

Charging capacitors from thermal fluctuations using diodes

P. M. Thibado ^{1,*}, J. C. Neu,² Pradeep Kumar ¹, Surendra Singh ¹ and L. L. Bonilla ³

¹*Department of Physics, University of Arkansas, Fayetteville, Arkansas 72701, USA*

²*Department of Mathematics, University of California, Berkeley, Berkeley, California 94720, USA*

³*G. Millán Institute for Fluid Dynamics, Nanoscience and Industrial Mathematics and Department of Mathematics, Universidad Carlos III de Madrid, 28911 Leganés, Spain*



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We theoretically consider a graphene ripple as a Brownian particle coupled to an energy storage circuit. When circuit and particle are at the same temperature, the second law forbids harvesting energy from the thermal motion of the Brownian particle, even if the circuit contains a rectifying diode. However, when the circuit contains a junction followed by two diodes wired in opposition, the approach to equilibrium may become ultraslow. Detailed balance is temporarily broken as current flows between the two diodes and charges storage capacitors. The energy harvested by each capacitor comes from the thermal bath of the diodes while the system obeys the first and second laws of thermodynamics.

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I. INTRODUCTION

Numerous sources of ambient energy including kinetic, solar, ambient radiation, acoustic, thermal, etc., are readily available for energy harvesting. Energy harvesting in a quiet, dark setting is the most challenging because only thermal energy is present. In such an environment the Brownian motion of electrons produces a stochastic alternating current [1,2]. If this signal is rectified, then energy could be harvested by charging a capacitor. Using a diode to rectify noise in thermal equilibrium was ruled out by Brillouin because it violates detailed balance [3]. Gunn added more insight by showing that diode nonlinearity generates an oppositely flowing current that cancels out the conventional rectified current [4,5]. Feynman popularized the notion that it is impossible to harvest thermal energy at a single temperature in his lecture series “Ratchet and pawl” [6].

Renewed interest in thermal energy harvesting emerged in the 1990s, when it was discovered that diodes can rectify stochastic signals provided long-time correlations (non-white noise) are present [7,8]. More recently, it was discovered that electrical circuits containing multiple loops can give rise to unusual correlations with vortex dynamics [9–11]. This fueled further interest in this problem.

The simplest nonlinear circuit that can potentially store charge has a diode and a capacitor. The master equation for this circuit was first derived and studied by van Kampen in 1960 [12]. He showed in equilibrium the capacitor has zero charge and developed an approximate Fokker-Planck equation (FPE) that does not satisfy the fluctuation-dissipation theorem. Later a diode-capacitor-resistor circuit was studied by Sokolov in the late 1990s [13,14]. He derived a FPE that satisfies the fluctuation-dissipation theorem. In his study, the resistor and diode are held at different temperatures and the steady-state heat engine efficiency is determined. What has

not been studied thus far is the full transient response of the charge on a storage capacitor for various diode-capacitor systems held at a single temperature.

In this study, we present a system capable of harvesting energy from thermal noise at a single temperature without violating the first or second law. Our system uses a small variable capacitor wired to two diodes and two storage capacitors using two current loops. Surprisingly, the nonlinearity of the diodes combined with the multiple current paths charges the capacitors with an ultraslow convergence to equilibrium. The harvested energy comes from the thermal baths of the diodes [15].

II. MODEL CIRCUIT

The two current loop circuit model used for this study is shown in Fig. 1(a). It includes a capacitor formed by a graphene membrane suspended near a STM tip and a dc bias voltage (V), which can be used to alter the average charge on the graphene capacitor. Due to thermal fluctuations, the distance between the graphene membrane and electrode changes, giving rise to a variable capacitance that can be written as $C(x) = C_0/(1 + \frac{x}{d})$, where $C_0 = \epsilon A/d$, ϵ is the permittivity, A is the effective area, d is the fixed distance between the membrane support and the tip, and $x(t)$ is the graphene position. The series combination of V and $C(x)$ acts as an ac power source, as charge must flow on and off the capacitor according to $q(t) = C(t)V$ [16,17]. Earlier experimental and theoretical studies of this circuit using scanning tunneling microscopy and Langevin equation found that graphene’s movement redistributes its thermal power to technologically important lower frequencies [18,19]. This, coupled with graphene’s unusual flexibility, makes it an ideal kinetic source of energy.

We have modified the earlier circuit [18] to include storage capacitors for energy harvesting. To achieve this, it is crucial to induce long-lived transients during which the capacitors

*Corresponding author: thibado@uark.edu

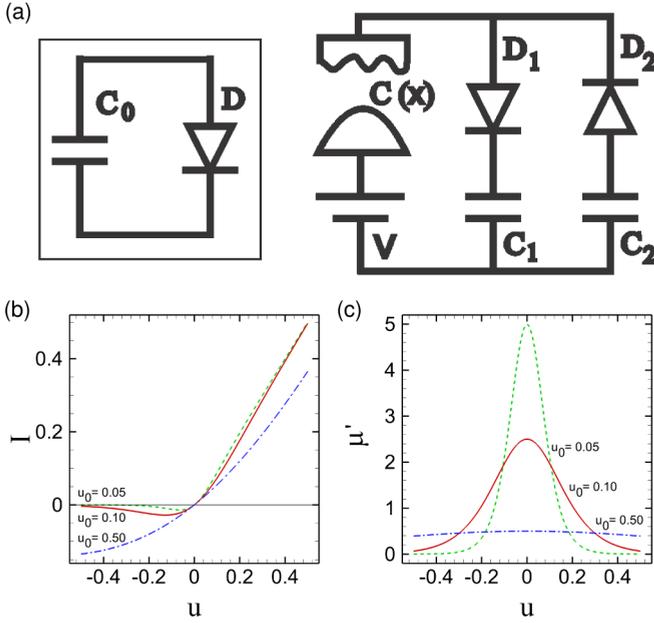


FIG. 1. Circuits and diode characteristics using $R = 1$. (a) Two-diode circuit with variable capacitance graphene membrane. The inset is a single diode with a capacitor. (b) Current-voltage characteristics of the diodes used in this study for three values of diode parameter u_0 . (c) Plots of the derivative of the diode conductance μ' are shown for the same three values of u_0 .

charge before they discharge to reach overall thermal equilibrium. The key to this is to connect the diodes to capacitors in the circuit such that the current passes only in one direction. This allows transient charging of capacitors. Of course, real diodes leak in the low conduction direction, and the system will eventually relax to thermal equilibrium. The circuit presented here maximizes the transient stage where the capacitors can charge and store energy amenable to harvesting.

At the graphene-diode junction in Fig. 1(a), current can either flow through diode D_1 and charge storage capacitor C_1 or flow through D_2 and charge C_2 . The diodes are wired in opposition. Forward current senses minimal resistance when flowing either from D_1 to C_1 or from C_2 to D_2 . This circuit with diodes, storage capacitors, and variable graphene capacitor is potentially a full-wave rectifying energy harvesting circuit. We model the harvested energy of the circuit using the Hamiltonian:

$$\mathcal{H}_q(q, q_1, q_2) = \frac{q^2}{2C(x)} + \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + qV, \quad (1)$$

where q , q_1 , and q_2 are the charges on the graphene and the two storage capacitors. The performance of the circuit depends on the current-voltage characteristics of the diodes. Each diode current I_i is related to diode conductance μ_i and voltage drop across the diode u_i via

$$I_1(u_1) = \mu_1(u_1)u_1, \quad I_2(u_2) = \mu_2(-u_2)u_2, \quad (2)$$

$$\mu(u) = \frac{1}{R} \frac{1}{1 + e^{-u/u_0}}, \quad (3)$$

where the minus sign in diode current I_2 aligns the forward bias direction to be opposite current I_1 , and the diode conductance $\mu(u)$ is modeled as a sigmoid with parameter u_0 , which controls how leaky the diodes are in reverse bias, as shown in the I - u curves in Fig. 1(b). The current-voltage curve is similar to an ideal diode in series with a resistor, which is more realistic, and except for the resistance R it depends on a single parameter u_0 [20]. It is not necessary to use the sigmoid function, as capacitor charging is found using a polynomial expansion of the ideal diode formula as well. Also shown in Fig. 1(c) is the derivative of the diode conductance $\mu' = d\mu/du$, which is a key function in the energy harvesting process. Note that while the current at zero volts is zero [Fig. 1(b)], the derivative of the diode conductance μ' is nonzero at zero volts and its value increases as u_0 decreases.

