simply the set of all finite ordinals. The order $(\omega, \in)$ is isomorphic to $(\mathbb{N}, <)$ and this also uniquely specifies ω as an ordinal. The ordinals provide the basis for *transfinite* constructions and in this sense they yield a generalization of the natural numbers into the infinite.

Formal notions of mathematical logic play a central role in Set Theory. Suppose X is a transitive set. Then a subset $Y \subseteq X$ is *logically definable* in $(X, \in)$ from parameters if there exist elements $a_1, \ldots, a_n$ of X and a logical formula $\phi(x_0, \ldots, x_n)$ in the formal language for Set Theory such that

$$Y = \{a \in X \mid (X, \in) \vDash \phi[a, a_1, \ldots, a_n]\}.$$

Let's look at two examples. If $X = \omega$ then a subset $Y \subseteq X$ is logically definable in $(X, \in)$ from parameters if and only if Y is finite or the complement of Y is finite. If X is the smallest transitive set which contains $\mathbb{R}$ then the projective sets $A \subseteq \mathbb{R}$ are exactly those sets $A \subseteq \mathbb{R}$ which are logically definable from parameters in X. In general if X is a finite transitive set then every subset of X is logically definable in $(X, \in)$. Assuming the Axiom of Choice, if X is an infinite transitive set then there must exist subsets of X which are not logically definable in $(X, \in)$ from parameters. The collection of *all* subsets of X is the *powerset* of X and is denoted by $\mathcal{P}(X)$.

Suppose X and Y are transitive sets. A function $\pi : X \to Y$ is an *elementary embedding* if for all formulas $\phi[x_0, \ldots, x_n]$ in the formal language for Set Theory and all $a_0, \ldots, a_n$ in X,

$$(X, \in) \vDash \phi[a_0, \ldots, a_n] \text{ iff } (Y, \in) \vDash \phi[\pi(a_0), \ldots, \pi(a_n)].$$

Note for example that if X is an ordinal and $\pi : X \to Y$ is an elementary embedding then Y must be an ordinal as well. In general elementary embeddings are simply functions which preserve logical truth (and so generalize the notion of isomorphism) and this makes sense for all mathematical structures (of the same logical type) not just the structures given by transitive sets which I am discussing here. However the case of transitive sets is quite special, if $\pi : X \to Y$ is both an elementary embedding and a surjection then π is the identity function.

Suppose X is a transitive set and U is a free ultrafilter over some index set I. Then one can form the ultrapower, X^I/U, to both define a new structure from X and an elementary embedding from X to this new structure–the points of X^I/U are equivalence classes $[f]_U$ of functions $f : I \to X$ where $f \sim g$ if $\{a \in I \mid f(a) = g(a)\} \in U$ and one defines $[f]_U \in_U [g]_U$ if $\{a \in I \mid f(a) \in g(a)\} \in U$. The elementary embedding sends a to $[f_a]_U$ where f_a is the constant function with value a.

For example, if $X = \omega$ then the ultrapower X^I/U is linear order. In general unless X is finite, the ultrapower X^I/U is not isomorphic to a transitive set. If however the ultrafilter U is closed under countable intersections then for each transitive set X, the ultrapower X^I/U is isomorphic to a transitive set Y and both Y and the isomorphism are unique. I note that in this situation, if the ultrapower is nontrivial (for example, if the ultrapower X^I/U is not isomorphic to X), then the set X must be very *large*. This is because if U is closed under countable intersections then U must be closed under intersections of cardinality δ for relatively large δ. This is the entry point for new notions of mathematical infinity which transcend the usual classical notions of infinity. It is such notions of infinity–completely unknown at the time of Luzin's questions and directly the result of the influence of mathematical logic within Set Theory–which are the key to resolving Luzin's questions. But surprisingly the explanation begins with yet another notion within Set Theory and this notion has nothing a priori to do with such (or any) strong axioms of infinity.

4. The hierarchy of large cardinals, determinacy, and the answers to Luzin's questions

Suppose $A \subseteq \mathbb{R}$. There is an associated infinite game involving two players. The players alternate choosing $\epsilon_i \in \{0, 1\}$. After infinitely many moves an infinite binary sequence $\langle \epsilon_i : i \in \mathbb{N} \rangle$ is defined. Player I wins this run of the game if

$$\Sigma_{i=1}^{\infty} \epsilon_i / 2^i \in A$$

otherwise Player II wins. Either player could choose to follow a *strategy* which is simply a function

$$\tau : \mathrm{SEQ} \to \{0, 1\}$$

where SEQ is the set of all finite binary sequences $\langle \epsilon_1, \ldots, \epsilon_n \rangle$. The strategy τ is a winning strategy for that player if by following τ, that player wins no matter how the other player moves. Trivially if $[0, 1] \subseteq A$ then every strategy is a winning strategy for Player I and if $A \cap [0, 1] = \emptyset$ then every strategy is a winning strategy for Player II. The set A is *determined* if there is a winning strategy for one of the players in the game associated to A.

Gale and Stewart [5] proved that if A is a closed set then A is determined and they asked whether this is also true when A is borel. Mycielski and Steinhaus [20] took a much bolder step and formulated 50 years ago the axiom AD.

Definition 1 (Mycielski, Steinhaus). *Axiom of Determinacy* (AD): Every set $A \subseteq \mathbb{R}$ is determined.□

The axiom AD is refuted by the Axiom of Choice and so it is false. But restricted versions have proven to be quite important and provide the answers (yes) to Luzin's questions, [18], [19], and [20].

Definition 2. *Projective Determinacy* (PD): Every projective set $A \subseteq \mathbb{R}$ is determined. □

Theorem 3. *Assume every projective set is determined.*

(1) (Mycielski, Steinhaus) *Every projective set has the property of Baire.*

(2) (Mycielski, Swierczkowski) *Every projective set is Lebesgue measurable.*

(3) (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*□

The axiom PD yields a rich structure theory for the projective sets and modulo notions of infinity no question about the projective sets is known to be unsolvable on the basis of ZFC + PD. But is PD even consistent and if consistent is PD true? The answers to both questions is yes but this involves another family of axioms, these are *large cardinal axioms* which are axioms of strong infinity. The basic modern form of these axioms is as follows where a class M is transitive if each element of M is a subset of M (just as for transitive sets). A cardinal κ is a *large cardinal* if there exists an elementary embedding,

$$j : V \to M$$

such that M is a transitive class and κ is the least cardinal such that $j(\alpha) \neq \alpha$. This is the *critical point* of j, denoted $\mathrm{CRT}(j)$. By requiring more sets to belong to M, possibly in a way that depends on action of j on the cardinals, one obtains a hierarchy of notions. The obvious maximum here, taking $M = V$, is not possible (it is refuted by the Axiom of Choice by a theorem of Kunen). In some cases the large cardinal axiom of interest holding at κ is specified by the existence of many elementary embeddings and possibly elementary embeddings with smaller critical points that the cardinal κ.

