On the Question of Absolute Undecidability*

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The incompleteness theorems show that for every sufficiently strong consistent formal system of mathematics there are mathematical statements undecided *relative* to the system.[1] A natural and intriguing question is whether there are mathematical statements that are in some sense *absolutely* undecidable, that is, undecidable relative to any set of axioms that are justified. Gödel was quick to point out that his original incompleteness theorems did not produce instances of absolute undecidability and hence did not undermine Hilbert's conviction that for every precisely formulated mathematical question there is a definite and discoverable answer. However, in his subsequent work in set theory, Gödel uncovered what he initially regarded as a plausible candidate for an absolutely undecidable statement. Furthermore, he expressed the hope that one might actually prove this. Eventually he came to reject this view and, moving to the other extreme, expressed the

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[1] Strictly speaking this is Rosser's strengthening of the first incompleteness theorem. Gödel had to assume more than consistency.

hope that there might be a generalized completeness theorem according to which there are no absolutely undecidable sentences.

In this paper I would like to bring the question of absolute undecidability into sharper relief by bringing results in contemporary set theory to bear on it. The question is intimately connected with the nature of reason and the justification of new axioms and this is why it seems elusive and difficult. It is much easier to show that a statement is not absolutely undecidable than to show either that a statement is absolutely undecidable or that there are no absolutely undecidable statements. For the former it suffices to find and justify new axioms that settle the statement. But the latter requires a characterization (or at least a circumscription) of what is to count as a justification and it is hard to see how we could ever be in a position to do this. Some would claim that the proliferation of independence results relative to the standardly accepted axioms ZFC already vindicate such a position and that we must be content with studying the consequences of ZFC and taking a relativist stance toward systems that lie beyond.[2] Others would go further and claim that the independence results lend credibility to a general skepticism about the transfinite and undercut the support for ZFC itself.[3] In this paper I will take a non-skeptical stance toward set theory and assume that as far as its basic features are concerned the enterprise is legitimate. This is not because I think that the subject is immune to criticism. There are many coherent stopping points in the hierarchy of increasingly strong mathematical systems, starting with strict finitism and moving up through predicativism to the higher reaches of set theory. One always faces difficulties in arguing across the divide between coherent positions. This occurs already at the bottom with strict finitism—for example, it is hard to give a strict finitist such as Nelson (1986) a non-circular justification of the totality of exponentiation. But it is of interest to spell out each position and this is what I will be doing here for strong systems of set theory.

Starting with a generally non-skeptical stance toward set theory I will argue that there is a remarkable amount of structure and unity beyond ZFC and that a network of results in modern set theory make for a compelling case for new axioms that settle many questions undecided by ZFC. I will argue that most of the candidates proposed as instances of absolute undecidability have been settled and that there is not currently a good argument to the

[2]See Shelah (2003).
[3]See Feferman (1999).

effect that a given sentence is absolutely undecidable.

The plan of the paper is as follows. In §1 I will introduce the themes of the paper through a historical discussion that focuses on three stages of Gödel's thought: (1) The view of 1939 according to which there is an absolutely undecidable sentence. (2) The view of 1946 where Gödel introduces the program for large cardinal axioms and entertains the possibility of a generalized completeness theorem according to which there are no statements undecidable relative to large cardinal axioms. (3) The mature view in which Gödel broadens the program for new axioms and gives his most forceful statements about the nature and power of reason in mathematics.[4] In the remainder of the paper these views will be clarified and assessed in light of modern developments in set theory.[5] In §2 I take up the view of 1939, where Gödel appears to have restricted his attention to "intrinsic" justifications of new axioms. I give a precise circumscription of the view in terms of "reflection principles" and state a theorem which shows that on this reconstruction the sentence Gödel proposed is indeed absolutely undecidable relative to the limited view he held. In §3 I turn to the later views which involve "extrinsic" justifications of new axioms. I argue that a network of theorems make for a compelling case for new axioms that settle many of the statements undecided by ZFC and, moreover, that there is a precise sense in which Gödel's program for large cardinals is a complete success "below" the sentence he proposed as a test case—the continuum hypothesis. In §4 I examine recent work of Hugh Woodin on the continuum hypothesis, which involves going "beyond" large cardinal axioms. Finally, in §5 I give a reconstruction of Gödel's view of 1946 in terms of the "logic of large cardinals", summarize where we now stand with regard to absolute undecidability and look at three possible scenarios for how the subject might unfold.[6]

I have tried to write the paper in such a way that the major ideas and arguments can be understood without knowing more than the basics of set theory. Most of the more technical material has been placed in parentheses or footnotes or occurs in the statements of various theorems. This material

[4]The mature view has its roots in 1944 but is primarily contained in texts that date from 1947 onward.

[5]For a related discussion, one that contains a more comprehensive account of the development of Gödel's views on absolute undecidability and that treats of a number of similar modern themes, see Kennedy & van Atten (2004).

[6]Gödel's views are also discussed from a contemporary perspective in Koellner (2003) although there the emphasis is on intrinsic rather than extrinsic justifications.

can be skimmed on a first reading since I have paraphrased most of it in non-technical terms in the surrounding text.[7]

1 Gödel on New Axioms

1.1. *Relative versus Absolute Undecidability.* The inherent limitations of the axiomatic method were first brought to light by the incompleteness theorems.[8] Consider the standard axiomatization PA of arithmetic and let Con(PA) be the arithmetical statement expressing the consistency of PA. In the context of this formal system the second incompleteness theorem states:

THEOREM 1. (Gödel, 1931) *Assume that* PA *is consistent. Then* PA $\nvdash$ Con(PA).[9]

If one strengthens the assumption to the *truth* of PA then the conclusion can be strengthened to the relative undecidability of Con(PA).[10] There is a sense, however, in which such instances of incompleteness are benign since to the extent that we are justified in accepting PA we are justified in accepting Con(PA) and so we know how to expand the axiom system so as to overcome the limitation.[11] The resulting system faces a similar difficulty but we know how to overcome that limitation as well, and so on.

There are other, more natural, ways of expanding the system in a way that captures the undecided sentence. Let us consider two. For the first note that implicit in our acceptance of PA is our acceptance of induction for any meaningful predicate on the natural numbers. So we are justified

[7]For unexplained notation and further background see Jech (2003) and Kanamori (1997).

[8]In my reading of Gödel's unpublished manuscripts I have benefited from Parsons (1995), the editors' introductions to Gödel (1995), and Kennedy & van Atten (2004). See the latter for a detailed discussion of the development of Gödel's views on absolute undecidability.

[9]Here the consistency predicate is assumed to be *standard* in that it derives from a formalized provability predicate that satisfies the Hilbert-Bernays-Löb derivability conditions and involves a Σ^0_1-enumeration of the axioms. This general form of the theorem is due to Feferman (1960), as is the result that the theorem can fail if the above conditions are not met.

[10]It suffices to strengthen the assumption to the 1-consistency of PA.

[11]To the skeptic who doubts that we are justified in accepting PA one can remark that if Con(PA) is indeed an instance of a limitation—that is, if it is independent—then it must be true since PA is Σ^0_1-complete and Con(PA) is a Π^0_1-statement.

in accepting the system obtained by expanding the language to include the truth predicate and expanding the axioms by adding the elementary axioms governing the truth predicate and allowing the truth predicate to figure in the induction scheme. The statement Con(PA) is provable in the resulting system. This procedure can be iterated into the transfinite in a controlled manner along the lines indicated in Feferman (1964) and Feferman (1991) to obtain a system which (in a slightly different guise) is known as *predicative analysis*.

The second approach is even more natural since it involves moving to a system that is already familiar from classical mathematics. Here we simply move to the system of next "higher type", allowing variables that range over subsets of natural numbers (which are essentially real numbers). This system is known as *analysis* or *second-order arithmetic*. It is sufficiently rich to define the truth predicate and sufficiently strong to prove Con(PA) and much more. One can then move to *third-order arithmetic* and so on up through the hierarchy of higher types. (For the purposes of this paper it will be convenient to use the *cumulative* hierarchy of types defined by letting $V_0 = \varnothing$, $V_{\alpha+1} = \mathscr{P}(V_\alpha)$, and $V_\lambda = \cup_{\alpha<\lambda} V_\alpha$ for limit ordinals λ. The universe of sets V is defined to be $\cup_{\alpha<\mathrm{ORD}} V_\alpha$ where ORD is the class of ordinals. The first infinite stage V_ω of this hierarchy is essentially the set of natural numbers and the theory of this stage is essentially first-order arithmetic; the next stage $V_{\omega+1}$ is essentially the set of real numbers and the theory of this stage is essentially second-order arithmetic, and so on. Thus to pass up through the higher orders of arithmetic is to pass through the stages of set theory.) This second approach is much stronger than the first. Already at the first stage of the process (i.e. in second-order arithmetic) one can prove the consistency of predicative analysis and settle fairly natural arithmetical sentences (such as Harvey Friedman's finite form of Kruskal's theorem) that are known to be beyond the reach of predicative analysis.

It is of interest to note that Gödel knew much of this quite early on and that in the second installment of his incompleteness paper (which never appeared) he had planed to take the second approach. Gödel alluded to this already in his original incompleteness paper but he was more explicit in unpublished manuscripts and in his correspondence. For example, Gödel (*1931?) says of his undecidable arithmetical sentence that it is

> not at all absolutely undecidable; rather, one can always pass to "higher" systems in which the sentence in question is decidable

> ... In particular, for example, it turns out that analysis is a system higher in this sense than number theory, and the axiom system of set theory is higher still than analysis. (p. 35)

That this involves the definition of 'truth' is made clear in two letters to Carnap. In a letter of Sept. 11, 1932 Gödel says that "in the second part of my work I will give a definition of 'truth'" and in a letter of Nov. 28, 1932 he continues, saying that "with its help one can show that undecidable sentences become decidable in systems which ascend farther in the sequence of types." Thus, although the above instances are undecidable *relative* to a system they are not *absolutely* undecidable.

1.2. *Candidates for Absolutely Undecidable Sentences.* The trouble is that once we move beyond arithmetic to analysis and set theory, the vastly greater expressive resources raise the possibility of sentences that are not decided at *any* level. We will focus on three candidates for absolutely undecidable sentences.

