Optimal Control Problems with Nonlinear Damped Viscous Wave Equations

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Abstract

This work focuses on the analysis and numerical solution of nonsmooth optimal control problems governed by a class of nonlinear damped viscous wave (NDVW) equations with both linear and nonlinear control strategies. These equations play a crucial role in modeling wave propagation in complex media, with significant applications in medical imaging and therapeutic interventions. Using advanced numerical techniques, we explore the complex interplay between damping, viscosity, and control strategies to enhance precision in control problems related to wave-like equations. The work provides valuable frameworks into optimizing wave dynamics, leading to improved methodologies in fields such as photoacoustic imaging, lithotripsy, and tissue elastography.

Introduction and mathematical contributions

- NDVW equations model nonlinear acoustic effects in engineering, geophysics, and medical imaging.
- Crucial in inverse problems like photoacoustic tomography.
- Used in piezoelectric lithotripsy to model acoustic pressure for non-invasive kidney stone treatment.

Research Focus:

- Solving optimal control problems governed by NDVW equations.
- Aims to develop accurate and efficient control strategies for nonlinear, dissipative wave systems.

Motivation:

- Improve precision and safety in lithotripsy.
- Enhance kidney stone fragmentation with minimal risk.
- Optimize wave propagation and control in practical applications.

Mathematical Contributions:

- Modeling and analyzing nonlinear control mechanisms in wave equations.
- Theoretical study of optimal control for NDVW in bounded n-D domains.
- Development of gradient-based numerical methods for 1D and 2D NDVW control problems.

Mathematical models of optimal control

Minimizing a cost functional $J(u,h)$:

$$J(u,h) := \frac{\alpha}{2}\|u-u_d\|^2_{L^2(Q)} + \frac{\beta}{2}\|u(\cdot,T)-u_T\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|h\|^2_{L^2(Q)} + \gamma\|h\|_{L^1(Q)}$$

Subject to the PDE constraint. The two cases are:

In Case 1:

$$\partial_{tt}u - \alpha_1\Delta\partial_t u - \Delta u + \gamma_1\partial_t u + f(u) = h(x,t),$$

where is $f(u)=|u|^2u$ or $f(u)=\sin(u)$, where the nonlinearity acts on the state u.

In Case 2:

$$\partial_{tt}u - \alpha_1\Delta\partial_t u - \Delta u + \gamma_1\partial_t u + f(u) = f(h),$$

where is $f(u)=|u|^2u$ or $f(u)=\sin(u)$, ,and $f(h)=|h|^2h$ or $f(h)=\sin(h)$,

With initial and boundary conditions:

$$u(x,0)=u_0(x), \quad \partial_t u(x,0)=u_1(x) \quad \text{in } \Omega,$$
$$u(x,t)=0 \quad \text{on } \Sigma.$$

Optimality system

The optimality system for the minimization problem is defined as follows:

$$\begin{aligned} &\partial_{tt}u - \alpha\Delta\partial_t u - \Delta u + \gamma\partial_t u + f(u) = h(x,t)\\ &\partial_{tt}q + \alpha\Delta\partial_t q - \Delta q - \gamma\partial_t q + f(u,q) = -\alpha(u-u_d)\\ &\nu h + \gamma\,\mathrm{sgn}(h) - q = 0\end{aligned} \tag{OPT1}$$

For linear control, Dirichlet boundary conditions $u(x,t)=0$ on $\Sigma=\partial\Omega\times(0,T)$ and initial conditions $u(x,0)=u_0(x)$, $\partial_t u(x,0)=u_1(x)$, in Ω. Where $f(u)=|u|^2u$ or $f(u)=sin(u)$,and $f(u,q)=3u^2q$ or $f(u,q)=cos(u)q$.

$$\begin{aligned} &\partial_{tt}u - \alpha\Delta\partial_t u - \Delta u + \gamma\partial_t u + f(u) = f(h)\\ &\partial_{tt}q + \alpha\Delta\partial_t q - \Delta q - \gamma\partial_t q + f(u,q) = -\alpha(u-u_d)\\ &\nu h + \gamma\,\mathrm{sgn}(h) - f(h,q) = 0\end{aligned} \tag{OPT2}$$

For nonlinear control, Dirichlet boundary conditions $u(x,t)=0$ on $\Sigma=\partial\Omega\times(0,T)$ and initial conditions $u(x,0)=u_0(x)$, $\partial_t u(x,0)=u_1(x)$, in Ω. Where $f(h)=|h|^2h$ or $f(h)=sin(h)$,and $f(h,q)=3h^2q$ or $f(h,q)=cos(h)q$.

NCG Scheme

Algorithm (Projected NCG Scheme)

1. Input: initial approximation h_0. Evaluate $d_0=-\nabla J(h_0)$, index $k=0$, maximum $k=k_{\max}$, tolerance tol.
2. While $k<k_{\max}$, do:
3. Set $h_{k+1}=h_k+\alpha_k d_k$, where α_k is obtained using a line-search algorithm.
4. Compute $g_{k+1}=\nabla J(h_{k+1})$.
5. Compute β_k^{DY} using $\beta_k^{\mathrm{DY}}=\frac{\|g_{k+1}\|^2}{d_k^T(g_{k+1}-g_k)}$.
6. Set $d_{k+1}=-g_{k+1}+\beta_k^{DY}d_k$.
7. If $\|\nabla J(h_k)\|_{l^2}<$ tol, terminate.
8. Set $k=k+1$.
9. End while.