The careful reader might object to the reference to classes but in all instances of interest one can require M and j be definable classes (from parameters but in a simple manner) and j need only be elementary with respect to rather simple formulas. The situation is analogous to that in Number Theory where one frequently refers to infinite collections such as the set of prime numbers. This

does not require that one work in a theory of infinite sets–similarly the reference to j and M here in all the relevant instances does not require in general that one work in a theory of classes.

In this scheme the simplest large cardinal notion is that of a *measurable cardinal*. A cardinal κ is measurable if there exists an elementary embedding $j : V \to M$ such that $\kappa = \mathrm{CRT}(j)$. This is not the usual definition (rather it is a theorem) but it is equivalent. The standard definition is that an uncountable cardinal κ is a measurable cardinal if there exists an ultrafilter μ on κ (more precisely on the complete boolean algebra given by $\mathcal{P}(\kappa)$) which is nonprincipal (i.e., free) and which is closed under intersections of cardinality δ for all $\delta < \kappa$. Given μ one can form the ultrapower of the universe of sets, V^κ/μ, show that this ultrapower is isomorphic to a transitive class, and so generates an elementary embedding as above. Conversely given an elementary embedding j with $\mathrm{CRT}(j) = \kappa$, define $\mu = \{A \subseteq \kappa \mid \kappa \in j(A)\}$. It follows that μ is a nonprincipal ultrafilter on κ which is closed under intersections of cardinality δ for all $\delta < \kappa$.

Beside measurable cardinals, there are *strong cardinals*, *Woodin cardinals*, *superstrong cardinals*, *supercompact cardinals*, *extendible cardinals*, *huge cardinals*, and much more. These I shall not define with exception of supercompact and extendible cardinals but I shall defer these particular definitions until Section 10. I refer the reader to the excellent exposition [10] for details and the history of the development of large cardinal axioms. I also note that all these large cardinal notions have equivalent reformulations in terms of ultrapowers or direct limits of ultrapowers.

The connection between Projective Determinacy and large cardinal axioms is given in the next two theorems the first of which is the seminal theorem of Martin and Steel [15]. These theorems proved in 1985 and 1987, respectively, brought to a close a chapter which began over 60 years earlier with the questions of Luzin. But as I hope to show in a convincing fashion, the real story was just beginning.

Theorem 4 (Martin, Steel). *Assume there are infinitely many Woodin cardinals. Then every projective set is determined.* □

Theorem 5 (Woodin). *The following are equivalent.*

(1) *Every projective set is determined.*

(2) *For each $n < \omega$ there is a countable (iterable) model M such that*

$$M \vDash \mathrm{ZFC} + \text{“There exist at least } n \text{ Woodin cardinals”}.$$ □

With these theorems one can assert that PD is both consistent and true and this represents a mathematical milestone since we now have the axioms for the structure,

$$\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle,$$

which are the correct extension of the Peano axioms for the structure $\langle \mathbb{N}, +, \cdot \rangle$.

Can this success be extended to the entire universe of sets?

5. Rank-universal sentences and the cumulative hierarchy

The cumulative hierarchy stratifies the universe of sets. The definition involves transfinite iterations of the operation of taking powersets. Recall that for each set X, the powerset of X, denoted $\mathcal{P}(X)$, is the set of all subsets of X. Now define by induction on the ordinal α a set V_α as follows.

1. $V_0 = \emptyset$.
2. (Successor step) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. (Limit step) $V_\alpha = \cup\{V_\beta \mid \beta < \alpha\}$.

It is a consequence of the axioms ZFC that for each ordinal α, V_α exists and moreover that for each set X there exists an ordinal α such that $X \in V_\alpha$. The set V_α is the *rank initial segment* of V determined by the ordinal α. This calibration of V suggests that to understand V one should simply proceed by induction on α, analyzing V_α.

The integers appear in V_ω, the reals appear in $V_{\omega+1}$, and all sets of reals appear in $V_{\omega+2}$. The projective sets in their incarnation as relations of Second Order Number Theory appear in effect in $V_{\omega+1}$ since $V_{\omega+1}$ is logically bi-interpretable with $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$. Given the amount of mathematical effort and development which was required to understand $V_{\omega+1}$ just to the point where one could identify the correct axioms for $V_{\omega+1}$, and noting that this is an infinitesimal fragment of the Universe of Sets, the prospects for understanding V to this same degree, or even just $V_{\omega+2}$ which would reveal whether the Continuum Hypothesis is true, is a daunting task.

I take a strong, perhaps unreasonable position, on this. The statement that Projective Determinacy is consistent is a new mathematical truth. It predicts facts about our world, for example that in the next 1000 years, so by ICM 3010, there will be no contradiction discovered from Projective Determinacy *by any means*. Of course one could respond with the observation that with each new theorem of mathematics comes such a prediction. For example from Wiles' proof of Fermat's Last Theorem, one has the superficially similar prediction that no counterexample to FLT will be discovered. But this prediction, while certainly a new prediction, is reducible by finite means (i.e. the proof) to a previous prediction–namely that the axioms (whatever they are) necessary for Wiles' proof will not be discovered to be contradictory. This is *not* the case for the prediction I have made above. That prediction is a genuinely new prediction which is not reducible by finite means to any previously held prediction (say from before 1960). This is the nature of the investigation of large cardinal axioms which sets it apart from other mathematical enterprises. But now there is a dilemma. The claim that a large cardinal axiom is consistent, such as the claim that the existence of Woodin cardinals is consistent, would seem ultimately to have to be founded on a conception of truth for the Universe of Sets which includes the existence of these large cardinals. But if our axioms for this Universe of Sets fail to resolve even the most basic questions about the Universe of Sets, such as that of the Continuum Hypothesis, then ultimately what sense is there to the claim that large cardinals exist? This is perhaps tolerable on a temporary basis during a period of axiomatic discovery but it certainly cannot be the permanent state of affairs.

The alternative position–that consistency claims can never be meaningfully made–is simply a rejection of the infinite altogether. And what if my prediction is correct and an instance of an evolving series of ever stronger similarly correct predictions? How will this skeptic explain that?

In any case an incremental approach might be prudent and so I shall restrict attention to sentences about the universe of sets of a particular form. A sentence ϕ is a *rank-universal* sentence if for some sentence ψ, ϕ asserts that

$$V_\alpha \vDash \psi$$

for all ordinals α. Similarly a sentence ϕ is a *rank-existential* sentence if for some sentence ψ, ϕ asserts that there exists an ordinal α such that $V_\alpha \vDash \psi$.