To describe the first candidate we will need to invoke the notion of a *projective* set of reals and the stratification of the projective sets of reals into the subclasses $\utilde{\Sigma}^1_1$, $\utilde{\Sigma}^1_2$, ..., $\utilde{\Sigma}^1_n$, ... The details of this classification will not be important. The important point is that these are "simple" sets of reals and that the stratification is one of increasingly complexity.[12] The early French and Russian analysts studied the projective sets and established

[12] Here are some further details: For the purposes of this paper we will regard the reals as elements of ω^ω, that is, as infinite sequences of natural numbers. We will also use the more familiar notation '$\mathbb{R}$' for ω^ω. As a topology on ω^ω we take the product topology of the discrete topology on ω. As a topological space ω^ω is homeomorphic to the standard space of irrationals. In addition to this space we will also be interested in the n-dimensional product spaces $(\omega^\omega)^n$. Given a subset A of $(\omega^\omega)^{n+1}$ the *complement* of A is just the set of elements not in A and the *projection* of A is the result of "projecting" A onto the space $(\omega^\omega)^n$ by "erasing" the last coordinate. The simplest sets of such a space are the closed sets. From these we can obtain more complex sets by iteratively applying the operations of complementation and projection. The $\utilde{\Sigma}^1_1$ sets are the projections of closed sets and the $\utilde{\Sigma}^1_{n+1}$ sets are the projections of complements of $\utilde{\Sigma}^1_n$ sets. The $\utilde{\Pi}^1_n$ sets are the complements of $\utilde{\Sigma}^1_n$ sets and a set is $\utilde{\Delta}^1_n$ if it is both $\utilde{\Sigma}^1_n$ and $\utilde{\Pi}^1_n$. The *projective* sets are the sets that are $\utilde{\Sigma}^1_n$ for some $n < \omega$. There is an equivalent classification in terms of definability. The projective sets of reals are the sets of reals that are definable (with real parameters) in second-order arithmetic. Here existential quantification over the reals corresponds to projection, negation corresponds to complementation, and the hierarchy $\utilde{\Sigma}^1_1$, $\utilde{\Sigma}^1_2$, ... parallels the classification of formulas in terms of quantifier complexity. We will also use the notation Σ^1_1, $\utilde{\Sigma}^1_1$ etc. to classify the corresponding sentences. When the symbol '$\sim$' is absent this indicates that real parameters are not allowed.

some of their basic properties. For example, in 1917 Luzin showed that the $\underset{\sim}{\Sigma}^1_1$ sets are Lebesgue measurable. However, it remained open whether all of the projective sets are Lebesgue measurable. Indeed this problem proved so intractable that Luzin (1925) was led to conjecture that it is absolutely undecidable, saying that "one does not know *and one will never know* whether it holds". Our first candidate for an absolutely undecidable statement is thus the statement PM that all projective sets of reals are Lebesgue measurable.

Our second candidate is as old as set theory itself. This is Cantor's continuum hypothesis (CH), which says that for every infinite set X of reals there is either a one-to-one correspondence between X and the natural numbers or between X and the real numbers. Cantor showed that there is no closed set of reals that is a counter example to CH and in 1917 Luzin improved this by showing that there is no $\underset{\sim}{\Sigma}^1_1$ counter-example. However, it remained open whether there is a projective counter-example and whether there is any counter-example whatsoever. This problem resisted the efforts of many people (including Hilbert) and Skolem (1923) conjectured that CH is relatively undecidable, saying that "it is quite probable that what is called the continuum problem ... is not solvable at all on this basis [that is, on the basis of Zermelo's axioms]". Many today have gone further in maintaining that CH is absolutely undecidable.

Skolem's conjecture was borne out by the following companion results of Gödel and Cohen:

THEOREM 2. (Gödel, 1938) *If* ZFC *is consistent then* ZFC + CH *is consistent.*

THEOREM 3. (Cohen, 1963) *If* ZFC *is consistent then* ZFC + ¬CH *is consistent.*

The first result is proved via the method of *inner models*, in this case by using the class L of constructible sets. This class is defined much like V except that in passing from one stage to the next, instead of taking all *arbitrary* subsets of the previous stage one takes only those which are *definable* with parameters. This brings us to our third candidate, namely, the statement $V = L$ asserting that all sets are constructible. Gödel showed that if ZFC is consistent then so is ZFC + $V = L$ and moreover that the latter implies CH. This gives the first result. The second result is proved via Cohen's more radical method of *forcing* or *outer models*. Here one uses a partial order $\mathbb{P}$ in V to approximate a generic object $G \subseteq \mathbb{P}$ and a generic extension $V[G]$. This is done in such a way that truth in $V[G]$ can be controlled in V and, by varying the choice of $\mathbb{P}$, one can vary the features of $V[G]$. Cohen used this method to construct models of ZFC + $V \neq L$ and ZFC + $\neg$CH, thus completing the proof that $V = L$ and CH are independent of ZFC. These dual methods have been used to show that a host of problems in mathematics are independent of ZFC. For example, Gödel showed that in L there is a Σ^1_2 well ordering of the reals and so this inner model satisfies ZFC + $\neg$PM; and (assuming an inaccessible) Solovay constructed an outer model satisfying ZFC + PM. Other notable examples of statements that are independent of ZFC are Suslin's hypothesis, Kaplanski's conjecture and the Whitehead problem in group theory. All of these statements are candidates for absolutely undecidable sentences.

The above statements of analysis and set theory differ from the early arithmetical instances of incompleteness in that their independence does *not* imply their truth. Moreover, it is not immediately clear whether they are settled at any level of the hierarchy. They are much more serious cases of independence. The question is whether they are instances of absolute undecidability and, if so, how one might go about showing this.

1.3. *The View of 1939: Absolute Undecidability.* Initially Gödel thought that it was "very likely" that $V = L$ is absolutely undecidable and he seems to have thought that one could show this. In his *1939b he says that

> the consistency of the proposition [$V = L$] (that every set is constructible) is also of interest in its own right, especially because it is very plausible that with [$V = L$] one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems, similar to Euclidean and non-Euclidean geometry. (p. 155)

Similar remarks appear in his *193? and *1940a. In *193? the discussion centers around Hilbert's conviction that "for any precisely formulated mathematical question a unique answer can be found", which Gödel elaborates informally as: "Given an arbitrary mathematical proposition A there exists a proof either for A or for not-A, where by "proof" is meant something which starts from evident axioms and proceeds by evident inferences". He then notes that "formulated in this way the problem is not accessible for mathematical treatment because it involves the non-mathematical notion of evidence." So one must render the notion of "proof" mathematically precise. He first does this in terms of provability in a given formal system and argues (as we have above) that when regimented in this way "the conviction about which Hilbert speaks remains entirely untouched" by his incompleteness results since the statements in question are "always decidable by evident inferences not expressible in the given formalism" (p. 164). However, he goes on to say that there are "[q]uestions connected with Cantor's continuum hypothesis" which "very likely are really undecidable." He concludes by saying: "So far I have not been able to prove their undecidability, but there are considerations which make it highly plausible that they really are undecidable" (175). In *1940a Gödel starts by saying that "A is very likely a really undecidable proposition (quite different from the undecidable proposition which I constructed some years ago and which can always be decided in logics of higher types)." Here 'A' is Gödel's abbreviation for "all reals are constructible" (and also for $V = L$). He then says that he can prove that "[e]ither A is absolutely undecidable or Cantor's continuum hypothesis is demonstrable" but that he has "not been able to determine which of these two possibilities is realized" (185). So it appears that Gödel thought that one might be in a position to establish that A and $V = L$ are absolutely undecidable.

It is difficult to see what he could have hoped to prove. The trouble is it would appear that any precise characterization of the notion of absolute provability would fall short of the full notion since one would be able to "diagonalize out" as in the construction of the Gödel sentence. This, however, does not rule the possibility of encompassing the notion. The strategy would be to give a precise characterization of a notion that encompassed the notion of absolute provability and then prove a theorem to the effect that $V = L$ is beyond the reach of this notion. In Section 2, I will suggest a reconstruction along these lines, one that seems to be faithful to the limited view that Gödel held at the time.

1.4. *The View of 1946: Generalized Completeness.* The notion of absolute provability (referred to by Gödel as 'absolute demonstrability') is revisited in his Princeton address of 1946. His model is Turing's analysis of computability which has the feature that "[b]y a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion" (p. 150). After noting that any particular formalism can be transcended and that "there cannot exist any formalism which would embrace all these steps", Gödel says that "this does not exclude that all these steps . . . could be described and collected together in some non-constructive way." He continues:

> In set theory, e.g., the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinatorial and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. (p. 151)

This is a natural idea. Earlier we saw that the arithmetical instances of undecidability that arise at one stage of the hierarchy are settled at the next. We then expressed the concern that there might be statements of analysis or set theory that are not settled at "any" stage of the hierarchy. In saying this we were not precise about just what stages of the hierarchy there are. Large cardinal axioms make this more precise by asserting that there are stages V_α with certain "largeness" properties. These axioms are intrinsically plausible and provide a canonical way of climbing the hierarchy of consistency strength. Some of the standard large cardinals (in order of increasing (logical) strength) are: inaccessible, Mahlo, weakly compact, indescribable, Erdös, measurable, strong, Woodin, supercompact, huge, etc.[13]

Gödel goes on to say of such a concept of provability that it "might have the required closure property, i.e., the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory (i.e., any proof involving the concept of truth which I just used) is replaceable by a proof from such an axiom of infinity." Furthermore, he entertains the possibility of a generalized completeness theorem:

> It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets. (Gödel (1946, p. 151))

Thus as an absolute concept of provability he proposes "provability from (true) large cardinal axioms". So Gödel went from thinking in 1939 that it was very likely that $V = L$ is absolutely undecidable (and that there was a bifurcation in set theory) to thinking that there might be *no* absolutely undecidable sentences.

[13]We will not be able to discuss large cardinal axioms in detail. See Kanamori (1997) for further details.

1.5. *The Program for New Axioms.* Let us call the program of using large cardinal axioms to settle questions undecided in ZFC the *program for large cardinal axioms*. If successful such a program would reduce all questions of set theory to questions concerning large cardinals. The question of how one might establish a "true assertion about the largeness of the universe" is touched on in his 1944 and taken up in the 1947/1964 paper on the continuum hypothesis. Gödel distinguishes between *intrinsic* and *extrinsic* justifications. In the first version of the paper intrinsic justifications are taken to involve an analysis of the concept of set and lead to "small" large cardinals such those asserting the existence of inaccessible cardinals and Mahlo cardinals. Regarding axioms asserting the existence of "large" large cardinals such as measurable cardinals he says it has not yet been made clear "that these axioms are implied by the general concept of set in the same sense as Mahlo's". However, he holds out hope that "there may exist, besides the usual axioms, the axioms of infinity, and the axioms mentioned in footnote 18, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts" (Gödel (1964), p. 261, revised footnote of September 1966). In the later version of the paper intrinsic justifications are elaborated in terms of rational intuition. In both versions the scope of intrinsic methods is held to be potentially broader than in his earlier writings and leads to the more general *program for new axioms*.

Extrinsic justifications are discussed in both versions of the paper. They were discussed already in Gödel (1944). Here Gödel embraces Russell's regressive method for discovering the axioms, according to which

> the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced. ... I think that ... this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. (p. 127)

This view is elaborated on in the paper on the continuum problem:

> ... even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely,

> inductively by studying its "success". Success here means fruitfulness in consequences, in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axioms, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. ... There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. (Gödel (1964, p. 261))

In his *1961/? Gödel upheld "the belief that for clear questions posed by reason, reason can also find clear answers" (p. 381). And in a letter of Sept. 29, 1966 to Church, he wrote:

> I disagree about the philosophical consequences of Cohen's result. In particular I don't think realists need expect any permanent ramifications ... as long as they are guided, in the choice of axioms, by mathematical intuition and by other criteria of rationality. (Gödel (2003, p. 372))

In the end, it was his belief in extrinsic justifications and the scope of reason that led Gödel to reject absolute undecidability and bifurcation in set theory.