The diode voltages follow from Kirchhoff's loop law (or derivatives of the Hamiltonian with respect to charges q_1 and q_2):

$$u_i = -\frac{\partial \mathcal{H}_q}{\partial q_i} = -\left[\frac{q_i}{C_i} + V + \frac{q_1 + q_2}{C(x)} \right], \quad (4)$$

where we have used $q = q_1 + q_2$ from Kirchhoff's junction law $I = I_1 + I_2$.

III. FOKKER-PLANCK EQUATION

The probability density $\rho(q_1, q_2, t)$ for capacitor charges q_1 and q_2 obeys the Fokker-Planck equation (FPE) derived from an electron master equation in the continuum limit [18]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_1}{\partial q_1} + \frac{\partial j_2}{\partial q_2} = 0, \quad (5)$$

where $j_i = \mu_i(u_i)(u_i\rho - k_B T \partial_i \rho)$. See Appendix A for the full system FPE. The relaxation time of graphene is much shorter than the circuit RC time; therefore, it reaches equilibrium much faster. In this limit, stochastic averaging [21,22] over the graphene dynamics allows us to replace $C(x)$ with C_0 provided the variation of $x(t)$ is small compared to d (see Appendix A). This is confirmed by numerical simulation of the Ito SDEs associated with the FPE (see Appendix F). We simplify this presentation further and set $V = 0$ to study energy harvesting solely from the thermal environment. We also studied the role of adding a nonzero bias voltage. The main outcome is to alter the final equilibrium charge on the storage capacitor to be $q = C_0 V$, instead of $q = 0$.

A. One diode-one capacitor circuit

To highlight the special features of the two-loop circuit, we first present numerical solutions for the one diode one capacitor circuit [inset of Fig. 1(a)]. The FPE for this circuit is given by (see Appendix B)

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial q} \left[\mu_0 \left(\frac{q}{C_0} \rho + k_B T \frac{\partial \rho}{\partial q} \right) \right], \quad \mu_0 = \mu(-q/C_0). \quad (6)$$

The average charge on the graphene capacitor in time is shown in Fig. 2(a) for three different u_0 values for fixed $C_0 = 4$, $k_B T = 1$, and $R = 1$. We have chosen parameters that allow the simulation to capture qualitatively the important physics. A quantitative comparison is made later. The unit of charge is

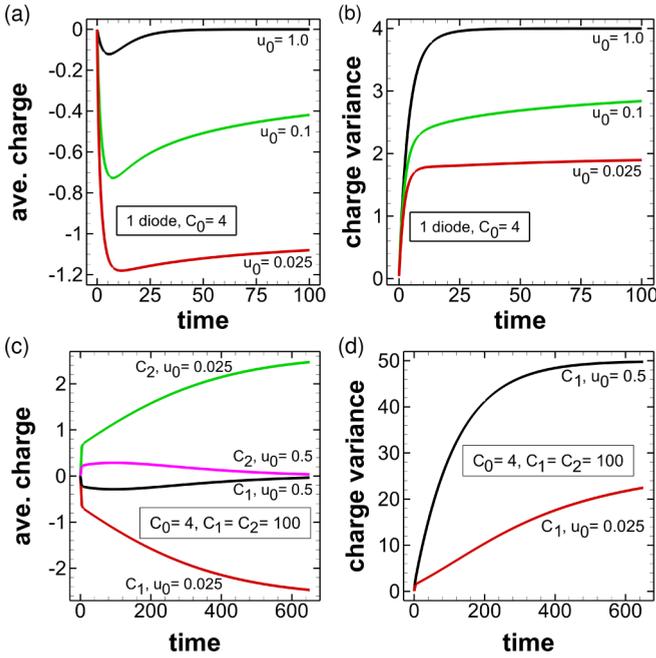


FIG. 2. Numerical solution of Fokker-Planck equation using $k_B T = 1$, $R = 1$, and various diode parameter values are displayed. (a) The average charge on the capacitor and (b) charge variance as functions of time for the single diode-capacitor circuit with different values of u_0 , as labeled. (c) The average charges on the capacitors and (d) their variance for the full two-diode circuit as functions of time for two different values of u_0 as labeled.

given by $\sqrt{k_B T C_0}$ and the unit of time is given by RC_0 . In all three cases, the average charge on the capacitor increases from zero to a negative maximum and then decays to equilibrium value zero. The relative sign of the charge reflects the choice of the positive direction for the current. The smallest value of the diode parameter ($u_0 = 0.025$) achieves the largest charge before slowly relaxing to zero. For the largest value of u_0 , the charge reaches the equilibrium in the shortest time.

The variance of the capacitor charge for the same diode parameters is shown in Fig. 2(b). It grows monotonically in time. For a perfect diode, the circuit would reach thermal equilibrium for $\langle q \rangle < 0$. This corresponds to probability density $\rho_\infty = e^{-q^2/(2C_0 k_B T)} \Theta(-q) \sqrt{\pi C_0 k_B T}/2$, where $\Theta(x)$ is the Heaviside unit step function. The average charge of this distribution is $\langle q \rangle = -\sqrt{2k_B T C_0/\pi}$ and the variance $\langle (q - \langle q \rangle)^2 \rangle = k_B T C_0 (1 - 2/\pi)$. The maximum average charge and variance in Figs. 2(a) and 2(b) tend to these values as $u_0 \rightarrow 0$. In the small temperature limit, $k_B T \ll C_0 u_0^2$, it is possible to describe analytically the evolution of the probability density from an initial condition (see Appendix B). The probability density is a skew Gaussian, where one side quickly reaches the equilibrium density. The other side advances as a front that slows down as it approaches equilibrium.

B. Full circuit

For the full circuit with two diodes and three capacitors, a new charging dynamic arises, as shown in Fig. 2(c). The graphene capacitance is kept at $C_0 = 4$, while the storage capacitances are set at $C_1 = C_2 = 100$. The average charges

on C_1 and C_2 as functions of time are shown in Fig. 2(c) for two different diode parameters. In the initial charging phase, we see a rapid increase in charge. Charges q_1 (negative) and q_2 (positive) are perfectly anticorrelated with each capacitor storing an equal amount of energy. However, after the initial rapid charging phase the charge does not monotonically decay to zero as in the one-diode-capacitor circuit. Depending on the value of u_0 , it may continue to increase before relaxing to zero. To illustrate dependence on the diode parameter u_0 , a second set of charging curves is shown in Fig. 2(c) with a larger u_0 . In this case, the charge decays more quickly to zero. The variance of the charge in time for both values of u_0 increases monotonically, as shown in Fig. 2(d). The variance eventually reaches the same equilibrium value for both.

To understand the origin of the initial rapid rise of the capacitor charge, we consider the equation for the average charge on the storage capacitor, which follows from Eq. (5),

$$\frac{d}{dt} \langle q_i \rangle = \langle u_i \mu_i(u_i) \rangle - k_B T \left(\frac{1}{C_0} + \frac{1}{C_i} \right) \langle \mu_i'(u_i) \rangle. \quad (7)$$

The first term on the right is the conventional Ohm's law current. The second term is the nonlinear thermal current proportional to the temperature. From this equation, the initial charge growth, with $\rho = \delta(q_1)\delta(q_2)$ is given by $\frac{d}{dt} \langle q_i \rangle|_{t=0} = -k_B T \left(\frac{1}{C_0} + \frac{1}{C_i} \right) \mu_i'(0)$. This shows that the initial charging of the capacitors is possible only for nonlinear resistive devices with nonzero conductance slope $\mu_i'(0) = (-1)^{i+1}/(4R u_0)$, which from Fig. 1(c) can be sizable for small u_0 . The initial diode current flows opposite to the conventional current and puts negative charges on storage capacitor C_1 . A detailed discussion of initial charging for both one diode and one capacitor as well as the three capacitor systems is provided in Appendices C and D.