For any sentence ψ, the assertion that

$$V_{\omega+2} \vDash \psi$$

is both rank-universal and rank-existential and so the Continuum Hypothesis is expressible as both a rank universal sentence and a rank existential sentence. There is nothing particularly special about the ordinal ω here or for that matter about 2 either. For example if δ_0 is the least Woodin cardinal then for any sentence ψ, the assertion that

$$V_{\delta_0+\omega} \vDash \psi$$

is both rank-universal and rank-existential, etc.

6. The effective cumulative hierarchy: L

Gödel's definition of L arises from restricting the successor step in the definition of the cumulative hierarchy. For each set X, let $\mathcal{P}_{\mathrm{Def}}(X)$ be the set of all $Y \subseteq X$ such that Y is logically definable in the structure $(X, \in)$ from parameters in X. If X is infinite and the Axiom of Choice holds then $\mathcal{P}_{\mathrm{Def}}(X)$ is never the set of all subsets of X. Now define L_α by induction on α as follows.

1. $L_0 = \emptyset$,
2. (Successor case) $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$,
3. (Limit case) $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

Definition 6. L is the class of all sets X such that $X \in L_\alpha$ for some ordinal α. □

The axiom "$V = L$" is Gödel's Axiom of Constructibility and this axiom is expressible by a rank-universal sentence.

Theorem 7. *Assume $V = L$.*

(1) (Gödel) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

(2) (Gödel) *There is a projective set which is not Lebesgue measurable (there is a projective wellordering of the reals).*

(3) (Scott) *There are no measurable cardinals.* □

Scott's Theorem was proved just before the seminal work of Cohen and so provided the first consistency proof that $V \neq L$. My own view is more extreme on the significance of Scott's Theorem:

Corollary 8. $V \neq L$. □

It is Scott's Theorem which shows that to find canonical models in which large cardinal axioms hold one must somehow enlarge L. This is the Inner Model Program.

7. $L(\mathbb{R})$ and AD

By *relativizing* L to $\mathbb{R}$ one obtains $L(\mathbb{R})$ which provides a transfinite extension of the projective sets. The formal definition proceeds by first defining $L_\alpha(\mathbb{R})$ by induction on α:

1. $L_0(\mathbb{R}) = \mathbb{R}$ (more precisely $L_0(\mathbb{R}) = V_{\omega+1}$),
2. (Successor case) $L_{\alpha+1}(\mathbb{R}) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(\mathbb{R}))$,
3. (Limit case) $L_\alpha(\mathbb{R}) = \cup\{L_\beta(\mathbb{R}) \mid \beta < \alpha\}$.

Definition 9. $L(\mathbb{R})$ is the class of all sets X such that $X \in L_\alpha(\mathbb{R})$ for some ordinal α. □

The projective sets are precisely the sets in

$$L_1(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$$

and

$$L_{\omega_1}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$$

is the smallest σ-algebra containing the projective sets and closed under images by continuous functions, $f : \mathbb{R} \to \mathbb{R}$. The collection $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a transfinite extension of the projective sets.

A natural axiom generalizing the axiom that all projective sets are determined is the axiom, "$L(\mathbb{R}) \vDash \mathrm{AD}$", which is simply the axiom which asserts that every set $A \in L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is determined.

Theorem 10 (Martin, Steel, Woodin). *Assume there are infinitely many Woodin cardinals with a measurable cardinal above. Then* $L(\mathbb{R}) \vDash \mathrm{AD}$. □

The proof of the theorem involves combining [15] with methods from previous results and something like the measurable cardinal is necessary but only just barely. The following theorem clarifies the situation by providing an exact match to the axiom "$L(\mathbb{R}) \vDash \mathrm{AD}$" within the hierarchy of large cardinals axioms and from the perspective of the formal consistency of theories.

Theorem 11 (Woodin). *The following theories are equiconsistent.*

(1) ZFC + "$L(\mathbb{R}) \vDash \mathrm{AD}$".

(2) ZFC + "There are infinitely many Woodin cardinals". □

The axiom, $L(\mathbb{R}) \vDash \mathrm{AD}$, gives a complete analysis of $L(\mathbb{R})$ extending the analysis that the axiom, all projective sets are determined, provides for the projective sets. For example Moschovakis's theorem on uniformization generalizes to show that for many ordinals α, assuming all sets in $L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ are determined, uniformization holds in $L_\alpha(\mathbb{R})$. This includes all countable α and quite a bit more. Subsequent work of Steel has exactly characterized these ordinals.

Of course assuming $V = L$, uniformization holds in $L(\mathbb{R})$ since in this case $L(\mathbb{R}) = L$. But if uniformization holds in $L(\mathbb{R})$ then the Axiom of Choice *must* hold in $L(\mathbb{R})$ and so in $L(\mathbb{R})$, uniformization implies that $L(\mathbb{R}) \nvDash \mathrm{AD}$. Thus there is mathematical tension between uniformization and the regularity properties such as Lebesgue measurability and having the property of Baire.

Theorem 12 (Woodin). *Suppose that uniformization holds in* $L_\alpha(\mathbb{R})$ *and that* $\alpha = \omega_1 \cdot \beta$ *for some limit ordinal* β*. Then the following are equivalent.*

(1) *Every set* $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *is Lebesgue measurable and has the property of Baire.*

(2) *Every set* $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *is determined.* □

The proof of the theorem uses rather elaborate machinery to construct given $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ and assuming (1), a countable transitive set M such that $A \cap M \in M$ and such that in M there are Woodin cardinals sufficient to establish that $A \cap M$ is determined. There is an additional requirement that $M \cap A$ be correct about whether a strategy is a winning strategy in the game associated to A and so the determinacy of $A \cap M$ within M yields the determinacy of A.

By a remarkable theorem of Steel the restriction on α is *necessary*, in particular Theorem 12 does not hold with $\alpha = \omega_1$ and this fact argues strongly that there is no elementary proof of the theorem even in specific cases such as $\alpha = \omega_1 \cdot \omega_1$ where the theorem does hold.

The previous theorem is now one of many analogous theorems which have been proved, including recent dramatic results of Sargsyan [21]. These theorems collectively confirm that the understanding of determinacy plays a central role in modern Set Theory. The ubiquity of Projective Determinacy in infinitary combinatorics is often cited as an independent confirmation of its truth.

8. The universally Baire sets

For any set E there is an associated enlargement of L, denoted $L[E]$, which is defined as follows. For each ordinal α, $L_\alpha[E]$ is first defined by induction on α:

1. $L_0[E] = \emptyset$
2. (Successor case) $L_{\alpha+1}[E] = \mathcal{P}_{\mathrm{Def}}(L_\alpha[E] \cup \{E \cap L_\alpha[E]\})$,
3. (Limit case) $L_\alpha[E] = \cup\{L_\beta[E] \mid \beta < \alpha\}$.