In what follows I will speak of reason and evidence in mathematics but I want to use these notions in as general and neutral a fashion as possible. I do not wish to present a theory of reason or even to commit to such a theory, such as one involving the notion of rational intuition. Instead I want to bring together what I regard as the strongest reasons that we currently have for new axioms and consider some new candidates. My aim will be to convince the reader that the particular reasons have force, that in many instances (for example, in the cases of definable determinacy) the case is compelling and, looking ahead, that there are scenarios in which we would have a compelling case with regard to CH. These are thus reasons that any general theory of reason will have to accommodate.[14]

[14] For more on the notion of reason in the neutral sense I intend see Parsons (2000).

2 Limitations of Intrinsic Justifications

According to the view of 1939 the statement $V = L$ is very likely to be absolutely undecidable. Establishing this would involve two things. First, one would have to render the problem amenable to mathematical treatment by giving a precise circumscription of the concept of absolute provability. Second, one would have to prove a theorem to the effect that neither $V = L$ nor its negation is absolutely provable on this reconstruction.

2.1. *Systems of Arbitrarily High Type.* In his *1940a Gödel states that his "proof [of the consistency of A] goes through for systems of arbitrarily high type" and that "[i]t is to be expected that also $\neg A$ will be consistent with the axioms of mathematics", the reason being that the inconsistency of $\neg A$ would "imply an inconsistency of the notion of a random sequence ... and it seems very unlikely that this notion should imply a contradiction" (pp. 184–185). Notice that Gödel slides from "systems of arbitrarily high type" to "the axioms of mathematics". He thus implicitly identifies the need for new axioms with the need for axioms asserting the existence of higher and higher types, that is, with the need for large cardinal axioms.

Gödel does not discuss extrinsic justifications until 1944 and there is some evidence that in 1939 he took all justifications to be intrinsic. For example, in his statement of Hilbert's conviction (quoted in §1.3) he identifies the possibility of settling a mathematical question with deducing it from "evident axioms" by "evident inferences". During this period his most extended discussion of new axioms is in his *1933o. Here, in motivating the axioms of set theory he uses a bootstrapping method to successively extend the hierarchy to levels that satisfy ZFC and much more. He does not spell out the details but the approach appears to be driven by the idea (implicit in the concept of set) that the totality of levels is "absolutely infinite" and hence "indefinable". Now, the most straightforward way of rendering precise the idea that V is "indefinable" is in terms of "reflection principles". Roughly speaking such principles assert that anything true in V falls short of characterizing V in that it is true within some earlier level. Schematically, a reflection principle has the form

$$V \models \varphi(A) \;\rightarrow\; \exists\alpha\, V_\alpha \models \varphi^\alpha(A^\alpha)$$

where $\varphi^\alpha(\cdot)$ is the result of relativizing the quantifiers of $\varphi(\cdot)$ to V_α and A^α is the result of relativizing an arbitrary parameter A to V_α.[15] Let us

[15] There are other principles that are called 'reflection principles' such as the principles

consider the view that the only justifications of new axioms are intrinsic justifications and that these involve spelling out the idea that the hierarchy of types is "absolutely infinite", an idea which in turn is rendered precise in terms of "reflection principles". Thus, the notion of absolute provability will be explicated in terms of the notion of being provable from ZFC and (true) reflection principles. I am not claiming that this is exactly what Gödel had in mind. There is too little textual evidence. I intend it only as a rational reconstruction of his view, one that coheres with what he says and puts us in a position to say something precise about the purported absolute undecidability of $V = L$.

2.2. *Extent of Reflection.* In order to render the general form of a reflection principle precise we have to specify the language, the nature of the parameters, and the method of relativization. Let us do this in stages. Consider first the case of *first-order reflection* where the language and parameters are first-order. For a first-order parameter $A \in V$ and a first-order formula φ, $\varphi^{\alpha}(A^{\alpha})$ is the result of taking $A^{\alpha} = A$ and interpreting the quantifiers in φ as ranging over V_{α}. This is how someone "living in V_{α}" would interpret $\varphi(A)$. If we let T be the axioms of ZFC with the axioms of Infinity and Replacement removed then it is a standard result that over T the first-order reflection scheme implies (and, in fact, is equivalent to) Infinity and Replacement, and so even these basic axioms of extent are subsumed by reflection principles.[16] Consider next the case of *second-order reflection*, where the language and parameters are second-order. For a second-order parameter $A \subseteq V$ and a second-order formula φ, $\varphi^{\alpha}(A^{\alpha})$ is the result of taking $A^{\alpha} = A \cap V_{\alpha}$ and interpreting the second-order quantifiers in φ as ranging over the subsets of V_{α}. Again, this is how someone "living in V_{α}" would interpret $\varphi(A)$. This principle yields inaccessible cardinals, Mahlo cardinals, weakly compact cardinals and more. One can continue up the higher-orders into the transfinite (while keeping the parameters of second-order) to obtain the so-called indescribable

of Reinhardt (which are more properly called 'extension principles') and modern "local reflection principles". Such principles are quite different than those discussed above. For a more comprehensive discussion see Koellner (2009a).

[16][Note added June 14, 2009. The version of the first-order reflection scheme involved in this standard result is different than the scheme discussed in the text. See Kanamori (1997), pp. 57–58. It is subsumed by the reflection principles discussed in the text when one allows higher-order parameters, which is our main focus.] In the second-order context one can formulate a theory that has as its models precisely the rank initial segments V_{α} of the universe; in this way *all* of the axioms of extent are subsumed by reflection principles. See Tait (2005a).

cardinals.[17] These principles exhaust those envisaged in Gödel's time.

Now, it is straightforward to show that such principles relativize to L and are preserved under small forcing extensions that violate $V = L$. Hence, if one takes the notion of absolute undecidability to be subsumed by these principles then, from this limited vantage point, $V = L$ really is absolutely undecidable.

One might try to go further and allow *parameters* of third and higher order but in doing so one immediately encounters inconsistency (assuming that one takes the natural course of inductively relativizing a higher-order parameter to the set consisting of the relativizations of its members). However, Tait has developed a workable theory with higher-order parameters by placing suitable restrictions on the language.[18] He shows that the resulting principles—the Γ_n-reflection principles—are stronger than those considered above (e.g. they imply the existence of ineffable cardinals) and are consistent relative to measurable cardinals. This leaves open the possibility that such principles might settle $V = L$. However, building on ideas of Reinhardt and Silver one can show the following:

THEOREM 4. *Assume that the Erdös cardinal $\kappa = \kappa(\omega)$ exists. Then there is a $\delta < \kappa$ such that V_δ satisfies Γ_n-reflection for all n.*[19]

Since such cardinals relativize to $V = L$ it follows that even with respect to this extended vantage point $V = L$ remains absolutely undecidable.

Perhaps intrinsic justifications of a different nature can overcome these limitations and secure axioms that violate $V = L$. This has not happened to date.[20] The kinds of justifications that have borne the most fruit and shown the greatest promise are *extrinsic* justifications.

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[17] There is a difficulty here in making sense of higher-order quantification over the entire universe of sets. Since Gödel's view of set theory involved an ontology of concepts (cf. Gödel (1964, fn. 18)) this would go some way to meeting this challenge. In any case, since our concern is with an upper bound on the view, let us take the liberal course of allowing such higher-order reflection principles.

[18] See Tait (1990), Tait (1998), and Tait (2005a).

[19] For $\alpha \geq \omega$ the *Erdös cardinal* $\kappa(\alpha)$ is the least κ such that $\kappa \to (\alpha)^{<\omega}_2$, that is, the least κ such that for each partition $P : [\kappa]^{<\omega} \to 2$ there is an $X \in [\kappa]^\alpha$ such that $\mathrm{Card}(P``[X]^n) = 1$ for all $n < \omega$.

[20] For more on the subject see Koellner (2009a).

3 Extent of the Program for Large Cardinals

In 1961 Scott showed that if one extends the axioms of ZFC by adding the axiom asserting the existence of a measurable cardinal then $V = L$ is refutable. This provided further hope that measurable cardinals might have some bearing on CH. This hope was soon dashed by a result of Levy and Solovay:

Theorem 5. (Levy and Solovay, 1967) *Suppose that κ is a measurable cardinal and $\mathbb{P}$ is a partial order such that $|\mathbb{P}| < \kappa$. Then if $G \subseteq \mathbb{P}$ is V-generic, then $V[G] \models$ "κ is measurable."*

Since the size of the continuum can be altered by forcing with such a "small" partial ordering $\mathbb{P}$ it follows that measurable cardinals cannot settle CH. Moreover, the argument generalizes to show that none of the familiar large cardinal axioms can settle CH.[21] Thus there can be no generalized completeness theorem of the sort Gödel entertained in 1946 and the program for large cardinals must be considered a failure at the level of CH.[22]

The remarkable fact is that the program for large cardinals has been a very successful "below CH" (in a sense to be made precise). So, in choosing CH as a test case for the program for large cardinals, Gödel put his finger on precisely the point where it breaks down. The first purpose of this section is to present a strong extrinsic case for new axioms. The second purpose is to make precise the above claim that the program for large cardinals has been a success "below" CH.[23]

[21] It is of interest to note that after learning of Cohen's method of forcing Gödel added a revised postscript in September 1966 to his 1947/1964 paper in which he says that "it seems to follow that the axioms of infinity mentioned in footnote 20 [which include axioms asserting the existence of measurable cardinals], to the extent to which they have so far been precisely formulated, are not sufficient to answer the question of the truth or falsehood of Cantor's continuum hypothesis" (p. 270).

[22] There might, however, be a new kind of large cardinal axiom that circumvents the result of Levy and Solovay and settles CH. In the final section we will discuss the notion of a large cardinal axiom in a more general setting and consider an axiom that has the flavour of a large cardinal axiom and may have the sensitivity to forcing necessary to have an influence on the size of the continuum.

[23] The approach I take owes much to Steel and Woodin—in particular, Steel (2000) and Woodin's Logic Colloquium 2000 lecture, published as Woodin (2005a)—and I am indebted to them for many helpful conversations. See also Hauser (2002). For alternative approaches see Foreman (2006) and Friedman (2006).