The dependence of maximum charge on various circuit parameters is explored in Fig. 3. The maximum average charge increases with the charging capacitance, as shown in Fig. 3(a) for two different diode parameters. The maximum charge as a function of the diode parameter for a fixed storage capacitance decreases as shown in Fig. 3(b). The time to reach maximum charge increases as u_0 decreases, as shown in Fig. 3(c). The time to reach maximum charge is very sensitive to the value of C_0 . Here the smaller the value of C_0 , the longer the capacitors remain charged, which is opposite to the the single diode case. In summary, better performance of the circuit in regard to energy harvesting is achieved for small u_0 and small values of the ratio C_0/C_1 . To understand the mechanism behind the charging of the storage capacitors, time evolution of the probability distributions of charges must be considered. For the plot of Fig. 2(c) with $u_0 = 0.025$, we present plots of the probability density of charges. Figure 4(a) shows the two-dimensional probability density, $\rho(q_1, q_2, t)$ for $t = 800$. The probability density is symmetric about $q_1 = -q_2$. The one-dimensional marginal charge distributions for each storage capacitor $\rho(q_1, 800)$ and $\rho(q_2, 800)$ are shown in Fig. 4(b). In the limit as the diode parameter u_0 is reduced to zero and the time increased, these distributions approach one sided Gaussian distributions. The two are mirror images of one another. Figure 4(c) shows the marginal charge distribution $\rho(q_1, t)$ every 100 time units. The distribution spreads out

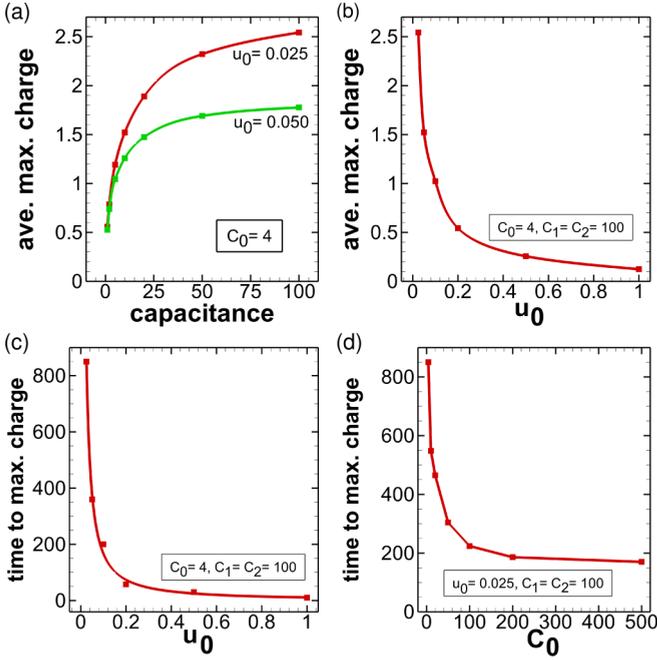


FIG. 3. Numerical solution of Fokker-Planck equation for our full circuit model using $k_B T = 1$ and $R = 1$. (a) Maximum average charge on the storage capacitors as a function of capacitance for two different values of the diode parameter u_0 . (b) Dependence of maximum average charge on storage capacitors on diode parameter u_0 . (c) Time to reach the maximum charge as a function of diode parameter u_0 . (d) The time to reach the maximum charge vs graphene capacitance parameter C_0 .

toward the thermal equilibrium distribution; however, it is apparent that the spread slows down. Just as the variance takes an extremely long time to reach equilibrium value [Fig. 2(d)], the right side of the distribution in Fig. 4(c) slows down. Thus, the two-diode circuit has created an ultraslow approach to equilibrium. From the time-dependent probability distributions, we can calculate the evolution of the Shannon entropy, which is shown in Fig. 4(d) (see Appendix D) [23]. The entropy monotonically increases in agreement with the second law and approaches the equilibrium value in the long-time limit.

Energy transferred to the storage capacitors came from the thermal bath. For the single ideal-diode-capacitor case, the energy harvested can be found analytically to be $k_B T / \pi$. More importantly, the power is found to be $k_B T / (\pi RC)$. A series of these units may be built on an integrated circuit with each using a space of less than 0.1 square microns, with the potential to produce a significant power density (see Appendix F) [24]. Surprisingly, a recent study found the average power density for wind and solar farms is relatively low, at 0.50 and 5.4 W/m^2 [25].

IV. DISCUSSION

The FPE (5) has a unique stable equilibrium solution corresponding to the minimum free energy and zero average stored charge (see Appendix D). However, before reaching thermal equilibrium, the circuit of Fig. 1(a) for a small value of u_0

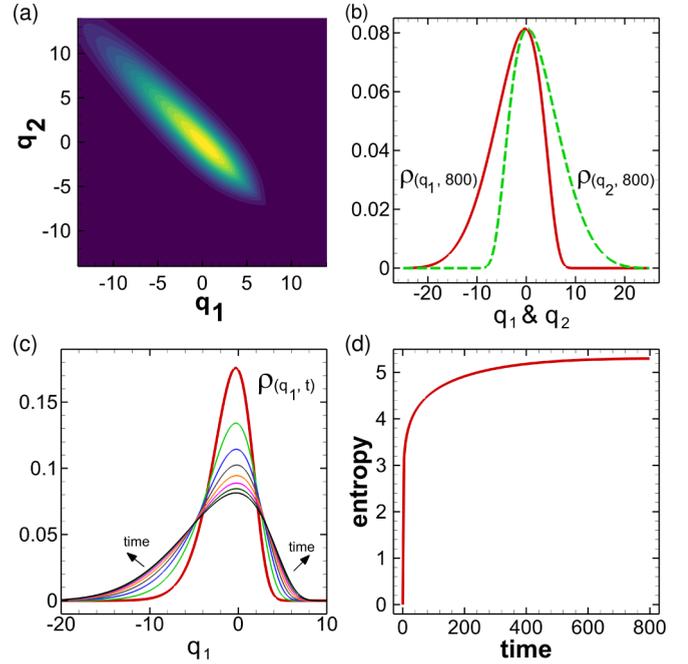


FIG. 4. Numerical solution of Fokker-Planck equation for our full circuit model using $k_B T = 1$ and $R = 1$. (a) Two-dimensional charge distribution $\rho(q_1, q_2, 800)$. (b) Charge distributions $\rho(q_1, 800)$ (full curve) and $\rho(q_2, 800)$ (dashed curve) for the storage capacitors at the time of maximum charge. (c) Time evolution of the charge distribution for the storage capacitors shown in equal time intervals from 100 to 800. (d) Time evolution of Shannon entropy.

produces a long-lived transient state in which the capacitors store charges proportional to their capacitance. Pushing off equilibrium for a sufficient time period allows the storage capacitors to be disconnected from the circuit and the energy taken from the thermal surroundings to be used. For energy harvesting, the circuit topology investigated here enhances the total charge and time to discharge, while providing additional handles for manipulating the outcome. It would be interesting to study other circuit designs, such as multiple stages of our circuit connected together in parallel or series. It is feasible to build a circuit array at a foundry as an integrated circuit using silicon fixed capacitors. This study demonstrates that energy can be harvested from the thermal surroundings at a single temperature without violating the laws of thermodynamics.

In an earlier study [17], we used a similar circuit to charge capacitors using a variable capacitor driven by a motor. The source of power charging the storage capacitors was the motor. In contrast, here we demonstrate that the ambient thermal environment can be a source of power for charging storage capacitors. With the circuit and environment at the same temperature, we prove that charging the capacitors does not violate the laws of thermodynamics. Furthermore, we have discovered a circuit topology that provides additional degrees of freedom which enable ultrafast charging of the storage capacitors combined with ultraslow convergence to equilibrium. These aspects have practical significance, as they allow time to disconnect the storage capacitors from the

TABLE I. Units for nondimensionalizing the equations of the model. $C_0V_0^2 = k_B T$, $R = 1/\mu_f$.

x	p	q, q_i	t	\mathcal{H}	V
l	\sqrt{mT}	C_0V_0	RC	$k_B T$	V_0

circuit for possible energy harvesting before they lose their charge.

In summary, we have studied theoretically the spontaneous thermal fluctuations of a circuit with diodes having nonlinear current-voltage characteristics and storage capacitors. If the storage capacitors have an initial charge of zero, then the circuit draws power from the thermal bath to charge them. Throughout the process, the system satisfies both the first and second laws of thermodynamics. From Brillouin, as mentioned earlier, we know that the diode nonlinearity generates an oppositely flowing current which exactly cancels out the rectified current to maintain detailed balance. However, this opposite current also initially charges the storage capacitors. The larger the storage capacitance, the more charge it can harvest. A smaller graphene capacitance provides a higher

initial rate of charging. In addition, a smaller graphene capacitance yields a longer charging time.