Then $L[E]$ is defined as the class of all sets X such that $X \in L_\alpha[E]$ for some ordinal α. One can also in analogous fashion modify the definition of $L(\mathbb{R})$ to define $L(\mathbb{R})[E]$. I caution that $L[\mathbb{R}] = L$ and so in general $L(\mathbb{R}) \neq L[\mathbb{R}]$.

Assuming the Axiom of Choice for *any* set X there exists a set E such that $X \in L[E]$ (this is equivalent to the Axiom of Choice). So for the Inner Model Program where one seeks structural generalizations of L one must somehow restrict the choices of E.

For the case of measurable cardinals there is an elegant solution to the choice of E. Suppose that κ is a measurable cardinal and let

$$j : V \to M$$

be an associated elementary embedding with $\mathrm{CRT}(j) = \kappa$. Define an ultrafilter μ on κ by $A \in \mu$ if $\kappa \in j(A)$. Then μ is a nonprincipal ultrafilter on κ closed under intersections of cardinality δ for all $\delta < \kappa$. The enlargement of L given by $L[\mu]$ turns out to be a true generalization of L.

Theorem 13 (Kunen). *Suppose κ is a measurable cardinal with associated ultrafilter μ.*

(1) *$L[\mu]$ and $\mu \cap L[\mu]$ each depend only on κ.*

(2) *$L[\mu] \cap \mathbb{R}$ is independent of both κ and μ.*

(3) *$L[\mu] \vDash$ "κ is the only measurable cardinal".* □

Theorem 14 (Silver). *Suppose κ is a measurable cardinal with associated ultrafilter μ. Then*

$$L[\mu] \vDash \text{"There is a projective wellordering of the reals"}.$$ □

One can also consider $L(\mathbb{R})[\mu]$, the associated enlargement of $L(\mathbb{R})$.

Theorem 15. *Suppose κ is a measurable cardinal with associated ultrafilter μ.*

(1) (Kunen) *$L(\mathbb{R})[\mu] \cap \mathcal{P}(\mathbb{R})$ is independent of both κ and μ.*

(2) (Woodin) *Suppose there are infinitely many Woodin cardinals with at least two measurable cardinals above. Then $L(\mathbb{R})[\mu] \vDash \mathrm{AD}$.* □

In general the enlargements of L produced by the Inner Model Program have companion enlargements of $L(\mathbb{R})$. If we cannot yet define an ultimate inner model perhaps we can nevertheless define the ultimate extension of $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. It turns out that assuming there is a proper class of Woodin cardinals, we can. Even more we can define the order in which these sets of reals are generated by that ultimate enlargement of L adapted to produce an enlargement of $L(\mathbb{R})$. The following definition is from [4].

Definition 16. A set $A \subseteq \mathbb{R}$ is *universally* Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}$, the preimage of A by π has the property of Baire in the space Ω. □

One can restrict to only those spaces Ω which are compact Hausdorff spaces and obtain an equivalent definition. The definition that a set $A \subseteq \mathbb{R} \times \mathbb{R}$ is universally Baire is identical. A partial function $f : \mathbb{R} \to \mathbb{R}$ is universally Baire if its graph is universally Baire. Given $A \subseteq \mathbb{R}$ one defines $L(A, \mathbb{R})$ following the definition of $L(\mathbb{R})$ except $L_0(A, \mathbb{R}) = L_0(\mathbb{R}) \cup \{A\}$.

Theorem 17. *Suppose that there is a proper class of Woodin cardinals.*

(1) (Martin-Steel) *Suppose $A \subseteq \mathbb{R}$ is universally Baire. Then A is determined.*

(2) (Woodin) *Suppose $A \subseteq \mathbb{R}$ is universally Baire. Then every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.*

(3) (Steel) *Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is universally Baire. Then A can be uniformized by a universally Baire function.* □

There is an ordinal measure of complexity for the universally Baire sets–this can be defined a number of ways and I define a somewhat coarse notion using a definition which is just for this account. Suppose A and B are subsets of $\mathbb{R}$. Define A to be *borel reducible* to B, written $A \leq_{\text{borel}} B$, if there is a borel function $\pi : \mathbb{R} \to \mathbb{R}$ such that either $A = \pi^{-1}[B]$ or $A = \mathbb{R}\backslash\pi^{-1}[B]$. Define $A <_{\text{borel}} B$ if $A \leq_{\text{borel}} B$ but $B \not\leq_{\text{borel}} A$. Finally define A and B to be *borel bi-reducible* if both $A \leq_{\text{borel}} B$ and $B \leq_{\text{borel}} A$. The *borel degree* of A is the equivalence class of all sets which are borel bi-reducible with A. The borel degree of a set $A \subseteq \mathbb{R}$ is analogous to the Turing degree of a set $A \subseteq \mathbb{N}$.

The following lemma is an immediate corollary of the rather remarkable Wadge's Lemma from the theory of determinacy together with the determinacy of the universally Baire sets. The subsequent theorem is similarly a corollary of a fundamental theorem of Martin on the Wadge order.

Lemma 18. *Assume there is a proper class of Woodin cardinals. Suppose that A and B are universally Baire subsets of $\mathbb{R}$.*

(1) *Either $A \leq_{\text{borel}} B$ or $B \leq_{borel} A$,*

(2) *Suppose $A <_{\text{borel}} B$. Then there is a borel function $\pi : \mathbb{R} \to \mathbb{R}$ such that $A = \pi^{-1}[B]$.* □

Theorem 19. *Assume there is a proper class of Woodin cardinals. There is no sequence $\langle A_i : i < \omega\rangle$ of universally Baire sets such that for all $i < \omega$, $A_{i+1} <_{\text{borel}} A_i$.* □

Thus, assuming there is a proper class of Woodin cardinals, the borel degrees of the universally Baire sets are linearly ordered by borel reducibility and moreover this linear order is a wellorder.

I illustrate the relevance of this to the Inner Model Program. Suppose that there is proper class of Woodin cardinals and consider the enlargement of $L(\mathbb{R})$ given by $L(\mathbb{R})[\mu]$ as discussed above. Then the sets in $L(\mathbb{R})[\mu] \cap \mathcal{P}(\mathbb{R})$ are all universally Baire. Suppose that $A, B \in L(\mathbb{R})[\mu] \cap \mathcal{P}(\mathbb{R})$ and that for some ordinal α, $A \in L_\alpha(\mathbb{R})[\mu]$ but $B \notin L_\alpha(\mathbb{R})[\mu]$. Then $A <_{\text{borel}} B$.