3.1. *Descriptive Set Theory.* The continuum hypothesis is a statement of third-order arithmetic—more precisely, it is a Σ^2_1 statement; it asserts the existence of a certain set of reals. The assessment of the program below CH will involve looking at sentences of lower complexity and (correspondingly) definable sets of reals. The most well known class of such sentences are those of second-order arithmetic, stratified into the hierarchy $\Sigma^1_0, \Sigma^1_1, \ldots, \Sigma^1_n, \ldots$. But this hierarchy can be continued into the transfinite while still remaining below Σ^2_1. This can be seen in terms of definable sets of reals. After the sets of reals definable (with real parameters) in second-order arithmetic—the projective sets—we have the sets of reals appearing at various levels of $L(\mathbb{R})$—the result of starting with the reals and iterating the definable powerset into the transfinite.[24]

The study of definable sets of reals is known as *descriptive set theory.* The central idea underlying the subject is that definable sets of reals are well behaved. Some notable results in the classical period that illustrate this idea are: $\utilde{\Sigma}^1_1$ sets are Lebesgue measurable (Luzin, 1917), $\utilde{\Sigma}^1_1$ sets have the property of Baire (Luzin, 1917), $\utilde{\Sigma}^1_1$ sets have the perfect set property (Suslin, 1917), and every $\utilde{\Sigma}^1_2$ subset of the plane can be uniformized by a $\utilde{\Sigma}^1_2$ set (Kondô, 1937). These results are provable in ZFC but as we noted above the early analysts ran into obstacles in extending them to higher levels of the projective hierarchy and this led Luzin to conjecture that one would never know whether the projective sets are Lebesgue measurable.

In the modern era of descriptive set theory it was discovered that the above regularity properties (at a given level of complexity) are unified by a single property—the property of *determinacy* (at roughly the same level). For a set of reals A consider the game G_A where two players take turns playing natural numbers:

$$\begin{array}{lcccccc} \mathrm{I} & a_0 & & a_1 & & \ldots & \\ \mathrm{II} & & b_0 & & b_1 & & \ldots \end{array}$$

When the game is over the players will have cooperated in producing the real number $\langle a_0, b_0, a_1, b_1, \ldots \rangle$. We say that player I wins a round of the game if this number is in the set A; otherwise player II wins the round. The game G_A is said to be *determined* if either player has a "winning strategy", that

[24] The projective sets of reals are those appearing in the first stage of this process. The sets of reals appearing at the successive levels of $L(\mathbb{R})$ thus forms a transfinite extension of the projective sets.

is, a strategy which ensures that the player wins a round regardless of how the other player plays. The *Axiom of Determinacy* (AD) is the statement that for every set of reals A the game G_A is determined. A straightforward argument shows that AD contradicts AC and for this reason the axiom was never really considered as a serious candidate for a new axiom. There is, however, an interesting class of axioms that are consistent with AC, namely, the axioms of *definable determinacy.* These axioms assert that all sets of reals at a given level of complexity are determined, notable examples being $\underset{\sim}{\Delta}^1_1$-determinacy (all Borel sets of reals are determined), PD (all projective sets of reals are determined) and $\mathrm{AD}^{L(\mathbb{R})}$ (all sets of reals in $L(\mathbb{R})$ are determined).

Martin showed that $\underset{\sim}{\Delta}^1_1$-determinacy is provable in ZFC. This single principle unifies the results from two paragraphs back and lies at the heart of the remarkably rich structure theory of definable sets of reals that can be established in ZFC. Furthermore, it was discovered that stronger forms of definable determinacy lift this structure theory to more complex sets of reals.

Our reason for concentrating on axioms of definable determinacy is twofold. First, since they knit together the results of classical descriptive set theory they serve as a focal point in assessing the program for large cardinals—if large cardinal axioms imply definable determinacy at a given level then they imply *all* of the statements of the corresponding level of the structure theory. Second, axioms of definable determinacy are plausible candidates for new axioms and, as we shall see, the considerations in their favour are quite strong. The two examples that we shall focus on are PD and $\mathrm{AD}^{L(\mathbb{R})}$.

Let me mention three such considerations before turning to the connection with large cardinals. For definiteness let us concentrate on PD. The first consideration is that PD yields the most natural and straightforward generalization to the projective sets of the structure theory that can be established in ZFC—in particular, it implies PM and so, if justified, refutes Luzin's conjecture. A second consideration is that PD implies results that were subsequently verified in ZFC, thus providing the kind of confirmation discussed in §1.5.[25] A third consideration is that PD appears to be "effectively complete" in that it settles any statement (apart from the inevitable (but benign) forms of arithmetic incompleteness) of second-order arithmetic not settled by ZFC—indeed PD appears to be more complete with respect to second-order arithmetic than PA is with respect to first-order arithmetic in that there are no known analogues of "natural" mathematical instances of in-

[25]See Martin (1998) for further discussion.

dependence such as the Paris-Harrington theorem and Friedman's finite form of Kruskal's theorem. These three features—generalization, verifiable consequences, and effective completeness—are strong considerations in support of PD. Similar considerations apply to higher grades of definable determinacy.[26]

3.2. *Definable Determinacy and Large Cardinals.* The case for axioms of definable determinacy is further strengthened by the fact that they are implied by large cardinal axioms. In 1970 Martin showed that if there is a measurable cardinal then all $\undertilde{\Sigma}^1_1$ sets of reals are determined. Martin (1980) then showed that under the much stronger assumption of a non-trivial iterable elementary embedding $j : V_\lambda \to V_\lambda$ all $\undertilde{\Sigma}^1_2$ sets of reals are determined. This was dramatically improved by Woodin who showed that if there is a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point less than λ then all sets of reals in $L(\mathbb{R})$ are determined (and hence AD is consistent). The bound was then lowered by Woodin, building on a groundbreaking result of Martin and Steel:

Theorem 6. (Martin and Steel) *Assume there are infinitely many Woodin cardinals. Then* PD*.*

Theorem 7. (Woodin) *Assume there are infinitely many Woodin cardinals and a measurable cardinal above them all. Then* $\mathrm{AD}^{L(\mathbb{R})}$*.*

The pattern persists: Stronger large cardinal axioms imply richer forms of definable determinacy and inherit their consequences—in particular, they refute Luzin's conjecture.

Conversely, definable determinacy implies (inner models of) large cardinals.

Theorem 8. (Woodin) *Assume* $\mathrm{AD}^{L(\mathbb{R})}$*. Then there is an inner model N of* ZFC + "*There are ω-many Woodin cardinals*"*.*

One can also recover $\mathrm{AD}^{L(\mathbb{R})}$ from its consequences.

Theorem 9. (Woodin) *Assume that every set of reals in $L(\mathbb{R})$ is Lebesgue measurable and has the property of Baire and assume Σ^2_1-uniformization holds in $L(\mathbb{R})$. Then* $\mathrm{AD}^{L(\mathbb{R})}$*.*

[26]See Moschovakis (1980), Maddy (1988a), Maddy (1988b) and Jackson (2009) for more on the structure theory and the manner in which definable determinacy axioms lift it to higher levels of complexity.

A very striking instance of this phenomenon is the following:

Theorem 10. (Woodin) *Assume that* "$\mathrm{PA}_S + \Sigma^1_2$*-determinacy*" *is consistent. Then* "$\mathrm{BGC} + \mathrm{ORD}$ *is Woodin*" *is consistent.*

Here PA_S is second-order arithmetic with schematic comprehension and choice and BGC is the schematic form of ZFC due to Bernays and Gödel. The theorem says that even in the context of analysis, significant large cardinal strength is required in order to establish Σ^1_2-determinacy. The situation here differs markedly from analogous results in arithmetic in that when one shifts from arithmetic to analysis the examples of statements requiring large cardinal strength both become more natural and require significantly greater large cardinal strength.

To summarize: Large cardinals are *sufficient* to prove definable determinacy and (inner models of) large cardinals are *necessary* to prove definable determinacy.

3.3. *Generic Absoluteness.* Definable Determinacy is not an isolated occurrence. As noted earlier, definable determinacy carries with it the entire structure theory; moreover, it appears to be "effectively complete"—for example, PD seems to be "effectively complete" with respect to statements of analysis. It is now time to make this precise and substantiate it. We shall do this in terms of generic absoluteness, the paradigm result being the following theorem of ZFC:

Theorem 11. (Shoenfield) *Suppose* φ *is a* Σ^1_2 *sentence,* $\mathbb{P}$ *is a partial order and* $G \subseteq \mathbb{P}$ *is* V*-generic. Then*

$$V \models \varphi \quad \textit{iff} \quad V[G] \models \varphi.$$

The theorem is proved by showing that there are certain tree representations for Σ^1_2 sets of reals that are robust under forcing and act as "oracles for truth". One consequence of the theorem is that the independence of Σ^1_2 statements can never be established by forcing. Another is that we have here a partial realization of the idea that consistency implies existence since if one establishes such a statement to be consistent via forcing then it must be true.

Under large cardinal assumptions the situation generalizes. Martin and Solovay showed that if there is a proper class of measurable cardinals then Σ^1_3 truth is *frozen* or *generically absolute* (in the sense indicated above). Woodin pushed this further:

THEOREM 12. (Woodin) *Assume there is a proper class of Woodin cardinals. Suppose φ is a sentence, $\mathbb{P}$ is a partial order and $G \subseteq \mathbb{P}$ is V-generic. Then*

$$L(\mathbb{R}) \models \varphi \;\; \textit{iff} \;\; L(\mathbb{R})^{V[G]} \models \varphi.$$

This can be pushed even beyond $L(\mathbb{R})$. To explain this we will need to invoke the notion of a *universally Baire* set of reals. The details of this notion will not be important. The important point is that under large cardinal assumptions sets of reals beyond $L(\mathbb{R})$ are universally Baire and such sets are well behaved. Let Γ^∞ be the collection of sets of reals that are universally Baire[27] and for κ an infinite regular cardinal let $H(\kappa)$ be the set of all sets x such that the cardinality of the transitive closure of x is less than κ.

THEOREM 13. (Woodin) *Suppose there is a proper class of Woodin cardinals and $A \in \Gamma^\infty$. Suppose $G \subseteq \mathbb{P}$ is V-generic. Then*

$$(H(\omega_1), \in, A)^V \prec (H(\omega_1)^{V[G]}, \in, A_G).$$

That is, we have generic absoluteness for "projective-in-A" where A is universally Baire. In fact, one has "$\Sigma^2_1(\Gamma^\infty)$-generic absoluteness":

THEOREM 14. (Woodin) *Suppose there is a proper class of Woodin cardinals and let φ be a sentence of the form*

$$\exists A \in \Gamma^\infty \; (H(\omega_1), \in, A) \models \psi.$$

Suppose $G \subseteq \mathbb{P}$ is V-generic. Then

$$V \models \varphi \;\; \textit{iff} \;\; V[G] \models \varphi.$$

[27] Here are some further details: For a cardinal δ, a set $A \subseteq \mathbb{R}$ is *δ-universally Baire* if for all partial orders $\mathbb{P}$ of cardinality δ, there exist trees S and T on $\omega \times \lambda$ (for some λ) such that $A = p[T]$ and, if $G \subseteq \mathbb{P}$ is V-generic, then $p[T]^{V[G]} = \mathbb{R}^{V[G]} - p[S]^{V[G]}$. A set $A \subseteq \mathbb{R}$ is *universally Baire* if it is δ-universally Baire for all δ. Universally Baire sets have canonical interpretations in generic extensions $V[G]$: Choose any $T, S \in V$ such that $p[T] = A$ and $p[T]^{V[G]} = \mathbb{R}^{V[G]} - p[S]^{V[G]}$ and set $A_G = p[T]^{V[G]}$. The point is that A_G is independent of the choice of T and S. For suppose $\bar{T}, \bar{S} \in V$ are two other such trees. And suppose $p[T]^{V[G]} \neq p[\bar{T}]^{V[G]}$, say $p[\bar{T}]^{V[G]} \cap p[S]^{V[G]} \neq \varnothing$. Then, by absoluteness of wellfoundedness, $p[\bar{T}] \cap p[S] \neq \varnothing$, which is a contradiction. Universally Baire sets of reals also have strong closure properties. For example, Woodin showed that if there is a proper class of Woodin cardinals and $A \in \Gamma^\infty$ then (1) $L(A, \mathbb{R}) \models \mathrm{AD}^+$ and (2) $\mathscr{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \Gamma^\infty$. Here AD^+ is a (potential) strengthening of AD designed for models of the form $L(\mathscr{P}(\mathbb{R}))$. See Woodin (1999).