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APPENDIX A: ELIMINATING GRAPHENE LOCAL EQUILIBRIUM FROM CIRCUIT

The FPE of the full system comprising graphene and circuit in Fig. 1 is [18]

$$\frac{\partial \rho}{\partial t} - \sum_{i=1}^2 \frac{\partial}{\partial q_i} \left[\mu_i(u_i) \left(\rho \frac{\partial \mathcal{H}}{\partial q_i} + k_B T \frac{\partial \rho}{\partial q_i} \right) \right] + \frac{p}{m} \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial p} \left[\rho \frac{\partial \mathcal{H}}{\partial x} + \eta \left(\frac{p}{m} \rho + k_B T \frac{\partial \rho}{\partial p} \right) \right] = 0, \quad (\text{A1a})$$

$$\mathcal{H} = \frac{p^2}{2m} + U(x) - \frac{C_0 V^2 x}{2d} + \frac{(q_1 + q_2)^2}{2C(x)} + \sum_{j=1}^2 \frac{q_j^2}{2c_j} + (q_1 + q_2)V. \quad (\text{A1b})$$

That the equilibrium $\rho_{\text{eq}} \propto e^{-\frac{\mathcal{H}}{k_B T}}$ is a globally stable solution of Eq. (A1a) can be proved by showing that the relative entropy,

$$\mathcal{F}[\rho](t) = -k_B \int \rho(x, p, q_1, q_2, t) \ln \left[\frac{\rho(x, p, q_1, q_2, t)}{\rho_{\text{eq}}(x, p, q_1, q_2)} \right] dx dp dq_1 dq_2, \quad (\text{A1c})$$

is a Lyapunov functional of Eq. (A1a); see Ref. [26]. Using the nondimensional units of Table I in Eqs. (A1), we obtain

$$\varepsilon \left\{ \frac{\partial \rho}{\partial t} - v \sum_{i=1}^2 \frac{\partial}{\partial q_i} \left[\mu_i \left(\rho \frac{\partial \mathcal{H}}{\partial q_i} + \frac{\partial \rho}{\partial q_i} \right) \right] \right\} + \delta p \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial p} \left[\delta \rho \frac{\partial \mathcal{H}}{\partial x} + p \rho + \frac{\partial \rho}{\partial p} \right] = 0, \quad (\text{A2a})$$

$$\mathcal{H} = \frac{p^2}{2} + U(x) - \frac{x}{2\lambda} + \frac{(q_1 + q_2)^2}{2\Gamma(v)} \left(1 + \frac{x}{\lambda} \right) + v \sum_{j=1}^2 q_j^2 + (q_1 + q_2)v, \quad (\text{A2b})$$

$$\varepsilon = \frac{m}{\eta RC}, \quad \delta = \frac{U_B \sqrt{m}}{\eta l \sqrt{k_B T}}, \quad \lambda = \frac{d}{l}, \quad v = \frac{C_0}{C}, \quad v = \frac{V}{V_0}, \quad V_0 = \sqrt{\frac{k_B T}{C_0}}. \quad (\text{A2c})$$

We now derive averaging formulas for the FPE and the Ito stochastic differential equations (SDEs) [27] assuming $\varepsilon \ll 1$. This is reasonable as m/η is the reciprocal of the phonon frequency (picosecond scale) and RC is typically on the nanosecond scale. In a fast timescale $\tau = t/\varepsilon$, the leading-order probability density evolves to the local equilibrium Υ below. Inserting

$$\rho = \Upsilon(x, p, q_1 + q_2) \tilde{\rho}(q_1, q_2, t) + \varepsilon f(x, p, \mathbf{q}), \quad (\text{A3a})$$

$$\Upsilon(x, p, q) = \frac{1}{Z(q)} e^{-\mathcal{H}_r}, \quad (\text{A3b})$$

$$\mathcal{H}_r = \frac{p^2}{2} + U(x) - \frac{x}{2\lambda} + \frac{q^2}{2\Gamma(v)} \left(1 + \frac{x}{\lambda} \right), \quad (\text{A3c})$$

$$\int \Upsilon(x, p, q) dx dp = 1, \quad \int \tilde{\rho}(q_1, q_2) dq_1 dq_2 = 1, \quad (\text{A3d})$$

into Eq. (A1a), we obtain

$$\frac{\partial}{\partial p} \left(f \frac{\partial \mathcal{H}}{\partial x} \delta + pf + \frac{\partial f}{\partial p} \right) - p \frac{\partial f}{\partial x} \delta = \Upsilon \frac{\partial \tilde{\rho}}{\partial t} - v \sum_{i=1}^2 \frac{\partial}{\partial q_i} \left[\mu_i(u_i) \left(\Upsilon \tilde{\rho} \frac{\partial \mathcal{H}}{\partial q_i} + \frac{\partial \Upsilon \tilde{\rho}}{\partial q_i} \right) \right], \quad (\text{A4})$$

plus higher-order terms. The solvability condition for the linear Eq. (A4) is that the integral of its right-hand side with respect to x , p be zero. Using Eqs. (A3) and after some algebra, this yields:

$$\frac{\partial \tilde{\rho}}{\partial t} = v \sum_{i=1}^2 \frac{\partial}{\partial q_i} \left\{ \langle \mu_i(u_i) \rangle_x \left[\frac{\partial \tilde{\rho}}{\partial q_i} - \frac{\langle \mu_i(u_i) u_i \rangle_x}{\langle \mu_i(u_i) \rangle_x} \tilde{\rho} \right] \right\}, \quad (\text{A5a})$$

$$\langle g(x) \rangle_x = \frac{\int g(x) \exp \left[-U(x) - \frac{q^2(x+\lambda) - \Gamma(v)x}{2\lambda\Gamma(v)} \right] dx}{\int \exp \left[-U(x) - \frac{q^2(x+\lambda) - \Gamma(v)x}{2\lambda\Gamma(v)} \right] dx}. \quad (\text{A5b})$$

The Ito SDEs corresponding to the averaged FPE (A5a) are

$$dq_i = \left[\frac{\partial \langle \mu_i(u_i) \rangle_x}{\partial q_i} + \langle \mu_i(u_i) u_i \rangle_x \right] d(vt) + \sqrt{2 \langle \mu_i(u_i) \rangle_x} dw_{q_i}(vt), \quad (\text{A6a})$$

$$u_i = - \left[\frac{q_1 + q_2}{\Gamma(v)} \left(1 + \frac{x}{\lambda} \right) + vq_i + v \right]. \quad (\text{A6b})$$

Equation (A6) are circuit equations in which the mobilities and currents are replaced by their stochastic averages for the local equilibrium of the graphene variables given by Eq. (A5b). These equations agree with the stochastic averaging theorem [21,22] and the numerical observation that the graphene is in local equilibrium with the instantaneous values of the charges in the circuit. In the limit $l \ll d$, $\lambda \rightarrow \infty$ and the averages $\langle \mu_i(u_i) \rangle$ and $\langle \mu_i(u_i) u_i \rangle$ coincide with $\mu_i(u_i)$ and $\mu_i(u_i) u_i$, respectively. Thus in this limit, we can replace $C(x) \approx C_0$ in the FPE (A1a).