In general, the ranking of the universally Baire sets given by borel reducibility must *refine* the order of generation of these sets in any possible enlargement of L adapted to define an enlargement of $L(\mathbb{R})$. The point here is that for any transitive set X, if $A \leq_{\text{borel}} B$ and $B \in \mathcal{P}_{\text{Def}}(X)$ then $A \in \mathcal{P}_{\text{Def}}(X)$.

In summary, the sets generated by any possible enlargement of L (subject only to very general constraints) adapted to define an enlargement of $L(\mathbb{R})$ defines an initial segment of the universally Baire sets relative to the order of borel reducibility. The extent of that initial segment is determined by the extent of the large cardinal axioms which hold in the initial segments of that enlargement.

9. Ω-logic and the Ω Conjecture

The foundational issues of truth in Set Theory arise because of Cohen's method of forcing and I shall refer in this paper to extensions obtained by the method of forcing as *Cohen extensions*. Cohen extensions are the source of the profound unsolvability of problems such as that of the Continuum Hypothesis which makes these problems seem so intractable. This is in contrast to Luzin's questions about the projective sets which we have seen are resolved by simply invoking strong notions of infinity. Perhaps then the best one can do is a multiverse conception of the universe of sets. To illustrate suppose that M is a countable (transitive) model of ZFC (of course one cannot prove such a set exists without appealing to large cardinal axioms). Let $\mathbb{V}(M)$ be the smallest collection of countable transitive models such that if (M_0, M_1) is any pair of countable transitive models with M_1 a Cohen extension of M_0, if either $M_0 \in \mathbb{V}(M)$ or $M_1 \in \mathbb{V}(M)$ then both models are in $\mathbb{V}(M)$. $\mathbb{V}(M)$ is the *generic multiverse* generated by M. Taking M to be V itself, this defines the generic-multiverse.

Of course one is interested in the corresponding notion of truth. So a sentence ϕ is *true* in the generic-multiverse generated by V if ϕ is true in *each* universe of the generic-multiverse generated by V. This can be made perfectly precise (without quantifying over classes) and I shall give a relatively simple reformulation at least for rank-universal sentences.

It turns out that for essentially all large cardinal axioms (and certainly all listed in this account), the existence of a proper class of κ witnessing the large cardinal axiom, is *invariant* across the generic-multiverse generated by V. For example the existence of a proper class of Woodin cardinals if true in *one* universe of the generic-multiverse generated by V, is true in *every* universe of the generic-multiverse generated by V [8]. Thus a generic-multiverse conception of the universe of sets could provide a framework for set theoretic truth which allows one to confirm the consistency of large cardinal axioms, even confirm that Projective Determinacy is true, and yet not resolve questions such as that of the Continuum Hypothesis. But only if such a conception of truth is compatible with the basic principles of infinity on which ZFC is founded.

There is a remarkable consequence of large cardinal axioms which gives a very simple equivalent definition (modulo the definition of Cohen's method of forcing) that a rank-universal sentence is true in the generic-multiverse generated by V.

Theorem 20. *Suppose that there is a proper class of Woodin cardinals and that ϕ is a rank-universal sentence. Then the following are equivalent.*

(1) *ϕ is true in the generic-multiverse generated by V.*

(2) *ϕ is true in all Cohen extensions of V.* □

To say that a rank-universal sentence is true in all Cohen extensions of V is itself a rank-universal sentence. So this theorem shows that in the context of a proper class of Woodin cardinals, the assertion that a rank-universal sentence is true in the generic-multiverse generated by V is itself a rank-universal sentence. I note that without the indicated large cardinal hypothesis, the previous theorem is *false*. For example the conclusion of the theorem is false if V is a Cohen extension of L and $\mathbb{R} \not\subseteq L$.

The generic-multiverse conception of truth is connected by the previous theorem to Ω-logic.

Definition 21. Suppose ϕ is a rank-universal sentence. Then ϕ is Ω-*valid*, written $\vDash_\Omega \phi$, if ϕ is true in all Cohen extensions of V. □

Is there a notion of proof for Ω-logic? This leads back to the universally Baire sets.

Definition 22. Suppose $A \subseteq \mathbb{R}$ is universally Baire and M is a countable transitive model of ZFC. Then M is *strongly A-closed* if $A \cap N \in N$ for all countable transitive sets N such that N is a Cohen extension of M. □

Definition 23. Assume there is a proper class of Woodin cardinals. Suppose ϕ is a rank-universal sentence. Then ϕ is Ω-*provable*, written $\vdash_\Omega \phi$, if there is a universally Baire set $A \subseteq \mathbb{R}$ such that if M is a countable transitive model of ZFC and M is strongly A-closed then $M \vDash$ "$\vDash_\Omega \phi$", or equivalently $N \vDash \phi$ for all countable transitive sets N such that N is a Cohen extension of M. □

The ordinal rank of complexity that I defined for universally Baire sets provides a very reasonable notion of length of proof and so Ω-logic shares many features with classical logic.

I now come to the Ω Conjecture and the issue is whether Ω-validity implies Ω-provability.

Definition 24 (The Ω Conjecture). Assume there is a proper class of Woodin cardinals and ϕ is a rank-universal sentence. Then ϕ is Ω-valid if and only if ϕ is Ω-provable. □

How does the Ω Conjecture impact the generic-multiverse conception of truth? There are two relevant theorems. The point is that for rank-universal sentences, truth in the generic-multiverse is equivalent to Ω-validity and so to Ω-provability if the Ω Conjecture holds (nontrivially).

Theorem 25. *Suppose that there is a proper class of Woodin cardinals and let δ_0 be the least Woodin cardinal. Assume the Ω Conjecture holds. Then the set of rank-universal sentences which are Ω-valid is definable in V_{δ_0+1}.* □

Let T_0 be the set of sentences ψ such that " $V_{\omega+2} \vDash \psi$" is a generic-multiverse truth and let T be the set of *all* rank-universal sentences which are generic-multiverse truths. Clearly T_0 is reducible to T. The second theorem shows that assuming the Ω Conjecture (and that there is a proper class of Woodin cardinals) then these two sets have the same computational complexity by showing that T is reducible to T_0 (and the proof gives the explicit reduction).

Theorem 26. *Suppose that there is a proper class of Woodin cardinals and assume the Ω Conjecture holds. Then T is recursively reducible to T_0.* □

Why is this a problem? Assuming the Ω Conjecture (and that there is a proper class of Woodin cardinals), then the second theorem shows that the whole hierarchy of rank-universal truth–in the generic-multiverse conception of truth–collapses to simply the truths of $V_{\omega+2}$. Moreover augmented by a second conjecture, the Ω Conjecture yields a strong form of the first theorem–namely that this set of sentences is actually definable in $V_{\omega+2}$.