Stronger large cardinal axioms imply that many sets of reals beyond $L(\mathbb{R})$ are universally Baire. Let us call a set *absolutely* Δ^2_1 if there are Σ^2_1 formulas which define complementary sets of reals in all generic extensions. Woodin showed that if there is a proper class of measurable Woodin cardinals then all absolutely Δ^2_1 sets of reals are universally Baire.[28] This is one precise sense in which CH was an unfortunate choice of a test case for the program for large cardinals—large cardinal axioms effectively settle all questions of complexity strictly below (in the above sense) that of CH.[29]

Moreover, just as large cardinals are necessary for definable determinacy, definable determinacy is necessary for generic absoluteness.

Theorem 15. (Woodin) *Suppose there is a proper class of strongly inaccessible cardinals. Suppose that the theory of $L(\mathbb{R})$ is generically absolute. Then* $\mathrm{AD}^{L(\mathbb{R})}$.

A convenient (but tendentious) way to summarize this and the companion result above is as follows: Call a theory 'good' if it freezes the theory of $L(\mathbb{R})$.

(1) There is a good theory.

(2) All good theories imply $\mathrm{AD}^{L(\mathbb{R})}$.

3.4. *Inner Model Theory and the Overlapping Consensus.* Definable determinacy is implicated in an even *more* dramatic fashion. In a certain sense it is *inevitable*. This comes about through its intimate connection with inner models of large cardinal axioms.

Theorem 16. (Harrington, Martin) *The following are equivalent:*

(1) $\utilde{\Pi}^1_1$*-determinacy.*

(2) *For all $x \in \mathbb{R}$, $x^{\#}$ exists.*[30]

Theorem 17. (Woodin) *The following are equivalent:*

[28]Under the same hypothesis one has that all of the "provably-Δ^2_1" sets of reals are universally Baire.

[29]One might worry that what is really going on here is that large cardinal axioms throw a wrench into the forcing machinery. But this is not so. Under large cardinal assumptions one has more generic extensions. What is really going on is that large cardinal axioms generate trees that are robust and act as oracles for truth.

[30]For a definition of $x^{\#}$ see Jech (2003) or Kanamori (1997).

(1) PD.

(2) *For each $n < \omega$, there is a transitive ω_1-iterable model M such that*

$$M \models \text{“ZFC} + \textit{there exist } n \textit{ Woodin cardinals”}.$$

The equivalence of definable determinacy and inner models for large cardinals generalizes to higher levels.

This is striking. We first saw that large cardinal axioms imply definable determinacy and then that definable determinacy implies inner models of large cardinal axioms. Ultimately, we see that definable determinacy is *equivalent* to the existence of certain inner models of large cardinal axioms. It should be stressed that whereas definable determinacy axioms are simple, the formulation of the relevant inner models for large cardinals is extraordinarily complex; moreover, as far as surface features are concerned the two have nothing to do with each other. This ultimate *convergence* of two entirely distinct domains is evidence that both are on the right track.

The connection between definable determinacy and inner models of large cardinals leads to a method—Woodin's core model induction—for propagating determinacy up the hierarchy of complexity. This machinery can be used to show that virtually *every* natural mathematical theory of sufficiently strong consistency strength actually implies $\mathrm{AD}^{L(\mathbb{R})}$. Here are two representative examples:

THEOREM 18. (Woodin) *Assume* ZFC + *there is an ω_1-dense ideal on ω_1. Then* $\mathrm{AD}^{L(\mathbb{R})}$.

THEOREM 19. (Steel) *Assume* ZFC + PFA. *Then* $\mathrm{AD}^{L(\mathbb{R})}$.

These two theories are incompatible[31] and yet both imply $\mathrm{AD}^{L(\mathbb{R})}$. There are many other examples. For instance, the axioms of Foreman (1998) (which imply CH) also imply $\mathrm{AD}^{L(\mathbb{R})}$. Definable determinacy is inevitable in that it lies in the overlapping consensus of all sufficiently strong natural mathematical theories.

3.5. *Summary.* The first goal of this section was to set forth some of the strongest extrinsic justifications of new axioms, in particular, axioms

[31] Todorĉević showed that PFA implies $2^\omega = \aleph_2$. Hence PFA implies MA + $\neg$CH which in turn implies that there is no ω_1-dense ideal on ω_1. Cf. Taylor (1979).

of definable determinacy. Let me bring together some of the main points, concentrating on $\mathrm{AD}^{L(\mathbb{R})}$ for definiteness:

(1) $\mathrm{AD}^{L(\mathbb{R})}$ lifts the structure theory that can be established in ZFC to the level of $L(\mathbb{R})$. This fruitful consequence provides extrinsic support for the axiom. The concern that there might be many axioms with the same fruitful consequence and that there is no reason for selecting one over the other is addressed by the recovery result (Theorem 9) which shows that $\mathrm{AD}^{L(\mathbb{R})}$ is necessary for this task.

(2) $\mathrm{AD}^{L(\mathbb{R})}$ is implied by large cardinals and so inherits the considerations in favour of the latter. Conversely, $\mathrm{AD}^{L(\mathbb{R})}$ implies the existence of inner models of large cardinals. Ultimately, $\mathrm{AD}^{L(\mathbb{R})}$ is equivalent to the existence of certain inner models of large cardinals. This sort of convergence of conceptually distinct domains is striking and unlikely to be an accident.

(3) $\mathrm{AD}^{L(\mathbb{R})}$ yields an "effectively complete" axiom for $L(\mathbb{R})$ in a sense explained in Theorem 12. Moreover, in the sense of Theorem 15, $\mathrm{AD}^{L(\mathbb{R})}$ is "necessary" if one is to have this sort of effective completeness.

(4) $\mathrm{AD}^{L(\mathbb{R})}$ in inevitable in that it lies in the overlapping consensus of all sufficiently strong, natural theories. This includes incompatible theories from radically distinct domains.[32]

All of this amounts to a compelling extrinsic case for $\mathrm{AD}^{L(\mathbb{R})}$ and a similar case holds for higher forms of definable determinacy.

The second goal of this section was to assess the extent of the program for large cardinals. We saw that the program fails at the level of CH and hence there can be no generalized completeness theorem of the sort Gödel entertained. But we also saw in §3.3 that there is a sense in which the program is a complete success below CH, viz. Theorem 14 combined with the result that large cardinals imply that absolutely Δ^2_1 sets of reals are

[32]It should be stressed that regularity properties, definable determinacy axioms and inner models of large cardinals are from conceptually distinct domains that have on their face nothing to do with one another. Their ultimate convergence is quite striking. It is made more striking by the fact that there is not even a direct proof of the recovery theorems in (1), (3) and (4) that connect these domains. The only known proofs proceed through inner model theory. This kind of convergence is quite different from the kind of convergence involved when two number theorists arrive at the same result. The latter arises from the fact that the number theorists are proceeding on the basis of the same assumptions, while in our present case we are dealing with steps beyond the currently accepted axioms. It is quite remarkable that steps in what appear to be completely different directions lead to the same place.

universally Baire.[33] (The case is further strengthened by combining this last fact with the considerations in footnote 27).

4 The Continuum Hypothesis

One must go beyond large cardinals in order to make an advance on CH and any case for the resolution of CH is going to look quite different than the above case for $\mathrm{AD}^{L(\mathbb{R})}$. For example, unlike $\mathrm{AD}^{L(\mathbb{R})}$, CH cannot be inevitable in the sense of being implied by every sufficiently strong natural theory.[34] Surprisingly, it is possible that in the case of CH one can have something parallel to the third point above, that is, it is possible that one can give a case of the form: there is a 'good' theory and all 'good' theories imply $\neg$CH. This approach is due to Woodin and it is grounded in a series of striking results of which I will give only the barest sketch in the hope of conveying the central ideas and illustrating the kind of justification it involves.[35]

4.1. Ω-*logic*. Woodin's approach is to extract the abstract features of the situation with regard to definable determinacy and put them to use in isolating an asymmetry between CH and its negation. This involves characterizing generic absoluteness in terms of a strong logic—Ω-logic.

Definition 1. Suppose there is a proper class of strongly inaccessible cardinals. Suppose T is a theory and φ is a sentence, both in the language of set theory. Let us write

$$T \models_\Omega \varphi$$

if whenever $\mathbb{P}$ is a partial order, α is an ordinal, and $G \subseteq \mathbb{P}$ is V-generic, then

$$\text{if } \; V[G]_\alpha \models T \; \text{ then } \; V[G]_\alpha \models \varphi.$$

Now in order for a logic to play a foundational role (from the point of view of its consequences) it must be robust in that the question of what implies

[33]In Sections 4 and 5 this will be reformulated in terms of a "logic of large cardinals", the result being that large cardinal axioms provide a "complete" theory of $L(\mathbb{R})$ (and beyond).

[34]Again, in all of this I am referring to large cardinal axioms which resemble those currently known in that they are invariant under small forcing.

[35]For more on the subject see Woodin (1999), Woodin (2005a), and Woodin (2005b). Also see Dehornoy (2004) for an overview and Bagaria, Castells & Larson (2006) for a detailed introduction (with proofs) to Ω-logic.

what cannot be altered by forcing. Fortunately, in the context of a proper class of Woodin cardinals, this is the case for Ω-logic.[36]

THEOREM 20. (Woodin) *Assume there is a proper class of Woodin cardinals. Suppose T is a theory, φ is a sentence, $\mathbb{P}$ is a partial order and $G \subseteq \mathbb{P}$ is V-generic. Then*

$$V \models \text{“}T \models_\Omega \varphi\text{”}$$

if and only if

$$V[G] \models \text{“}T \models_\Omega \varphi\text{”}.$$

When $T \models_\Omega \varphi$ we say that φ is *Ω_T-valid* and when $T \not\models_\Omega \neg\varphi$ we say that φ is *Ω_T-satisfiable*. For a collection Γ of sentences we say that T is *Ω-complete for* Γ if for all $\varphi \in \Gamma$ either $T \models_\Omega \varphi$ or $T \models_\Omega \neg\varphi$. Two cases of interest are when Γ is the set of sentences of the form $H(\omega_2) \models \varphi$ and when Γ is the set of sentences of the form $L(\mathbb{R}) \models \varphi$. We will use $\Gamma(H(\omega_2))$ to abbreviate the former and $\Gamma(L(\mathbb{R}))$ to abbreviate the latter. Using this terminology we can rephrase Theorem 12 by saying that in the presence of a proper class of Woodin cardinals ZFC is Ω-complete for $\Gamma(L(\mathbb{R}))$. This is a partial realization of Gödel's conjectured completeness theorem for large cardinals. We will return to the subject in Section 5.