APPENDIX B: ONE DIODE IN THE LIMIT AS $k_B T \ll C u_0^2$: PROPAGATION OF EQUILIBRIUM FRONT

The substitution

$$\rho(q, t) = g(q, t) e^{-q^2/(2C_0 k_B T)}, \quad (\text{B1})$$

transforms Eq. (6) into

$$\frac{\partial g}{\partial t} + \frac{q}{C_0} \mu_0 \frac{\partial g}{\partial q} - k_B T \frac{\partial}{\partial q} \left(\mu_0 \frac{\partial g}{\partial q} \right) = 0. \quad (\text{B2})$$

Provided $k_B T \ll C_0 u_0^2$ in Eqs. (B1), the last term on the right-hand side of this equation can be ignored and we obtain the solution

$$g \propto \Theta(Q(t) - q), \quad \text{where} \quad (\text{B3a})$$

$$\dot{Q} = \frac{Q}{C_0} \mu_0 = \frac{1}{RC_0} \frac{Q}{1 + e^{Q/(C_0 u_0)}}. \quad (\text{B3b})$$

The velocity of characteristics is exponentially small as $Q \rightarrow \infty$, so the front slows down dramatically as it advances. The time it takes the front to advance from $q = Q$ to $q = Q + \delta Q$, $0 < \delta Q \ll Q$, is approximated by

$$\delta t \sim RC \frac{1 + e^{Q/(C_0 u_0)}}{Q} \delta Q. \quad (\text{B3c})$$

This time becomes exponentially large as $Q \rightarrow \infty$. The decay of the ensemble-averaged charge slows down due to the slowing of the front. If we approximate

$$\rho = \frac{\Theta[Q(t) - q]}{\int_{-\infty}^Q e^{-q^2/(2C_0 k_B T)} dq} e^{-q^2/(2C_0 k_B T)}, \quad (\text{B4})$$

then the uniform value of g behind the front is not exactly time independent, consistent with the advection equation Eq. (B2). This is an error associated with the step function approximation to g . The error in normalization is exponentially small for $Q \gg \sqrt{C_0 k_B T}$ and has negligible effect on the estimate of ensemble-averaged charge, which is now

$$\langle q \rangle = \frac{\int_{-\infty}^Q q e^{-\frac{q^2}{2C_0 k_B T}} dq}{\sqrt{2\pi C_0 k_B T}} = -\sqrt{\frac{C_0 k_B T}{2\pi}} e^{-\frac{Q^2}{2C_0 k_B T}}. \quad (\text{B5})$$

The relative increase of average charge over the time in which the front advances from $q = Q$ to $q = Q + \delta Q$ is

$$\frac{\delta \langle q \rangle}{\langle q \rangle} \sim \frac{Q \delta Q}{C_0 k_B T}. \quad (\text{B6a})$$

From Eqs. (B3c) and (B6), the time required for a given relative increase of charge, $a = \delta \langle q \rangle / \langle q \rangle$, is

$$\delta t \sim RC_0^2 k_B T \frac{1 + e^{Q/(C_0 u_0)}}{Q^2} a. \quad (\text{B6b})$$

This time becomes exponentially large as $Q \rightarrow \infty$.

The diffusion in Eq. (B2) smooths out the front at $q = Q(t)$. To see how this works, we examine the equation for the gradient of g near the front, $r = \partial g / \partial q$. Equation (B2) becomes

$$\frac{\partial r}{\partial t} + \frac{\partial}{\partial q} \left[q \mu_0 r - k_B T \frac{\partial(\mu_0 r)}{\partial q} \right] = 0. \quad (\text{B7})$$

This is a diffusion-convection equation (different from the FPE) and therefore r is locally conserved. Hence, the total change in g across the front is conserved. We represent r in ‘‘traveling wave’’ form,

$$r = \mathcal{R}(\zeta, t), \quad \zeta = \frac{q - Q(t)}{\sqrt{C_0 k_B T}}. \quad (\text{B8})$$

\mathcal{R} satisfies

$$\frac{\partial \mathcal{R}}{\partial t} + \frac{1}{\sqrt{C_0 k_B T}} \frac{\partial}{\partial \zeta} \left\{ \left[\frac{Q + \sqrt{C_0 k_B T} \zeta}{C} \mu \left(-\frac{Q + \sqrt{C_0 k_B T} \zeta}{C_0} \right) - \dot{Q} \right] \mathcal{R} - \sqrt{\frac{k_B T}{C_0}} \frac{\partial}{\partial \zeta} \left[\mu \left(-\frac{Q + \sqrt{C_0 k_B T} \zeta}{C_0} \right) \mathcal{R} \right] \right\} = 0,$$

or, evoking Eq. (B3b) for \dot{Q} ,

$$\frac{\partial \mathcal{R}}{\partial t} + \frac{\partial}{\partial \zeta} \left\{ \frac{\frac{Q + \sqrt{C_0 k_B T} \zeta}{C_0} \mu \left(-\frac{Q + \sqrt{C_0 k_B T} \zeta}{C_0} \right) - \frac{Q}{C_0} \mu \left(-\frac{Q}{C_0} \right)}{\sqrt{C_0 k_B T}} \right\} \mathcal{R} - \frac{\partial}{\partial \zeta} \left[\mu \left(-\frac{Q + \sqrt{C_0 k_B T} \zeta}{C_0} \right) \frac{\mathcal{R}}{C_0} \right] = 0. \quad (\text{B9})$$

In the limit as $C_0 k_B T \rightarrow 0$,

$$C_0 \frac{\partial \mathcal{R}}{\partial t} - \frac{\partial}{\partial \zeta} \left\{ (u\mu)'(u)\zeta \mathcal{R} + \frac{\partial}{\partial \zeta} [\mu(u)\mathcal{R}] \right\} = 0, \quad (\text{B10})$$

where $u = -Q/C_0$ and $\mu(u)$ is given by Eq. (6). This equation has a Gaussian as solution,

$$\mathcal{R} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\zeta^2/(2\sigma)}, \quad (\text{B11})$$

whose variance satisfies

$$C_0 \dot{\sigma} - 2(u\mu)'(u)\sigma = 2\mu(u). \quad (\text{B12})$$

From Eq. (B3b), this equation becomes

$$\frac{d\sigma}{dQ} - \frac{2(Q\mu)'(Q)}{Q\mu(Q)} \sigma = \frac{2}{Q}. \quad (\text{B13})$$

Suppose the front has advanced far enough so $Q \gg C_0 u_0$, in which case $\mu(Q) \sim e^{-Q/(C_0 u_0)}$, and Eq. (B13) reduces to

$$\frac{d\sigma}{dQ} + \left(\frac{2}{C_0 u_0} - \frac{2}{Q} \right) \sigma = \frac{2}{Q}. \quad (\text{B14})$$

As $Q \rightarrow \infty$, there is an asymptotic solution

$$\sigma \sim \frac{C_0 u_0}{Q} + \frac{3C_0^2 u_0^2}{2Q^2} + \frac{3C_0^3 u_0^3}{Q^3} + \dots, \quad (\text{B15})$$

and the front thickness narrows as it propagates further to the right.

APPENDIX C: SCALINGS FOR THE INITIAL TIME STAGE AND FOR LONGER TIMES

We now change variables to scaled charge sums and differences in Eq. (5) for the complete circuit in Fig. 1 according to the definitions:

$$\chi = \frac{q_1 - q_2}{C_0 V_0}, \quad \eta = \frac{q_1 + q_2}{C_0 V_0} (1 + \epsilon), \quad s = \frac{t}{RC_0}, \quad (\text{C1a})$$

$$\epsilon = \frac{C_0}{2C}, \quad V_0 = \sqrt{\frac{k_B T}{C_0}}. \quad (\text{C1b})$$

This scaling is appropriate for an initial time stage or layer provided $\epsilon \ll 1$. Equation (5) becomes

$$\frac{\partial \rho}{\partial s} = \mathcal{L}_0 \rho + \epsilon \mathcal{N}_1 \rho + \epsilon^2 \mathcal{N}_2 \rho, \quad (\text{C2a})$$

$$\mathcal{L}_0 = \frac{\partial}{\partial \eta} \left[(\mu_1 + \mu_2) \left(\frac{\partial}{\partial \eta} + \eta \right) + (\mu_1 - \mu_2) \frac{\partial}{\partial \chi} \right] + \frac{\partial}{\partial \chi} \left[(\mu_1 - \mu_2) \left(\frac{\partial}{\partial \eta} + \eta \right) + (\mu_1 + \mu_2) \frac{\partial}{\partial \chi} \right], \quad (\text{C2b})$$

$$\mathcal{N}_1 = \frac{\partial}{\partial \eta} \left[(\mu_1 - \mu_2) \left(\frac{\partial}{\partial \chi} + \chi \right) + (\mu_1 + \mu_2) \left(2 \frac{\partial}{\partial \eta} + \eta \right) \right] + \frac{\partial}{\partial \chi} \left[(\mu_1 - \mu_2) \left(2 \frac{\partial}{\partial \eta} + \eta \right) + (\mu_1 + \mu_2) \frac{\partial}{\partial \chi} \right], \quad (\text{C2c})$$

$$\mathcal{N}_2 = \frac{\partial}{\partial \eta} \left[(\mu_1 + \mu_2) \frac{\partial}{\partial \eta} + (\mu_1 - \mu_2) \chi \right], \quad (\text{C2d})$$

$$\mu_i = \mu(-\eta + (-1)^i \epsilon \chi). \quad (\text{C2e})$$

For longer times, after the initial time stage, we have to change χ and the time s to

$$\xi = \epsilon \chi = \frac{q_1 - q_2}{2C V_0}, \quad \tilde{t} = \epsilon s = \frac{t}{2RC}. \quad (\text{C3})$$