This collapse is completely counter to the fundamental principles concerning infinity on which Set Theory is founded. Moreover since $V_{\omega+2}$ is in essence just the standard structure for Third Order Number Theory, this collapse shows that the generic-multiverse conception of truth (for rank-universal sentences) is simply a version of third order formalism. If the Ω Conjecture is true then the generic-multiverse conception of truth is untenable.

No viable alternative multiverse conception of truth is known that survives the challenge posed by the Ω Conjecture and this seems to argue for a multiverse of one universe which leads us back to searching for generalizations of the axiom $V = L$ and the Inner Model Program.

Perhaps this all is simply evidence that the Ω Conjecture is false. The Ω Conjecture is invariant across the generic multiverse generated by V and so a reasonable conjecture is that if the Ω Conjecture can fail then it must be refuted by some large cardinal axiom. But the Ω Conjecture *holds* in all the enlargements of L produced by the Inner Model Program and so to the extent this program succeeds in analyzing large cardinal axioms, *no large cardinal axiom can refute the Ω Conjecture*.

10. Extender models, supercompact cardinals, and HOD

It is Scott's theorem that if $V = L$ then there are no measurable cardinals which necessitates the search for generalizations of the definition of L in which large cardinal axioms can hold. This is reinforced by Gödel's theorem that shows that if $V = L$ then one cannot have the true theory of the projective sets: projective determinacy must fail and moreover there are pathological projective sets.

But how should one enlarge L? The enlargements are of the form $L[\tilde{E}]$ for some set (or class) $\tilde{E}$. The problem is to identify sets $\tilde{E}$ for which $L[\tilde{E}]$ is a generalization of L from the perspective of definability. Since the issue is large cardinal axioms, these sets should somehow be derived from large cardinals. The relevant key notion is that of an *extender*, the modern formulation is due to Jensen and based on an earlier formulation due to Mitchell. There are precursors due to Powell (in a model theoretic context) and to Jensen, see [10] for more details. To simplify this exposition I deviate from the standard definition of an extender and use a definition which is in some ways more restricted, in other ways more general, but in all ways less technical to state.

Definition 27. A function, $E : \mathcal{P}(\gamma) \to \mathcal{P}(\gamma)$ where γ is an ordinal, is an *extender* of length γ if there exists an elementary embedding $j : V \to M$ such that

1. $\mathrm{CRT}(j) < \gamma$ and $V_{\gamma+\omega} \subseteq M$,
2. for all $A \subseteq \gamma$, $E(A) = j(A) \cap \gamma$. □

If E is an extender it is convenient to define $\mathrm{CRT}(E) = \mathrm{CRT}(j)$ where $j : V \to M$ witnesses that E is an extender. This is well-defined and $\mathrm{CRT}(E)$ is easily computed from E itself.

Let's look at an example. Suppose $j : V \to M$ is an elementary embedding with $\mathrm{CRT}(j) = \kappa$ and with $V_{\kappa+\omega} \subset M$. Recall that the associated ultrafilter μ on κ is defined by $A \in \mu$ if $\kappa \in j(A)$. Let

$$E : \mathcal{P}(\kappa + 1) \to \mathcal{P}(\kappa + 1)$$

be the associated extender of length $\kappa + 1$, this is the shortest possible extender defined from j. Then $L[E] = L[\mu]$. Unfortunately if $\kappa < \gamma \leq j(\kappa)$, $V_{\gamma+\omega} \subset M$, and if E is the associated extender of length γ, then nothing changes; $L[E]$ is closed under E and $L[E] = L[\mu]$. Therefore $L[E]$ where E is a single extender is not a rich enough enlargement of L. For this reason (and others) one must use sequences $\tilde{E}$ of extenders and the sequence must be suitably repackaged so that for example $L[\tilde{E}]$ is actually closed under the extenders on the sequence.

Definition 28. Suppose $\tilde{E} \subset \mathrm{Ord} \times V$ and for each $\alpha \in \mathrm{dom}(\tilde{E})$ let $E_\alpha = \{b \mid (\alpha, b) \in \tilde{E}\}$. Then $L[\tilde{E}]$ is an *extender model* if for each $\alpha \in \mathrm{dom}(\tilde{E})$ there exists an extender E such that

$$E_\alpha \cap L[\tilde{E}] = \{(A, \eta) \mid A \in \mathrm{dom}(E) \text{ and } \eta \in E(A)\} \cap L[\tilde{E}]. \qquad \square$$

In general the difficulty in constructing an extender model $L[\tilde{E}]$ is arranging that for enough $\alpha \in \mathrm{dom}(\tilde{E})$, there exists an extender E *of the inner model* $L[\tilde{E}]$ (so satisfying the definition that I have given of an extender but as interpreted within $L[\tilde{E}]$) such that

$$E_\alpha \cap L[\tilde{E}] = \{(A, \eta) \mid A \in \mathrm{dom}(E) \text{ and } \eta \in E(A)\},$$

while simultaneously arranging that $L[\tilde{E}]$ is canonical. It is the tension between these two goals which generates the difficulties. I have not defined what it means for $L[\tilde{E}]$ to be canonical but there is an easily stated variation of this requirement that modulo possibly passing to a Cohen extension of V can be imposed with no additional difficulty, it is the requirement that the sets in $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})[\tilde{E}]$ be universally Baire or even just that $L(\mathbb{R})[\tilde{E}] \vDash \mathrm{AD}$.

In fact the generalizations of L which have been constructed and which are evidently true generalizations of L are not exactly in the form of extender models as defined here but they contain such extender models in the sense that if N is the constructed enlargement of L then there is an extender model $L[\tilde{E}]$ such that $L[\tilde{E}] \subseteq N$ and such that the large cardinal axioms witnessed to hold in $L[\tilde{E}]$ by $\tilde{E}$ closely approximate the large cardinal axioms for which N was constructed to realize. This justifies using the simplified definition here which is really just for this exposition. The actual definitions are rather complicated and they involve an elaborate induction in which one has to show that the definitions are not vacuous. I note that this machinery underlies quite a number of theorems (including the equivalence theorem, Theorem 12) and so there are compelling mathematical reasons to pursue such developments besides the foundational reasons I have focused on.

The critical large cardinal notion is due to Reinhardt and Solovay from nearly 40 years ago. The definition below is based on a reformulation due to Magidor.

Definition 29. A cardinal δ is a *supercompact cardinal* if for all $\gamma > \delta$ there exist $\bar{\gamma} < \delta$ and an elementary embedding $j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$ such that $j(\bar{\delta}) = \delta$ where $\bar{\delta} = \mathrm{CRT}(j)$. $\square$

This definition of a supercompact cardinal is closely related to the definition of an extendible cardinal which is due to Reinhardt and which is a much stronger large cardinal notion.