4.2. *The Continuum Hypothesis.* The interest of the structure $H(\omega_2)$ is that CH is equivalent to a statement in $\Gamma(H(\omega_2))$. The main conjecture concerning CH is the following:

CH CONJECTURE. *Assume there is a proper class of Woodin cardinals.*

(1) *There is an axiom A such that*
  - (*i*) *A is Ω_{ZFC}-satisfiable and*
  - (*ii*) ZFC + A *is Ω-complete for* $\Gamma(H(\omega_2))$.

(2) *Any such axiom A has the feature that*

$$\mathrm{ZFC} + A \models_\Omega \text{“}H(\omega_2) \models \neg\mathrm{CH}\text{”}.$$

A convenient (and tendentious) way to rephrase this is as follows: Call an axiom A ‘good’ if it satisfies (1) above. Then the conjecture says:

[36] It is of interest to note that second-order logic does not meet this requirement under *any* large cardinal assumptions.

(1) There is a good axiom.

(2) All good axioms Ω-imply $\neg$CH.

Woodin has proved the CH Conjecture assuming a conjecture which for the purposes of this exposition we will call the Strong Ω Conjecture. The Strong Ω Conjecture is a conjunction of two other conjectures—the Ω Conjecture and the statement that the AD^+ Conjecture is Ω-valid. We shall now describe these terms.[37]

Recall that validity for first order logic is Π_1 in the universe of sets and the Gödel completeness theorem reduces this to a finitary notion. Now, validity for Ω-logic is Π_2 in the universe of sets and the Ω Conjecture reduces this to an "Ω-finitary" notion, one where the proofs are sets of reals that are sufficiently robust (i.e. universally Baire). The "syntactic" notion of proof for Ω-logic is defined as follows:

Definition 2. Let $A \in \Gamma^\infty$ and M be a countable transitive model of ZFC. M is A-*closed* if for all set generic extensions $M[G]$ of M,

$$A \cap M[G] \in M[G].$$

Definition 3. Let T be a set of sentences and φ be a sentence. Then $T \vdash_\Omega \varphi$ if there is a set $A \subseteq \mathbb{R}$ such that

(1) $L(A, \mathbb{R}) \models \mathrm{AD}^+$,

(2) $\mathscr{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \Gamma^\infty$, and

(3) for all countable transitive A-closed M,

$$M \models \text{“}T \models_\Omega \varphi\text{”}.$$

This notion of provability (like the semantic notion of consequence) is sufficiently robust:

Theorem 21. (Woodin) *Assume there is a proper class of Woodin cardinals. Suppose T is a set of sentences, φ is a sentence, $\mathbb{P}$ is a partial order, and $G \subseteq \mathbb{P}$ is V-generic. Then*

$$V \models \text{“}T \vdash_\Omega \varphi\text{”}$$

[37]Woodin originally thought that he could prove the CH Conjecture assuming only the Ω Conjecture but he recently discovered that the proof needed the additional assumption.

if and only if

$$V[G] \models \text{“}T \vdash_\Omega \varphi\text{”}.$$

Furthermore, the soundness theorem for Ω-logic is known to hold:

THEOREM 22. (Woodin) *Suppose T is a set of sentences and φ is a sentence. If $T \vdash_\Omega \varphi$ then $T \models_\Omega \varphi$.*

The corresponding completeness theorem is open:

Ω CONJECTURE. (Woodin) *Assume there is a proper class of Woodin cardinals. Then for each sentence φ,*

$$\varnothing \models_\Omega \varphi$$

if and only if

$$\varnothing \vdash_\Omega \varphi.$$

To define the Strong Ω Conjecture we need to introduce the AD^+ Conjecture:

AD^+ CONJECTURE (Woodin). *Suppose that A and B are sets of reals such that $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ satisfy AD^+. Suppose every set*

$$X \in \mathscr{P}(\mathbb{R}) \cap \big(L(A, \mathbb{R}) \cup L(B, \mathbb{R})\big)$$

is ω_1-universally Baire. Then either

$$(\utilde{\Delta}^2_1)^{L(A,\mathbb{R})} \subseteq (\utilde{\Delta}^2_1)^{L(B,\mathbb{R})}$$

or

$$(\utilde{\Delta}^2_1)^{L(B,\mathbb{R})} \subseteq (\utilde{\Delta}^2_1)^{L(A,\mathbb{R})}.$$

STRONG Ω CONJECTURE (Woodin). *Assume there is a proper class of Woodin cardinals. Then the Ω Conjecture holds and the AD^+ Conjecture is Ω-valid.*

We are now in a position to say what is known about the CH Conjecture. First, we need a candidate for a ‘good’ axiom.

DEFINITION 4. Let I_{NS} be the non-stationary ideal on ω_1. Let $(*)_0$ be the sentence:

For each projective set A and for each Π_2-sentence φ, if

$$\text{“}\langle H(\omega_2), \in, I_{\text{NS}}, A\rangle \models \varphi\text{”}$$

is Ω_{ZFC}-consistent, then

$$\langle H(\omega_2), \in, I_{\text{NS}}, A\rangle \models \varphi.$$

(A statement φ is said to be Ω_{ZFC}-consistent if its negation is not Ω_{ZFC}-provable, that is, if $\text{ZFC} \nvdash_\Omega \neg\varphi$.) The axiom $(*)_0$ states a “maximum property” for $H(\omega_2)$ of the kind entertained by Gödel:

> ... from an axiom in some sense opposite to $[V = L]$, the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas $[V = L]$ states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ... (Gödel (1964, fn. 23, pp. 262–3))

THEOREM 23. (Woodin) *Assume there is a proper class of Woodin cardinals. Then*

(*i*) $(*)_0$ *is* Ω_{ZFC}*-consistent and*

(*ii*) *for every sentence* φ *either*

$$\text{ZFC} + (*)_0 \vdash_\Omega \text{“}H(\omega_2) \models \varphi\text{”}$$

or

$$\text{ZFC} + (*)_0 \vdash_\Omega \text{“}H(\omega_2) \models \neg\varphi\text{”}$$

It follows from Ω-soundness that $(*)_0$ freezes the theory of $H(\omega_2)$. Thus to prove the first part of the CH Conjecture it suffices to show that $(*)_0$ is Ω_{ZFC}-satisfiable. This is open. (It is known that $(*)_0$ can be forced over $L(\mathbb{R})$ under suitable large cardinal assumptions. The question is whether it can be forced over V.) So, we *almost* have that $(*)_0$ is good. Moreover, this maximum property settles CH.

THEOREM 24. (Woodin) *Assume there is a proper class of Woodin cardinals and that* $(*)_0$ *holds. Then* $2^{\aleph_0} = \aleph_2$.

Finally, we *almost* have that all good axioms refute CH.

THEOREM 25. (Woodin) *Assume there is a proper class of Woodin cardinals that the* AD^+ *Conjecture is* Ω*-provable in* ZFC. *Suppose* A *is an axiom such that*

(*i*) A *is* Ω_{ZFC}*-consistent and*

(*ii*) *for every sentence* φ *either*

$$\mathrm{ZFC} + A \vdash_\Omega \text{“}H(\omega_2) \models \varphi\text{”}$$

or

$$\mathrm{ZFC} + A \vdash_\Omega \text{“}H(\omega_2) \models \neg\varphi\text{”}.$$

Then

$$\mathrm{ZFC} + A \vdash_\Omega \neg\mathrm{CH}.$$

If one replaces the syntactic notions in Theorems 23 and 25 with the semantic notions then one has the CH Conjecture. Thus:

COROLLARY 1. *The Strong* Ω *Conjecture implies the* CH *Conjecture.*

So, granting the Strong Ω Conjecture, all good axioms refute CH.

A possible worry is that the Strong Ω Conjecture is as intractable as CH. But this is unlikely in light of the following result.

THEOREM 26. *Assume there is a proper class of Woodin cardinals. Suppose* $\mathbb{P}$ *is a partial order and* $G \subseteq \mathbb{P}$ *is* V*-generic. Then*

$$V \models \text{“}\textit{Strong } \Omega \textit{ Conjecture}\text{”}$$

if and only if

$$V[G] \models \text{“}\textit{Strong } \Omega \textit{ Conjecture}\text{”}.$$

To summarize:

(1) The Strong Ω Conjecture implies that there is a good axiom and all good axioms Ω-imply $\neg$CH.

(2) The Strong Ω Conjecture is unlikely to be as intractable as CH.

The above case for $\neg$CH is weaker than the case for $\mathrm{AD}^{L(\mathbb{R})}$ in that $\neg$CH lacks the inevitability had by $\mathrm{AD}^{L(\mathbb{R})}$. This, however, is simply an inevitable consequence of the fact that CH is not settled by large cardinal axioms. With CH one reaches a transition point in the kind of justification that can be given—the case is necessarily going to have to be more subtle. As a symptom of this consider the following scenario: Suppose that inner model theory reaches "L-like" models $L[E]$ that can accommodate all large cardinals and have much of the rich combinatorial structure of current inner models. An axiom of the form $V = L[E]$ would then be a plausible new axiom—it could not be refuted in the way that $V = L$ was and it would have the virtue of settling many undecided questions, in particular, it would imply CH. If one could also force $(*)_0$ over $L[E]$ then $V = L[E]$ and $V = L[E][G]$ would close competitors.

To strengthen the case for $\neg$CH we need a proof of the Strong Ω Conjecture and an analysis of the structure theory of $H(\omega_2)$ under $(*)_0$. It is hard to resist quoting the words with which Gödel closed his paper on the continuum problem: "I believe that adding up all that has been said one has good reason for suspecting that the role of the continuum problem in set theory will be to lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture." (Gödel (1964, p. 264))

5 Three Prospects

We have seen that there is a compelling case for axioms that settle $V = L$ and PM and that there is a good case for axioms settling CH. There is at present not a strong case for absolute undecidability. I want now to consider three scenarios for how the subject might unfold.

5.1. *The Ω Conjecture.* Suppose it turns out that the Ω Conjecture is true. In this case, Woodin has shown (as we shall see below) that Ω-logic is essentially the "logic of large cardinals". It thus provides a precise explication of the version of absolute provability that Gödel proposed in 1946.