Then Eq. (5) becomes

$$\epsilon \frac{\partial \rho}{\partial \tilde{t}} = \mathcal{L} \rho + \epsilon (2\mathcal{L} + \mathcal{M}_1) \rho + \epsilon^2 (\mathcal{L} + \mathcal{M}_2) \rho, \quad (\text{C4a})$$

$$\mathcal{L} = \frac{\partial}{\partial \eta} \left[(\mu_1 + \mu_2) \left(\frac{\partial}{\partial \eta} + \eta \right) + (\mu_1 - \mu_2) \xi \right] \quad (\text{C4b})$$

$$\mathcal{M}_1 = \frac{\partial}{\partial \eta} \left[(\mu_1 - \mu_2) \left(\frac{\partial}{\partial \xi} - \xi \right) - (\mu_1 + \mu_2) \eta \right] + \frac{\partial}{\partial \xi} \left[(\mu_1 - \mu_2) \left(\frac{\partial}{\partial \eta} + \eta \right) + (\mu_1 + \mu_2) \xi \right], \quad (\text{C4c})$$

$$\mathcal{M}_2 = \frac{\partial}{\partial \eta} \left[(\mu_1 - \mu_2) \left(\frac{\partial}{\partial \xi} - \xi \right) - (\mu_1 + \mu_2) \eta \right] + \frac{\partial}{\partial \xi} \left[(\mu_1 + \mu_2) \frac{\partial}{\partial \xi} + (\mu_1 - \mu_2) \frac{\partial}{\partial \eta} \right], \quad (\text{C4d})$$

$$\mu(u) = \frac{1}{1 + e^{-u/w}}, \quad w = \frac{u_0}{V_0}, \quad \frac{u_j}{V_0} = (-1)^j \xi - \eta, \quad j = 1, 2, \quad (\text{C4e})$$

$$\frac{\mathcal{H}}{k_B T} = \frac{\eta^2}{2(1 + \epsilon)} + \frac{\xi^2}{2\epsilon}, \quad \rho_{\text{eq}}(\eta, \xi) = \frac{1}{2\pi\sqrt{\epsilon(1 + \epsilon)}} \exp \left[- \left(\frac{\eta^2}{2(1 + \epsilon)} + \frac{\xi^2}{2\epsilon} \right) \right]. \quad (\text{C4f})$$

in which we have dropped the tilde in \tilde{t} . It is straightforward to check that ρ_{eq} in Eq. (C4f) is a stationary solution of Eq. (C4a). However, as $\epsilon \rightarrow 0$ and for diodes conducting in opposite directions, we shall find in Appendix E an approximate stationary solution that is different from equilibrium to the order in ϵ we consider.

APPENDIX D: INITIAL LAYER

For diodes conducting in opposite directions, the arguments of the mobility functions in Eq. (C2e) are $\pm\eta - \epsilon\chi$,

$$\mu_1 + \mu_2 = 1 + O(\epsilon), \quad \mu_1 - \mu_2 = -\tanh \frac{\eta}{2w} + O(\epsilon). \quad (\text{D1})$$

Then the leading order of Eq. (C2a) is

$$\begin{aligned} \frac{\partial \rho^{(0)}}{\partial s} &= \frac{\partial}{\partial \eta} \left[\left(\frac{\partial}{\partial \eta} + \eta \right) - \tanh \frac{\eta}{2w} \frac{\partial}{\partial \chi} \right] \rho^{(0)} \\ &\quad - \frac{\partial}{\partial \chi} \left[\tanh \frac{\eta}{2w} \left(\frac{\partial}{\partial \eta} + \eta \right) - \frac{\partial}{\partial \chi} \right] \rho^{(0)}. \end{aligned} \quad (\text{D2a})$$

We now substitute $\rho^{(0)} = e^{-\eta^2/2} R^{(0)}(\chi, \eta, s)$ in this equation and integrate the result with respect to η to obtain an equation for the χ -dependent reduced probability density. We find

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \frac{\partial R^{(0)}}{\partial s} \frac{d\eta}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \eta} \left[e^{-\frac{\eta^2}{2}} \left(\frac{\partial R^{(0)}}{\partial \eta} - \tanh \frac{\eta}{2w} \frac{\partial R^{(0)}}{\partial \chi} \right) \right] \right. \\ &\quad \left. + e^{-\frac{\eta^2}{2}} \frac{\partial}{\partial \chi} \left[-\tanh \frac{\eta}{2w} \left(\frac{\partial R^{(0)}}{\partial \eta} - \tanh \frac{\eta}{2w} \frac{\partial R^{(0)}}{\partial \chi} \right) \right] \right. \\ &\quad \left. + \text{sech}^2 \frac{\eta}{2w} \frac{\partial R^{(0)}}{\partial \chi} \right\} \frac{d\eta}{\sqrt{2\pi}}. \end{aligned} \quad (\text{D2b})$$

Assuming that $R^{(0)}$ is a function of s and of the new variable $\sigma = \chi + 2w \ln \cosh \frac{\eta}{2w}$, all terms on the right-hand side of Eq. (D2b) cancel except for the last one. Then we obtain the heat equation:

$$\frac{\partial R^{(0)}}{\partial s} = a \frac{\partial^2 R^{(0)}}{\partial \sigma^2}, \quad (\text{D3a})$$

$$\sigma = \chi + 2w \ln \cosh \frac{\eta}{2w}, \quad (\text{D3b})$$

$$a = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \text{sech}^2 \frac{\eta}{2w} d\eta. \quad (\text{D3c})$$

For a delta-function initial condition corresponding to the initial zero charge in the circuit, the Gaussian kernel solves this equation and produces the normalized solution,

$$\rho^{(0)}(\chi, \eta, s) = \frac{e^{-\frac{\eta^2}{2}}}{2\pi\sqrt{2as}} \exp \left[- \frac{(\chi + 2w \ln \cosh \frac{\eta}{2w})^2}{4as} \right]. \quad (\text{D4})$$

This probability density yields the averages:

$$\langle \eta \rangle = 0, \quad \langle \sigma^2 \rangle = 2as, \quad (\text{D5a})$$

$$\langle \chi \rangle = - \frac{2w}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \ln \cosh \frac{\eta}{2w} d\eta \underset{w \rightarrow 0}{\sim} - \sqrt{\frac{2}{\pi}}. \quad (\text{D5b})$$

According to Eq. (D5a), the average charge at the capacitor C_0 is zero, capacitor 1 has negative average charge, and capacitor 2 has positive average charge of the same magnitude (the opposite signs to charges due to a battery if we ignore thermal fluctuations). This is a surprising result: The system does not evolve to the equilibrium $e^{-\eta^2/2}/\sqrt{2\pi}$. Instead, this initial layer builds up opposite charges at the capacitors and the variance of the state (D5a) increases linearly with time. Note that, in dimensional units, the charge (D5b) yields $\langle q_1 \rangle \sim -(1/2)\sqrt{2k_B T C_0/\pi}$, which is half the charge for the case of a single perfectly conducting diode (with piecewise linear current-voltage curve), except that the capacitor in series with the diode has been replaced by the small capacitance C_0 .

We can calculate the average energy rate and the entropy production from Eqs. (D2a) and (D4) using integration by parts. The results are as follows:

$$\frac{d}{ds} \langle \mathcal{H} \rangle = O(\epsilon),$$

$$\frac{dS}{ds} = - \frac{d}{ds} \int \rho \ln \rho d\eta d\chi = \frac{1}{as} + O(\epsilon). \quad (\text{D6})$$

The production of entropy declines as time elapses. Thus, the entropy increases to a large value after $t = 0$ and then it increases logarithmically as $s \rightarrow \infty$ at the end of the initial stage. Direct numerical simulations of the stochastic equations show an initial buildup of entropy followed by stabilization in Fig. 4(d).