Definition 30. A cardinal δ is an *extendible cardinal* if for all $\gamma > \delta$ there exist $\bar{\gamma} > \delta$ and an elementary embedding $j : V_{\gamma+1} \to V_{\bar{\gamma}+1}$ such that $\mathrm{CRT}(j) = \delta$ and $j(\delta) > \gamma$. $\square$

If E is an extender then E can be used to construct an elementary embedding $j_E : V \to M_E$ as a direct limit of elementary embeddings given by ultrapowers. If $\mathcal{E}$ is a collection of extenders then $\mathcal{E}$ *witnesses* that δ is a supercompact cardinal if the elementary embeddings $j_E|V_\alpha$ where $E \in \mathcal{E}$ suffice to witness the definition above that δ is supercompact. I can now state the recent theorems which show that by the level of exactly one supercompact cardinal something remarkable happens [29].

Suppose N is a transitive class. Then $\mathcal{E}(N{:}V)$ denotes the class of all $F \cap N$ such that F is an extender, $F \cap N \in N$, and such that $F \cap N$ is an extender in N.

Theorem 31. *Suppose $L[\tilde{E}]$ is an extender model such that δ is supercompact in $L[\tilde{E}]$ and this is witnessed in $L[\tilde{E}]$ by $\mathcal{E}(L[\tilde{E}]{:}V)$. Suppose F is an extender of strongly inaccessible length such that $L[\tilde{E}]$ is closed under F and such that* $\mathrm{CRT}(F) \geq \delta$. *Then $F \cap L[\tilde{E}] \in L[\tilde{E}]$.* □

I note that L is closed under *all* extenders and more generally if N is any transitive class which contains the ordinals and if sufficient large cardinals exist in V then necessarily N is closed under F for a rich class of extenders F of strongly inaccessible length. Therefore the requirements that $L[\tilde{E}]$ be closed under F and F have strongly inaccessible length are not a very restrictive requirements.

To date the basic methodology of extender models is such that if F is an extender in $L[\tilde{E}]$, then F is given by an intial segment of an extender on the sequence $\tilde{E}$, [22]. This has always seemed an essential feature of the detailed analysis of $L[\tilde{E}]$ and it is closely related to why each new construction of an extender model has come with an associated generalization of Scott's Theorem (that there are no measurable cardinals in L). The previous theorem easily yields a complete reversal of this at the level[1] of one supercompact cardinal. For example if $\kappa > \delta$ is an extendible cardinal then κ must be a supercompact cardinal in $L[\tilde{E}]$ and this generalizes to essentially all large cardinal notions.

The next theorem–which is also closely related to the previous theorem–gives yet another measure of the transcendence of extender models at the level of one supercompact cardinal. These extender models, at least in a background universe of sufficient large cardinal strength, must correctly compute the proof relation for Ω-logic.

Theorem 32. *Suppose there is a proper class of extendible cardinals. Suppose $L[\tilde{E}]$ is an extender model such that δ is supercompact in $L[\tilde{E}]$ and this is witnessed in $L[\tilde{E}]$ by $\mathcal{E}(L[\tilde{E}]{:}V)$. Then for all rank-universal sentences ϕ the following are equivalent.*

(1) $\vdash_\Omega \phi$.

(2) $L[\tilde{E}] \vDash$ "$\vdash_\Omega \phi$". □

I require another definition due to Gödel. This definition is of the class HOD of all hereditarily ordinal definable sets and here I give an equivalent reformulation of Gödel's definition which highlights it as some sort of merge of the definitions of the cumulative hierarchy and that of L.

Definition 33. HOD is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $A \subseteq \alpha$ such that A is definable in V_α without parameters and such that $X \in L[A]$. □

If $V = L$ then $\mathrm{HOD} = L$ but if for example $L(\mathbb{R}) \vDash \mathrm{AD}$ then $\mathrm{HOD} \neq L$. The class HOD is not in general canonical, for example by passing to a Cohen extension of V one can arrange that *any* designated set of V be an element of HOD as defined in the extension.

There is a remarkable theorem of Vopenka which connects HOD and Cohen's method of forcing, see [10]. This theorem illustrates why Cohen's method is so central in Set Theory and for reasons other than simply establishing independence results. If $G \subset \mathrm{Ord}$ then HOD_G is simply HOD defined allowing G as a parameter (so $G \in \mathrm{HOD}_G$).

Theorem 34 (Vopenka). *For each set $G \subset \mathrm{Ord}$, if $G \notin \mathrm{HOD}$ then HOD_G is a Cohen extension of* HOD. □

[1]There are some very recent developments which show that the critical stage is actually below the level of one supercompact cardinal and which plausibly identify exactly the critical stage.

With these definitions I can pose two questions any positive solution to which will likely involve the successful extension of the Inner Model Program to the level of one supercompact cardinal–in the sense of producing (subject to the requirements of the program) an extender model $L[\tilde{E}]$ such that for some δ, δ is supercompact in $L[\tilde{E}]$ and this is witnessed in $L[\tilde{E}]$ by $\mathcal{E}(L[\tilde{E}]{:}V)$. For the first question it is entirely possible that there be a positive solution obtained by other means but not (I believe) for the second question.

> *Suppose that there is a proper class of Woodin cardinals and that δ is an extendible cardinal. Must $\mathcal{E}(\mathrm{HOD}, V)$ witness in* HOD *that δ is a supercompact cardinal?*

A positive solution to this question would have significant consequences in Set Theory independent of whether the solution has anything to do with the Inner Model Program.

I give an example. Consider the ultimate large cardinal axiom–that of the existence of a Reinhardt cardinal–which asserts there is a nontrivial elementary embedding $j : V \to V$. This axiom as I have noted is refuted by the Axiom of Choice. But is the axiom consistent with ZF? The axiom AD is refuted by the Axiom of Choice and yet as we have seen, not only is ZF + AD consistent (the theory holds in $L(\mathbb{R})$), this theory is of fundamental interest (again because it holds in $L(\mathbb{R})$).

But the existence of a Reinhardt cardinal can be shown to *imply* the consistency *with* the Axiom of Choice of all the other large cardinal axioms I have mentioned [29]. So what can possibly provide the basis for the claim that the existence of Reinhardt cardinals is consistent with ZF? Certainly not their existence since that would deny the Axiom of Choice.

A positive answer to the question above would yield as a corollary that in ZF if there is a proper class of supercompact cardinals then *there are no Reinhardt cardinals* and very likely yield the outright nonexistence of Reinhardt cardinals–this would resolve a key foundational issue. More fundamentally and in an ironic twist since one early motivation of the study of the projective sets was a rejection of the Axiom of Choice, the positive answer will reveal deep connections between large cardinal axioms and *proving* instances of the Axiom of Choice [29].