Results

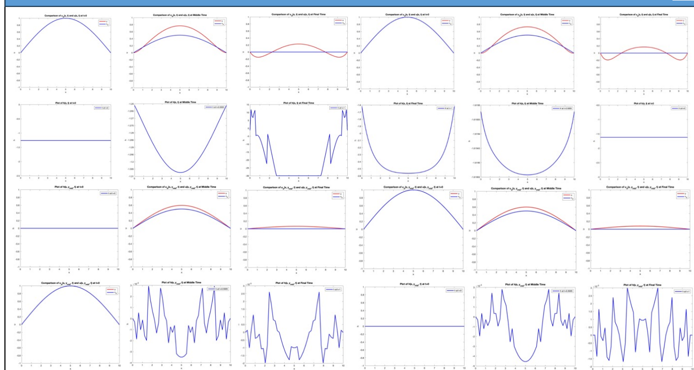

Figure in case 1: Evolution of state $u(x,t)$ and control $h(x,t)$ over time.

The figure shows how $u(x,t)$ evolves from $t=0$ to $t=1$, approaching the desired state $u_d(x,t)$. Boundary plots of $h(x,t)$ illustrate effective control at both ends, confirming the strategy guides $u(x,t)$ towards $u_d\,(x,t)$ with stability and accuracy.

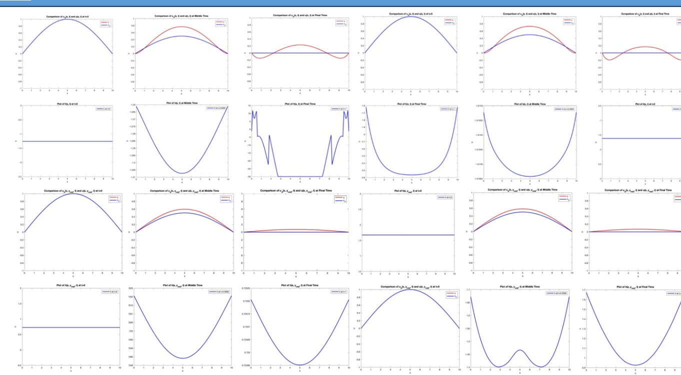

Figure in case 2: Evolution of state $u(x,t)$ and control $h(x,t)$ over time.

The figure shows how $u(x,t)$ evolves from $t=0$ to $t=1$, approaching the desired state $u_d(x,t)$. Boundary plots of $h(x,t)$ illustrate effective control at both ends, confirming the strategy guides $u(x,t)$ towards $u_d\,(x,t)$ with stability and accuracy.

Conclusion

In this work, we explored optimal control problems governed by Nonlinear Damped Viscous Wave (NDVW) equations, incorporating both linear and nonlinear control mechanisms. We studied the theoretical properties of these control problems and implemented them numerically. The results demonstrate that the control strategies are effective in driving the NDVW equation toward the desired state. These findings confirm the potential of using optimal control to manage complex, nonlinear, and dissipative wave behavior. Future work will focus on applying these control models to practical problems, particularly in medical and engineering applications such as lithotripsy and photoacoustic imaging.

Looking ahead, the control strategies developed in this work can be extended to tackle real-world challenges. In particular, medical fields such as non-invasive lithotripsy and photoacoustic imaging can benefit from improved wave control to enhance safety, precision, and treatment efficacy. Further research may also explore higher-dimensional models, more advanced numerical techniques, and the incorporation of uncertainty or real-time feedback into the control process.

References

[1] L. Ammann and I. Yousept. Acoustic full waveform inversion via optimal control: First- and second-order analysis. SIAM Journal on Control and Optimization, 61(4):2468–2496, 2023.

[2] Z. Bai, H. Zhao, and S. Wang. Parameter inversion of the diffusive–viscous wave equation based on gaussian process regression. Journal of Geophysics and Engineering, 20(6):1291–1304, 2023.

[3] L. Biegler, G. Biros, O. Ghattas, M. Heinkenschloss, D. Keyes, B. Mallick, Y. Marzouk, L. Tenorio, B. van Bloemen Waanders, and K. Willcox. Large-scale inverse problems and quantification of uncertainty. Large-Scale Inverse Problems and Quantification of Uncertainty, 2010.

[4] L. Bjørnø. Applied Underwater Acoustics. Elsevier, 2017. Neighbors, T. and Bradley, D. (Eds.).

5] T. Bui-Thanh, O. Ghattas, J. Martin, and G. Stadler. A computational framework for infinite-dimensional bayesian inverse problems part I: The linearized case, with application to global seismic inversion. SIAM Journal on Scientific Computing, 35(6):A2494–A2523, 2013.

[6] Tuan, Nguyen Huy, Vo Van Au, and Nguyen Huu Can. "Regularization of initial inverse problem for strongly damped wave equation." Applicable Analysis 97.1 (2018): 69-88.