In the long-time scaling (C3) with the variables ξ and t , Eq. (D4) becomes

$$\begin{aligned} \rho^{(0)}(\xi, \eta, t) &= \frac{e^{-\frac{\eta^2}{2}}}{2\pi\sqrt{2a\epsilon t}} \exp\left[-\frac{(\xi + 2w\epsilon \ln \cosh \frac{\eta}{2w})^2}{4a\epsilon t}\right] \\ &\sim \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}} \delta(\xi), \end{aligned} \quad (\text{D7a})$$

as $\epsilon \rightarrow 0$ and $t = O(1)$. Equations (D5) become

$$\langle \eta \rangle = 0, \quad \langle \xi \rangle = -\frac{2\epsilon w}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \ln \cosh \frac{\eta}{2w} d\eta, \quad (\text{D7b})$$

$$\begin{aligned} \langle (\xi - \langle \xi \rangle)^2 \rangle &= \frac{2\epsilon t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \text{sech}^2 \frac{\eta}{2w} d\eta \\ &+ \frac{4\epsilon^2 w^2}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \left(\ln \cosh \frac{\eta}{2w} \right)^2 d\eta \right. \\ &\left. - \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2}} \ln \cosh \frac{\eta}{2w} d\eta \right)^2 \right]. \end{aligned} \quad (\text{D7c})$$

APPENDIX E: LONG-TIME SCALING AND QUASISTATIONARY PROBABILITY DENSITY

After the initial layer described in Appendix D, and for appropriate small values of ϵ and w , the numerical solution of the FPE indicates that the probability density produces a very flat maximum of the average capacitor charge before decreasing to zero (thermal equilibrium); see Fig. 2. For sufficiently small values of ϵ and w , the flat maximum corresponds to a quasistationary solution of the FPE, which we derive in this section.

If the diodes are conducting in opposite directions, then we use the scaling (C3) producing Eqs. (C4). The stationary solution of $\mathcal{L}\rho = 0$ is

$$\begin{aligned} E(\xi, \eta) &= e^{-\frac{\eta^2}{2}} \exp\left[-\xi \int_0^\eta \frac{\mu(-s - \xi) - \mu(s - \xi)}{\mu(-s - \xi) + \mu(s - \xi)} ds\right] \\ &= e^{-\frac{\eta^2}{2}} \left(\frac{e^{-\frac{\xi}{w}} + \cosh \frac{\eta}{w}}{1 + e^{-\frac{\xi}{w}}} \right)^{\xi w}, \end{aligned} \quad (\text{E1})$$

which is integrable in the variable η . Moreover, the solution of the leading univariate FPE in fast timescale t/ϵ , $\partial\rho/\partial(t/\epsilon) = \mathcal{L}\rho$, tends to a normalized version of (E1) as $t/\epsilon \rightarrow \infty$:

$$\rho^{(0)} = \hat{E}(\xi, \eta) P(\xi, t; \epsilon) = \frac{E(\xi, \eta) P(\xi, t; \epsilon)}{\int_{-\infty}^{\infty} E(\xi, \eta) d\eta}, \quad (\text{E2a})$$

$$\int_{-\infty}^{\infty} P(\xi, t; \epsilon) d\xi = 1. \quad (\text{E2b})$$

Clearly, the average of η using Eq. (E2a) is zero because $E(\xi, \eta)$ is even in η . So the charge at the capacitor C_0 is zero.

1. Reduced FPE by the Chapman-Enskog method

To find the reduced equation for the slowly varying probability density $P(\xi; \epsilon)$, we use the Chapman-Enskog method:

$$\rho = \rho^{(0)}(\xi, \eta; P) + \sum_{j=1}^2 \epsilon^j \rho^{(j)}(\eta; P) + O(\epsilon^3), \quad (\text{E3a})$$

$$\frac{\partial P}{\partial t} = F^{(0)} + \epsilon F^{(1)} + O(\epsilon^2), \quad (\text{E3b})$$

$$\int_{-\infty}^{\infty} \rho^{(j)}(\eta; P) d\eta = 0, \quad (\text{E3c})$$

where the $F^{(j)}$ are functionals of P selected so that the hierarchy of linear equations for the $\rho^{(j)}$ have bounded solutions. The result is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \xi} \left[M(\xi; \epsilon) P + \epsilon D(\xi) \frac{\partial P}{\partial \xi} \right], \quad (\text{E4a})$$

$$\int_{-\infty}^{\infty} P(\xi, t; \epsilon) d\xi = 1.$$

The stationary solution of Eq. (E4a) is

$$P_s(\xi; \epsilon) = \frac{1}{Z} \exp\left[-\int \frac{M(\xi; \epsilon)}{\epsilon D(\xi)} d\xi\right]. \quad (\text{E4b})$$

This probability density is a globally stable solution of the reduced equation (E4a) because the following relative entropy is a Lyapunov functional:

$$\begin{aligned} \mathcal{F}[P] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{E} P \ln \left(\frac{\hat{E} P}{\hat{E} P_s} \right) d\eta d\xi \\ &= \int_{-\infty}^{\infty} P(\xi, t; \epsilon) \ln \left[\frac{P(\xi, t; \epsilon)}{P_s(\xi; \epsilon)} \right] d\xi. \end{aligned} \quad (\text{E4c})$$

Thus, the reduced probability density evolves towards $P_s(\xi; \epsilon)$ from the initial condition $P_s(0; \epsilon) = \delta(\xi)$, which is compatible with the initial layer of Eq. (D7a).

The drift and diffusion coefficients in the reduced FPE (E4a) are

$$\begin{aligned} M(\xi; \epsilon) &= 4\xi \int_{-\infty}^{\infty} \frac{\mu_1 \mu_2 \hat{E}}{\mu_1 + \mu_2} d\eta + \epsilon \int_{-\infty}^{\infty} \frac{\hat{E}}{\mu_1 + \mu_2} \left\{ 4(\mu_1 - \mu_2) \frac{\Psi}{\hat{E}} + 4\mu_1 \mu_2 \frac{\partial \ln \hat{E}}{\partial \xi} \right. \\ &+ (\mu_1^2 - \mu_2^2) \eta + (\mu_1 - \mu_2)^2 \xi - (\mu_1 + \mu_2) \frac{\partial(\mu_1 - \mu_2)}{\partial \eta} + 4\xi \mu_1 \mu_2 \\ &\left. \times \left[\frac{\eta^2}{2} - \int_{-\infty}^{\infty} \frac{\eta^2}{2} \hat{E} d\eta + \int_0^\eta \frac{\frac{4\Psi}{\hat{E}} + (\mu_1 - \mu_2) \left(\xi - \frac{\partial \ln \hat{E}}{\partial \xi} \right)}{\mu_1 + \mu_2} d\eta' - \int_{-\infty}^{\infty} d\eta \hat{E} \int_0^\eta \frac{\frac{4\Psi}{\hat{E}} + (\mu_1 - \mu_2) \left(\xi - \frac{\partial \ln \hat{E}}{\partial \xi} \right)}{\mu_1 + \mu_2} \right] \right\} d\eta, \end{aligned} \quad (\text{E5a})$$

$$\Psi(\xi, \eta) = \frac{\partial}{\partial \xi} \left(\xi \int_{\eta}^{\infty} \frac{\mu_1 \mu_2 \hat{E} d\eta'}{\mu_1 + \mu_2} \right) - \left(\int_{\eta}^{\infty} \hat{E} d\eta' \right) \frac{\partial}{\partial \xi} \left(\xi \int_{-\infty}^{\infty} \frac{\mu_1 \mu_2 \hat{E} d\eta'}{\mu_1 + \mu_2} \right), \quad (\text{E5b})$$

$$D(\xi) = \int_{-\infty}^{\infty} \frac{4\mu_1 \mu_2 \hat{E}}{\mu_1 + \mu_2} \left\{ 1 + \xi \left(\frac{1}{\mu_2} - \frac{1}{\mu_1} \right) \Phi + 4\xi^2 \left[\int_0^{\eta} \frac{\Phi d\eta'}{(\mu_1 + \mu_2) \hat{E}} - \int_{-\infty}^{\infty} \hat{E} \int_0^{\eta} \frac{\Phi d\eta'}{(\mu_1 + \mu_2) \hat{E}} \right] \right. \\ \left. - \xi \left(\int_0^{\eta} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} d\eta' - \int_{-\infty}^{\infty} d\eta \hat{E} \int_0^{\eta} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} d\eta' \right) \right\} d\eta, \quad (\text{E5c})$$

$$\Phi(\xi, \eta) = \int_{\eta}^{\infty} \frac{\mu_1 \mu_2 \hat{E} d\eta'}{\mu_1 + \mu_2} - \left(\int_{-\infty}^{\infty} \frac{\mu_1 \mu_2 \hat{E} d\eta'}{\mu_1 + \mu_2} \right) \int_{\eta}^{\infty} \hat{E} d\eta, \quad \Phi(\xi, -\infty) = 0. \quad (\text{E5d})$$