The second question is a variant of the first question and the formulation involves the universally Baire sets. The positive solution to this question arguably must involve the successful extension of the Inner Model Program to the level of one supercompact cardinal.

> *Suppose that there is a proper class of Woodin cardinals and that there is a supercompact cardinal. Must there exist δ and an extender model $L[\tilde{E}]$ such that for all $x \in \mathbb{R} \cap L[\tilde{E}]$ there is a universally Baire set $A \subseteq \mathbb{R}$ such that:*
>
> (1) *δ is supercompact in $L[\tilde{E}]$ and this is witnessed in $L[\tilde{E}]$ by $\mathcal{E}(L[\tilde{E}]{:}V)$.*
>
> (2) *$x \in N$ where N is* HOD *as defined in $L(A, \mathbb{R})$?*

The answer to both questions is yes for Woodin cardinals. These are difficult constructions which evolved over 20 years and involved substantial contributions from quite a number of mathematicians. The basic definitions in their current form are due primarily to Mitchell and Steel over the period 1988-1999, [17] with revisions [26]. This work was based on earlier work of Martin and Steel [16], Dodd and Jensen (and others), and there is an alternative scheme which has subsequently been developed by Jensen, [9].

11. The axiom for ultimate-*L*

Even if one has identified the *construction* of ultimate-L this does not obviously yield the *axiom* that V is ultimate-L. This is in part because not all the extenders used in the construction survive as extenders in the inner model. The isolation of the axiom requires a much deeper understanding of the construction and this is an important issue in the whole program which I have ignored until now.

By combining the three notions of universally Baire sets, relative constructibility, and HOD, I can formulate what I conjecture will be the axiom that V is ultimate-L. I do this in the context that there is a supercompact cardinal and a proper class of Woodin cardinals though the latter is ultimately irrelevant.

The formulation of this axiom involves one last definition. Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

The connection between the determinacy of the projective sets and Woodin cardinals generalizes to a structural connection illustrated by the following theorem where $\mathrm{HOD}^{L(A,\mathbb{R})}$ denotes HOD as defined within $L(A, \mathbb{R})$.

Theorem 35 (Woodin). *Suppose that there is a proper class of Woodin cardinals and that A is universally Baire. Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in* $\mathrm{HOD}^{L(A,\mathbb{R})}$. □

The connection runs much deeper as indicated by the following theorem of Steel and I now refer to extender models in their true form (and moreover expanded to include elementary substructures) and not the simple approximation that I have defined previously. The Mitchell-Steel extender models are the solutions of the Inner Model Program at the level of Woodin cardinals which I alluded to in the discussion after the two test questions.

Theorem 36 (Steel). *Suppose that there is a proper class of Woodin cardinals. Let $\delta = \Theta^{L(\mathbb{R})}$. Then* $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ *is a Mitchell-Steel extender model.* □

Theorem 37 (Woodin). *Suppose that there is a proper class of Woodin cardinals. Then* $\mathrm{HOD}^{L(\mathbb{R})}$ *is* not *a Mitchell-Steel extender model.* □

But then what is $\mathrm{HOD}^{L(\mathbb{R})}$? It belongs to a different, previously unknown, class of extender models, these are the *strategic extender models*. For a significant initial segment of the universally Baire sets, $\mathrm{HOD}^{L(A,\mathbb{R})}$ has been verified to be a strategic extender model and there is very strong evidence that this will be true for all universally Baire sets. Until recently it was not clear at all what large cardinal axioms could hold in these models. But on the basis of the foundational questions which I have been discussing combined with associated mathematical developments, [29], there is compelling evidence (to me) that these inner models $\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\delta$ where $\delta = \Theta^{L(A,\mathbb{R})}$ *cannot* be limiting in any way: the only issue (assuming these are strategic extender models) is whether strategic extender models can exist at the level of one supercompact cardinal for then just as is the case for extender models, they are transcendent for large cardinals. There is absolutely compelling evidence that strategic extender models exist which *are* transcendent for Ω-logic in the sense of Theorem 32 and from this perspective it seems perhaps obvious that there must exist strategic extender models at the level of one supercompact cardinal as well. The underlying point here is that the family of inner models $\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\delta$ where $\delta = \Theta^{L(A,\mathbb{R})}$ and A is universally Baire *are* collectively transcendent for Ω-logic. Therefore if these inner models are strategic extender models then strategic extender models are transcendent for Ω-logic as well.

Extending the theory of extender models to the level of one supercompact cardinal seems difficult enough, why should there be any optimism that this can be done for strategic extender models the theory of which has generally been more difficult. There is a key and fundamental difference. The structure and theory of strategic extender models will be fully revealed by the inner models $\mathrm{HOD}^{L(A,\mathbb{R})}$ where A is universally Baire. So the mathematical problem is not one of finding the correct definition to satisfy a possibly vague goal, but rather of the analysis of structures *we can already define*. Moreover we have a rich framework provided by determinacy in which to undertake that analysis. I should emphasize that prior to the proof of Theorem 37, it was not known if strategic extender models could exist in any reasonable form.

I now come to my main conjecture which is the conjecture that the following axiom is the axiom that V is ultimate-L. The formulation of the axiom involves rank-existential sentences as opposed to rank-universal sentences. However it can be shown that the axiom is expressible as a rank-universal sentence modulo the indicated large cardinal hypothesis. There are natural refinements of the axiom, for example one can allow more complicated sentences and one can enlarge $L(A, \mathbb{R})$ to $L(\Gamma, \mathbb{R})$ where Γ is a suitable initial segment of the universally Baire sets.

Axiom. There is a proper class of Woodin cardinals. Further for each rank-existential sentence ϕ, if ϕ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \vDash \phi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$. □

Of course one could have made this conjecture independent of the recent results about the maximality of the inner model for one supercompact cardinal. But it is precisely these results which make this conjecture plausible and which provide a realistic scenario for proving that the axiom above is in fact the axiom that V is ultimate-L. In [28] and [29] a number of partial results concerning this conjecture are proved.

Far more speculative is the conjecture which I also make for all of the reasons discussed at length in [29]: *The axiom above is true*. By this I mean that the axiom will eventually be validated on the basis of accepted and compelling principles of infinity exactly as the axiom of Projective Determinacy has been validated.

This axiom implies the Continuum Hypothesis as well as the Ω Conjecture and together with its natural refinements will arguably reduce all questions of Set Theory to axioms of strong infinity and so banish the specter of undecidability as demonstrated by Cohen's method of forcing.

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