We first need to render precise the notion of a large cardinal axiom. Following Woodin let us say that a *large cardinal axiom* is a statement of the form $\exists x\, \varphi(x)$ where $\varphi(x)$ is Σ_2 and (as a theorem of ZFC) if κ is a cardinal and $V \models \varphi[\kappa]$ then κ is strongly inaccessible and for all partial orders $\mathbb{P} \in V_\kappa$ and all V-generics $G \subseteq \mathbb{P}$, $V[G] \models \varphi[\kappa]$.[38] For a large cardinal axiom $\exists x\, \varphi(x)$

[38] This directly captures most of the standard large cardinal axioms—for example, "κ is

we say that V *is* φ*-closed* if for every set X there is a transitive set M and an ordinal κ such that $X \in V_\kappa^M$, $M \models \mathrm{ZFC}$, and $M \models \varphi[\kappa]$. Notice that if $\exists x\, \varphi(x)$ is a large cardinal axiom and $\varphi[\kappa]$ holds for a proper class of inaccessible cardinals then V is φ-closed.

LEMMA 5. (Woodin) *Assume there is a proper class of Woodin cardinals. Suppose that ψ is Π_2. Then $\mathrm{ZFC} \vdash_\Omega \psi$ iff there is a large cardinal axiom $\exists x\, \varphi(x)$ such that*

(*i*) $\mathrm{ZFC} \vdash_\Omega$ "*V is φ-closed*"

(*ii*) $\mathrm{ZFC} +$ "*V is φ-closed*" $\vdash \psi$.

(Notice that the statement "V is φ-closed" is Π_2.) One can show from this that (assuming a proper class of Woodin cardinals) the Ω Conjecture is equivalent to the statement that if V is φ-closed for some large cardinal axiom φ then $\mathrm{ZFC} \vdash_\Omega$ "V is φ-closed".

So, assuming the Ω Conjecture and a proper class of Woodin cardinals, if V is φ-closed for some large cardinal axiom $\exists x\, \varphi(x)$, then $\mathrm{ZFC} \vdash_\Omega$ "V is φ-closed"; and if ψ is a Π_2 sentence that is a *first-order* consequence of $\mathrm{ZFC} +$ "V is φ-closed", then $\mathrm{ZFC} \vdash_\Omega \psi$. Thus, under the Ω Conjecture and a proper class of Woodin cardinals, Ω-logic is simply the logic of large cardinal axioms under which V is φ-closed. It is therefore a reasonable regimentation of Gödel's 1946 proposal of absolute provability (with respect to Π_2 sentences). But can it really be considered absolute?

In the case of the view of 1939 we provided a characterization of absolute provability in terms of reflection principles and we saw that on this conception $V = L$ is indeed absolutely undecidable. Gödel came to think that the notion of absolute provability outstripped this notion and we saw that there are strong extrinsic justifications for axioms of definable determinacy and these, of course, imply inner models of large cardinals that violate $V = L$. We now have a partial reconstruction of his 1946 notion of absolute provability (one that accommodates *all* large cardinals) in terms of Ω-logic. We know that CH is beyond its reach (just as $V = L$ is beyond the reach of the earlier notion). But there are two views one can have on the matter. First, in parallel with the view of 1939, one can hold onto the idea that the notion

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measurable", "κ is a Woodin cardinal", "κ is the critical point of a non-trivial elementary embedding $j : V_\lambda \to V_\lambda$". It does not capture "$\kappa$ is supercompact" directly but one can remedy this by considering "$\exists\delta\, V_\delta \models \kappa$ is supercompact".

of provability really is absolute and maintain that CH is absolutely undecidable and signals a bifurcation in set theory. Second, one can reject the absoluteness of the notion, maintaining that there are extrinsic justifications that outstrip provability in Ω-logic.

There are a number of difficulties with the first position even in this richer context. First, Woodin has shown that the Strong Ω Conjecture and the assumption of a proper class of Woodin cardinals implies that $\{\varphi \mid \varnothing \models_\Omega \varphi\}$ is definable in $\langle H(\mathfrak{c}^+), \in\rangle$, where $\mathfrak{c}$ is the cardinality of the continuum.[39] So the view in question amounts to a rejection of the transfinite beyond the continuum. As Woodin puts it, such a view is just formalism "two steps up". Second, it overlooks the fact that there might be arguments that enable us to leverage certain asymmetries and provide reasons for a statement despite the fact that neither it nor its negation is Ω_{ZFC}-valid. An example of this is the argument against CH presented in the last section, an argument in which the very notion of Ω_{ZFC}-validity plays a central role. Furthermore, there might be other arguments. We will consider one in §5.3.

5.2. *Incompatible Ω-complete Theories.* The above discussion was conditioned on the truth of the Strong Ω Conjecture. But it could turn out to be false and in this case there is another approach to CH.

The paradigm result in this direction is the following early result of Woodin:

THEOREM 27. (Woodin, 1985) *Assume there is a proper class of measurable Woodin cardinals. Then* ZFC + CH *is Ω-complete for* Σ^2_1.

Thus, under large cardinals we have that ZFC + $(*)_0$ is Ω-complete for $\Gamma(H(\omega_2))$ and ZFC+CH is Ω-complete for Σ^2_1. Two questions naturally arise. First, are there recursive theories with higher degrees of Ω-completeness? Second, is there a unique such theory (with respect to a given level of complexity)? Regarding the first question, Abraham and Shelah have shown:

THEOREM 28. (Abraham-Shelah, 1993) ZFC + CH *is not Ω-complete for* Σ^2_2.

It is open whether there is a strengthening of CH that is Ω-complete for Σ^2_2.[40]

[39] Contrast this with the case of second-order logic where, by a result of Väänänen (2001), the set of valid sentences is Π_2-complete *over* V. Of course, this could be the case with Ω-logic if the Strong Ω Conjecture fails.

[40] A conjectured candidate is the statement $\Diamond_G$ asserting $H(\omega_1) \equiv H(\omega_1)^{\mathrm{Coll}(\omega_1,\mathbb{R})}$.

However, if the Strong Ω Conjecture is true then a recursive theory that is Ω-complete for Σ^2_2 is the most that one could hope for.

THEOREM 29. (Woodin) *If there is a proper class of Woodin cardinals and the Strong Ω Conjecture holds then there is no recursive theory that is Ω-complete for Σ^2_3.*

But if the Strong Ω Conjecture fails then there might exist recursive theories $T_n \subseteq T_{n+1}$ such that $\mathrm{ZFC} + T_n$ is Ω-complete for Σ^2_n for each $n < \omega$, that is, for third-order arithmetic. Steel (2004) conjectures that this is the case (and hence that the Strong Ω-conjecture is false). He maintains that if (i) all large cardinals are preserved under small forcing, (ii) every interesting theory can be forced relative to large cardinals, and (iii) the theories T_n are extendible to T_α for arbitrarily large α, then one would have solved the continuum problem. Now, if there were a unique such sequence of theories (in the sense that all such theories agreed on their Ω-consequences for third-order arithmetic) and they implied, say, CH then this would make a very strong case for CH.

But there might be two such sequences—say T_α and S_α—that are incompatible. For example, one might imply CH while the other implies $\neg$CH. Would this amount to the absolute undecidability of CH? There are two views that one might have of the scenario. On the first view the generic intertranslatability of the two theories shows that there is no meaningful difference between them.[41] On the second view a meaningful difference remains. On neither view do we have a clear case of an absolutely undecidable sentence. This is because on the first view CH is not a genuine instance of absolute undecidability since it is not even a meaningful statement, while on the second view a meaningful difference remains and this opens up the possibility that there might be considerations that one could advance in favour of one theory over the other. In the next section I will present a scenario for how this might happen.[42]

[41] This is Steel's view. See Steel (2004) and Maddy (2005) for further discussion.

[42] Although it is not necessary for my purposes here to determine which view is correct, the question is of independent interest and has bearing on the search for new axioms. For example, it has bearing on whether in the scenario discussed at the end of §4 there is a substantive issue in deciding between $V = L[E]$ and $V = L[E][G]$. So let me say something to bring out the issues involved. Our background assumptions imply that large cardinal axioms will not distinguish between the S_α-sequence and the T_α-sequence. So there is no help from above. They also imply that the two sequences have the same arithmetical consequences. So there is no help from below. One might try looking at other

5.3. *The Structure Theory of* $L(V_{\lambda+1})$. Recall that $\mathrm{AD}^{L(\mathbb{R})}$ was first proved from the assumption of a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point less than λ. It turns out that there is a striking parallel between the structure theory of $L(\mathbb{R})$ under the assumption of determinacy and that of $L(V_{\lambda+1})$ under the embedding assumption. Here λ is the analogue of ω, λ^+ is the analogue of ω_1 and fragments of the embedding are analogues of game strategies. Some examples that flesh out the parallel are the following:

(1) Measurability: (Woodin) λ^+ is measurable in $L(V_{\lambda+1})$.

(2) Perfect Set Property: (Woodin) Suppose $X \subseteq \mathscr{P}(\lambda)$ is "projective", i.e. definable with parameters in $\langle H(\lambda^+), \in\rangle$. Then either $|X| \leq \lambda$ or $|X| = 2^\lambda$ and X contains a "perfect set".

(3) Periodicity: (Martin) Suppose $j : V_\lambda \to V_\lambda$ is a non-trivial elementary embedding. If j is Π^1_{2n+1} elementary then j is Π^1_{2n+2} elementary. (Here the superscript refers to quantification over subsets of V_λ.)

(4) Coding: (Woodin) For each $\delta < \Theta^{L(V_{\lambda+1})}$ there exists $\pi \in L(V_{\lambda+1})$ such that
$$\pi : V_{\lambda+1} \xrightarrow{\text{onto}} \mathscr{P}(\delta) \cap L(V_{\lambda+1}).$$
Hence $\Theta^{L(V_{\lambda+1})}$ is weakly inaccessible in $L(V_{\lambda+1})$.

consequences. For example, the T_α-sequence might have illuminating consequences for the theory of $L(\mathbb{R})$ that we can subsequently verify in a weaker theory. But since the T_α-sequence and the S_α-sequence have the same consequences for $L(\mathbb{R})$, the advocate of the S_α-sequence can incorporate anything the advocate of the T_α-sequence does by first forcing T_α and then applying an absoluteness argument. Moreover, should it turn out that the T_α-sequence leads to a much simpler and elegant account of the universe, one can say that the advocate of the S_α-sequence recognizes these virtues through the generic interpretation. All of this might incline one to the first view. But the generic interpretation—regarded either through Boolean valued models or countable models—is non-standard and since both parties recognize this they are not taking each other's statements at face value. In analogous situations we would not be inclined to conclude that mutual interpretability implies that there is no substantive disagreement. For example, consider (a) HA and PA, (b) Euclidean geometry and hyperbolic geometry, and (c) two physical theories that are mutually interpretable, have the same empirical consequences, and yet such that one is simple and elegant while the other is cumbersome and ad hoc. These pairs of theories are mutually interpretable and yet there seems to be a substantive difference in each case. Why should anything be different in the present context? The difference between the two views ultimately rests on differing views concerning the nature of real mathematical content and how it is determined. It is more than we can hope to answer here.