If we have only one diode, say, $\mu_2 = 0$, then $M = D = 0$ as the scaling (C3) does not make sense. Note that $(\mu'_1 - \mu'_2)$ is large and therefore the dominant terms in the drift coefficient (E5a) are

$$M(\xi; \epsilon) \approx 4\xi \int_{-\infty}^{\infty} \frac{\mu_1 \mu_2 \hat{E}}{\mu_1 + \mu_2} d\eta - \epsilon \int_{-\infty}^{\infty} \frac{\partial(\mu_1 - \mu_2)}{\partial \eta} \hat{E} d\eta. \quad (\text{E5e})$$

The reduced stationary probability density has to be calculated numerically because it cannot be approximated by a Gaussian function with a large second derivative about its maximum. Then the integrals entering the stationary averages over ξ cannot be approximated by the usual expansion about the maximum of the integrand. Figure 5 depicts the reduced stationary probability density $P_s(\xi)$ of Eq. (E4b) for two different values of w . Lowering w shifts the maximum of $P_s(\xi)$ to the left, which yields a negative average charge $\langle q_1 \rangle = -C\sqrt{T/c} |\langle \xi \rangle|$ at the first capacitor and a positive charge $\langle q_2 \rangle = C\sqrt{T/c} |\langle \xi \rangle|$ at the second capacitor. Notice that the signs of these charges coincide with those provided by the initial layer of Appendix D and are contrary to those of the charges produced by a noiseless circuit with a battery as in Fig. 1(a).

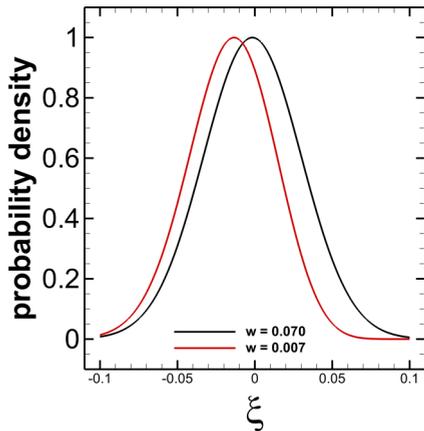


FIG. 5. Reduced stationary probability density $P_s(\xi)$ for two different values of w . Lowering w shifts the maximum of $P_s(\xi)$ thereby yielding nonzero average charge $\langle \xi \rangle$.

2. Calculation of the coefficient functions in the reduced FPE

How do we find the coefficients in Eq. (E4a)? Inserting Eqs. (E3a) and (E3b) into (C4a), we obtain Eq. (E2a) and the hierarchy

$$\mathcal{L}\rho^{(1)} = F^{(0)}\hat{E} - (2\mathcal{L} + \mathcal{M}_1)\rho^{(0)}, \quad (\text{E6a})$$

$$\mathcal{L}\rho^{(2)} = F^{(1)}\hat{E} + \frac{\delta\rho^{(1)}}{\delta P} F^{(0)} - (2\mathcal{L} + \mathcal{M}_1)\rho^{(1)} \\ - (\mathcal{L} + \mathcal{M}_2)\rho^{(0)}, \quad (\text{E6b})$$

etc. The solvability condition for Eq. (E6a) is that the integral of its right-hand side over $\eta \in \mathbb{R}$ vanish. This yields

$$F^{(0)} = 4 \frac{\partial}{\partial \xi} \left[\xi P \left(\frac{\mu_1 \mu_2 \hat{E}}{\mu_1 + \mu_2} \right) \right], \quad (\text{E7a})$$

$$\overline{f(\eta)} = \int_{-\infty}^{\infty} f(\eta) d\eta. \quad (\text{E7b})$$

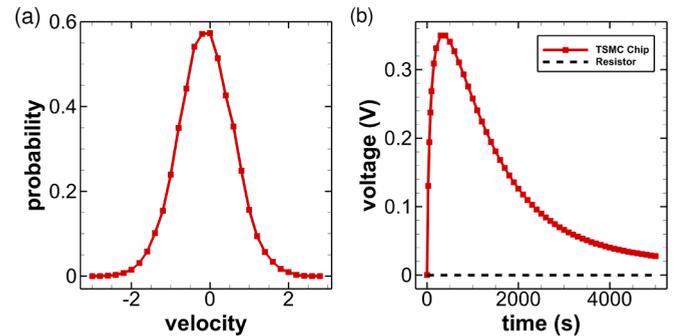


FIG. 6. (a) Measured histogram of the numerically simulated velocity distribution for graphene. (b) Measured voltage versus time plot for a chip comprising a single diode-capacitor circuit (as control, same plot for a resistor).

The solution of Eq. (E6a) that satisfies (E3c) is

$$\rho^{(1)} = \hat{E}(R - \overline{\hat{E}R}), \quad (\text{E8a})$$

where

$$\frac{\partial R}{\partial \eta} = \frac{4\xi \frac{\Phi}{\hat{E}} - \mu_1 + \mu_2}{\mu_1 + \mu_2} \frac{\partial P}{\partial \xi} + \left[\eta + \frac{4\frac{\Psi}{\hat{E}} + (\mu_1 - \mu_2)(\xi - \frac{\partial \ln \hat{E}}{\partial \xi})}{\mu_1 + \mu_2} \right] P, \quad (\text{E8b})$$

$$\begin{aligned} \rho^{(1)} = & \left(\int_0^\eta \frac{4\xi \frac{\Phi}{\hat{E}} - \mu_1 + \mu_2}{\mu_1 + \mu_2} - \overline{\hat{E} \int_0^\eta \frac{4\xi \frac{\Phi}{\hat{E}} - \mu_1 + \mu_2}{\mu_1 + \mu_2}} \right) \hat{E} \frac{\partial P}{\partial \xi} \\ & + \left[\frac{\eta^2}{2} - \frac{\overline{\eta^2 \hat{E}}}{2} + \int_0^\eta \frac{4\frac{\Psi}{\hat{E}} + (\mu_1 - \mu_2)(\xi - \frac{\partial \ln \hat{E}}{\partial \xi})}{\mu_1 + \mu_2} - \overline{\hat{E} \int_0^\eta \frac{4\frac{\Psi}{\hat{E}} + (\mu_1 - \mu_2)(\xi - \frac{\partial \ln \hat{E}}{\partial \xi})}{\mu_1 + \mu_2}} \right] \hat{E} P. \end{aligned} \quad (\text{E8c})$$

After integration by parts, the solvability condition for Eq. (E6b) yields

$$F^{(1)} = \frac{\partial}{\partial \xi} \left\{ \overline{(\mu_1 + \mu_2) \hat{E}} \frac{\partial P}{\partial \xi} + \left[\overline{(\mu_1 + \mu_2) \frac{\partial \hat{E}}{\partial \xi}} - (\mu'_1 - \mu'_2) \hat{E} \right] P + \frac{4\mu_1 \mu_2 \xi}{\mu_1 + \mu_2} \rho^{(1)} + \overline{(\mu_1 - \mu_2) \hat{E}} \frac{\partial R}{\partial \eta} \right\}. \quad (\text{E9})$$

We now insert Eqs. (E7a) and (E9) in (E3b) and use Eqs. (E8), thereby obtaining the drift and diffusion coefficients of Eqs. (E5) after some algebra.

APPENDIX F: DATA FROM STOCHASTIC SIMULATIONS AND FROM EXPERIMENTS

Figure 6(a) shows the simulated velocity distribution of graphene after a short time (1% of the total simulation time). It is a Gaussian corresponding to local equilibrium as explained in Appendix A.

Figure 6(b) depicts the time evolution of the voltage across the capacitor (proportional to stored charge) measured using the single diode-capacitor circuit. We observe that it follows the theoretical curve for the inset of Fig. 1(a).

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