(5) STABILITY: (Woodin) Let $\delta = (\underset{\sim}{\delta}_1^2)^{L(V_{\lambda+1})}$ be the least γ such that $L_\gamma(V_{\lambda+1}) \prec_{\Sigma_1} L(V_{\lambda+1})$. Then δ is measurable in $L(V_{\lambda+1})$.

The analogue of each of these statements is known to hold in $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$. There are many more examples and many are sure to follow.[43]

Some things are known to hold in $L(V_{\lambda+1})$ under the embedding assumption that are conjectured for $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$. For example, for each δ such that $\lambda < \delta < \Theta^{L(V_{\lambda+1})}$ and δ is regular in $L(V_{\lambda+1})$

$$(\mathscr{P}(\delta)/\mathrm{NS}_\delta)^{L(V_{\lambda+1})}$$

is atomic.

Some things are known to hold in $L(\mathbb{R})$ under AD that are plausible candidates for $L(V_{\lambda+1})$. For example,

(A) for each infinite regular cardinal $\kappa < \lambda^+$, the club filter in $L(V_{\lambda+1})$ is an ultrafilter on $\{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \kappa\}$ and

(B) every club $A \in \mathscr{P}(\lambda^+) \cap L(V_{\lambda+1})$ is definable from parameters in $\langle H(\lambda^+), \in \rangle$.

The parallel between the structure theories is already rich and remarkable. The understanding of one structure theory provides insight into the other and in this way the two hypotheses are mutually supporting. The development of the parallel that can be established under existing axioms (namely, $\mathrm{AD}^{L(\mathbb{R})}$ and the embedding axiom) provides evidence that the parallel extends. And as we establish further theorems to this effect the case becomes stronger. But it could be the case that the embedding axiom is insufficient to flesh out the parallel just as it is the case that ZFC is insufficient to lift the structure theory of $\underset{\sim}{\Delta}_1^1$ sets to the projective level. Suppose it turns out that the embedding axiom is not the full analogue of $\mathrm{AD}^{L(\mathbb{R})}$ but that if we supplement it with new axioms then we "complete the picture" and "round out the analogy". This would provide an extrinsic justification of the new axioms. For definiteness let us suppose that filling in the missing pieces of the puzzle involves the addition of (A) and (B)—the analogues of which hold in $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$. Suppose further that the development of the parallel under the new axioms provides insight into the structure theory of $L(\mathbb{R})$. Of course, since the theory of $L(\mathbb{R})$ is generically invariant under

[43] See Woodin (2005a) for more on $L(V_{\lambda+1})$.

large cardinal assumptions it is unlikely that the new axioms would have new consequences but they might have abundant "verifiable" consequences, that is, "consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs" (Gödel (1964, p. 261)). All of this would make a strong case for the new axioms (A) and (B).

The following question (asked by Woodin) is open: Does the embedding axiom in conjunction with (A) and (B) settle CH? Notice that we do not here mean the *analogue* of CH but rather CH itself.[44] This is one way in which Woodin's case against CH could be strengthened. But it also has bearing on the scenario considered in §5.2. This is because if, say, T_α includes (A) and (B) while S_α does not, then the two theories are not on a par—S_α is ignoring the structural parallel. Although the two theories are generically intertranslatable we have here a case where there is further structure that we can leverage to provide reason for favouring one theory over the other.

I do not want to place too much weight on the particulars of this possible scenario. The purpose of the discussion is twofold. First, to isolate a new kind of reason that might be given in support of new axioms—one involving the rounding out of an almost complete structural parallel. Second, to argue that one might be able to distinguish between incompatible theories that are Ω-complete for third-order arithmetic.

We have seen that a compelling case can be made for new axioms that settle many of the proposed candidates for absolutely undecidable sentences. This is true of $V = L$ and PD and the advances on CH are promising. There is at present no solid argument to the effect that a given statement is absolutely undecidable. We do not even have a clear scenario for how such an argument might go.

Postscript

Added June 14, 2009. In this postscript I would like to briefly discuss some recent developments that bear on the topics treated in this paper. They concern (1) general reflection principles, (2) the prospect of incompatible Ω-complete theories, (3) the prospect of an ultimate inner model, and (4) the

[44]The axioms (A) and (B) appear to interfere with the standard ways of altering the value of the continuum via forcing.

structure theory of $L(V_{\lambda+1})$.

1. *Reflection Principles.* In Section 2, I state a theorem to the effect that a certain class of Tait's general reflection principles are weak; more precisely, Γ_n-reflection (for each n) is consistent relative to the existence of the Erdös cardinal $\kappa(\omega)$ and hence such reflection principles are compatible with $V = L$. The reflection principles covered by this theorem are only a small fragment of a broad class of general reflection principles introduced by Tait and my reason for focusing on them in the paper is that the remaining reflection principles were not known to be consistent relative to large cardinal axioms. It turns out that the theorem in the paper is optimal. For the remaining reflection principles in Tait's hierarchy turn out to be inconsistent; moreover, one can refine Tait's hierarchy and prove a dichotomy theorem to the effect that the refined hierarchy of general reflection principles neatly divides into those that are weak (in that they are consistent relative to the Erdös cardinal $\kappa(\omega)$) and those that are inconsistent. See Koellner (2009a).

2. *Incompatible Ω-Complete Theories.* In Section 5.2 I discuss a very optimistic scenario for supplementing large cardinal axioms. According to this scenario, for each specifiable fragment V_λ of the universe of sets (such as $V_{\omega+2}$ or V_κ, where κ is the least inaccessible cardinal) there is a large cardinal axiom L and a recursively enumerable sequence of axioms $\vec{T}$ such that $\mathrm{ZFC} + L + \vec{T}$ is Ω-complete for the theory of V_λ; moreover, there is a unique such theory in that any other theory $\mathrm{ZFC} + L + \vec{S}$ with this feature agrees with $\mathrm{ZFC} + L + \vec{T}$ on the Ω-computation of the theory of V_λ. Were this to be the case there would be a "unique Ω-complete picture" of V_λ. It is now known that uniqueness must fail: If there is one such theory then there *must* be another with the same degree of Ω-completeness but which gives a different "Ω-complete picture" of V_λ; in particular, one can arrange that the two theories differ on CH and many other statements. Thus, should there exist one such theory there would be many and one would have a radical bifurcation of Ω-complete theories (a possibility entertained in the last paragraph of Section 5.2). One way to rule out such a bifurcation is to prove the Strong Ω Conjecture. See Koellner & Woodin (2009) and Koellner (2009b) for more on this subject.

3. *The Prospect of an Ultimate Inner Model.* In the penultimate paragraph of Section 4, I consider the prospect of an ultimate inner model, one that is "L-like" and yet compatible with all large cardinals. Until quite recently such a prospect seemed quite far-fetched. To see why let us briefly re-

call the general pattern of inner model theory: Given a certain initial stretch of the large cardinal hierarchy one defines an "L-like" inner model that is able to accommodate large cardinals in this initial stretch by "absorbing" them from V. But for every such model, there are slightly stronger large cardinals that cannot be accommodated by the model and which, moreover, imply that the model is a poor approximation to V. To accommodate these additional large cardinals one must define a *new* inner model. But it too will be transcended by other large cardinal axioms. Thus, on this picture, there is a succession of inner models that provide better and better approximations to the universe of sets but there is no single model that is "close to V" and can accommodate all large cardinal axioms.

Recent developments of Hugh Woodin indicate that this picture could change dramatically. One of the main outcomes of his recent work (contained in his forthcoming *Suitable Extender Sequences*) is the following dichotomy theorem: Either there is no "L-like" inner model for one supercompact cardinal (which would amount to a failure of inner model theory) or there is an "L-like" inner model that is both "close to V" and able to accommodate *all* large cardinal axioms in the traditional hierarchy (and, in fact, in a recently discovered extension of this hierarchy). The precise details of this theorem—in particular, the minimal conditions required to count as "L-like", the notion of being "close to V", and the transfer theorems that describe the extent of the large cardinal axioms that are accommodated—are spelled out in *Suitable Extender Sequences*. Thus, if inner model theory (in anything like its present form) succeeds in producing an inner model that reaches one supercompact cardinal, then this model, call it L^Ω, will be (a) "close to V", (b) able to accommodate *all* large cardinals that have been investigated to date, and (c) such that its inner structure is very well understood (in particular, it would satisfy CH and, for any traditional statement of set theory, φ, one would generally be able to determine whether or not φ held in L^Ω). This would make $V = L^\Omega$ a compelling axiom, one that along with large cardinal axioms would (arguably) provide the ultimate completion of the axioms of ZFC.

However, there are also competing candidates for the ultimate inner model. To begin with, there is a "strategic" version L^Ω_S of L^Ω, one that is modeled on the analysis of HOD in determinacy models. The possibility of this model is opened up by the oversight mentioned in the introductory footnote to this paper. In addition to L^Ω and L^Ω_S there are also the models obtained by forcing $(*)_0$ over these models. All of these models would share the virtues of L^{Ω} but they would give different answers to certain questions. For example, L^{Ω}_{S} would have information about L^{Ω} that L^{Ω} could not have about itself and while both of these models would satisfy CH the $(*)_0$-extensions of these models would satisfy ¬CH. The question then arises as to how one would sort between them.

4. *The Structure Theory of* $L(V_{\lambda+1})$. In Section 5.3, I discuss a structural parallel between the theory of $L(\mathbb{R})$ under the assumption of AD and the theory of $L(V_{\lambda+1})$ under the assumption of a non-trivial elementary embedding from $L(V_{\lambda+1})$ into itself with critical point below λ. On the basis of the existing parallel and guided by axioms of determinacy stronger than AD, Woodin (in *Suitable Extender Sequences*) has recently discovered an entire hierarchy of much stronger large cardinal axioms. Moreover, guided by the analogy, he has isolated a series of conjectures concerning the structure theory of $L(V_{\lambda+1})$ that may (like the axioms A and B mentioned in the text) settle CH. The models L^{Ω}, L^{Ω}_{S} and their $(*)_0$-extensions, should they exist, will be able to accommodate the embedding axioms for $L(V_{\lambda+1})$ and, within this context, one will have answers to questions concerning the structure theory of $L(V_{\lambda+1})$. In this way, by isolating the correct structure theory for $L(V_{\lambda+1})$, one may be able to select from among L^{Ω}, L^{Ω}_{S} and their $(*)_0$-extensions and find the true candidate for V. Indeed, it is already known that under reasonable assumptions a very optimistic analogue of the structure theory of $L(\mathbb{R})$ under AD cannot hold in L^{Ω} or the $(*)_0$-extensions. However, it may hold in L^{Ω}_{S}. Should this be the case it would be striking affirmation of the axiom $V = L^{\Omega}_{S}$.

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