Set theory

Readings: C&C, Ch. 2.1–2

1. Sets

1.1. Introducing sets and their basic properties

- *Sets* are abstract unordered collections of distinct objects:

(1) $\{2, 4, 11\}$

(2) $\{4, 2, 11\}$

- Note the word *unordered*. The two sets above are the same.
- The objects in a set are called *members* or *elements* of that set:

(3) $2 \in \{2, 4, 11\}$

(4) $3 \notin \{2, 4, 11\}$

- Anything can be an element of a set, including other sets:

(5) $\{8, \{\pi, \text{Mary}\}, \text{John}, \Diamond\}$

- A set can be infinite:

(6) {1, 3, 5, 7, ...}

- *Singleton sets* contain only one element:

(7) {6}

- The *empty set* contains no elements; the common notation for the empty set is:

(8) {}

or

(9) $\varnothing$

- The *cardinality* of a set is the number of elements it contains:

(10) $|\{5, 0, \text{Mary}\}| = 3$

- Sometimes we can't list all the elements of the set, because there are too many of them, or because we don't know what they all are. But as long as we know the unique distinctive feature of a given set, we can use *predicate notation* to describe that set. E.g., instead of listing all novels, one can use predicate notation to define the set of all novels:

(11) a. {Emma, 1984, Frankenstein, Middlemarch, Trainspotting, ...}
b. $\{x \mid x \text{ is a novel}\}$

In-class Exercise 1

- What is the cardinality of each of the following sets?

(12) a. $\{13, \varnothing\}$
b. $\{404, \{7, 66, 1\}\}$
c. $\{a, b, b, c\}$
d. $\{a, \{b\}, b, c\}$

- Are the following statements true?

(13) a. $\{3, 3, 2\} = \{3, 2\}$
b. {Mary, {John}} = {John, Mary}
c. $\{\varnothing\} = \varnothing$
d. $\{x \mid x \in A\} = A$
e. Mary $\in \{x \mid 1 = 1\}$
f. Mary $\in \{x \mid 1 = 0\}$

1.2. Relations between sets

- One set can be a *subset* of another:

(14) $\{9, 11\} \subseteq \{9, 11, 13\}$

(15) $\{9, 10\} \nsubseteq \{9, 11, 13\}$

- Formal definition of subsethood:

(16) $A \subseteq B$ iff for all x : if $x \in A$, then $x \in B$
($A \subseteq B$ is true if and only if every element of A (if any) is also an element of B)

- By the definition in (16), every set is a subset of itself. If A is a subset of B, but not equal to it, A is a *proper subset*:

(17) $\{9, 11\} \subset \{9, 11, 13\}$

(18) $\{9, 11, 13\} \not\subset \{9, 11, 13\}$

- Formal definition of proper subsethood:

(19) $A \subset B$ iff $A \neq B$ and for all x : if $x \in A$, then $x \in B$

- The empty set is a subset of every set.
- The reverse of subset is *superset*. Supersets can be proper, too.

(20) $\{42, 3, 11\} \supset \{42, 3\}$

(21) $\{42, 3, 11\} \supseteq \{42, 3, 11\}$

(22) $\{42, 3, 11\} \not\supset \{42, 3, 11\}$

In-class Exercise 2

- Are the following statements true?

(23) a. $\{d, f\} \subseteq \{d, e, f\}$
b. $\varnothing \subseteq \{\text{John}, 14\}$
c. $\varnothing \in \{a, 0\}$
d. $\{6, 9\} \subseteq \{6, \{9\}, 7\}$
e. $\{55, \{99\}\} \not\subseteq \{55, 77, \{99\}\}$

1.3. Operations on sets

- The *intersection* of A and B, written $A \cap B$, is the set of all entities x such that x is an element of A and x is an element of B, e.g.:

(24) $\{w, x, y\} \cap \{x, y, z\} = \{x, y\}$

- If the intersection of two sets is empty, those two sets are called *disjoint*.
- The *union* of A and B, written $A \cup B$, is the set of all entities x such that x is an element of A or x is an element of B, e.g.:

(25) $\{w, x\} \cup \{y, z\} = \{w, x, y, z\}$

- We can also *subtract* one set from another. The *difference* of A and B, written $A - B$ or $A \setminus B$, is the set of all entities x such that x is an element of A and x is not an element of B, e.g.:

(26) $\{w, x, y, z\} - \{v, x, y\} = \{w, z\}$

- The result of subtracting B from A is often called the *complement set* of B relative to A.
- The *absolute complement* of A, written $\overline{A}$, is the set of all things that are not elements of A.

In-class Exercise 3

- Draw diagrams for intersection, union, and difference (shade the resulting areas). Do these diagrams remind you of anything?

2. Ordered pairs, relations and functions

2.1. Ordered pairs

- *Ordered pairs* are abstract pairs of objects, but, unlike sets, they are... ordered:

(27) $\{19, 91\} = \{91, 19\}$

(28) $\langle 19, 91\rangle \neq \langle 91, 19\rangle$

- Like with sets, anything can be an element of an ordered pair, including sets and other ordered pairs:

(29) $\langle\{1984, \dagger\}, \langle m, e\rangle\rangle$

In-class Exercise 4

- Are the following statements true?

(30) a. $\langle\{x, y\}, \{y, z\}\rangle = \langle\{y, z\}, \{x, y\}\rangle$
b. $\langle 0, \{88, a\}\rangle = \langle 0, \{a, 88\}\rangle$
c. $\{\langle a, b\rangle, \langle b, a\rangle\} = \{\langle b, a\rangle, \langle a, b\rangle\}$

2.2. Relations

- *Relations* are sets of ordered pairs. For example, (31) is the relation that relates each positive integer to the one following it.

(31) $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle, ...\}$

- The set from which the first elements of a relation are drawn is called its *domain*, and the set from which the second elements are drawn is called *range*.
- If you take two sets A and B and pair each element of A with each element of B, the result is the *Cartesian product* of A and B:

(32) $\{a, b\} \times \{0, 1, 2\} = \{\langle a, 0\rangle, \langle a, 1\rangle, \langle a, 2\rangle\ \langle b, 0\rangle, \langle b, 1\rangle, \langle b, 2\rangle\}$

In-class Exercise 5

- What English word could be represented by the relation in (33)?

(33) {⟨Michelle Obama, Malia Obama⟩, ⟨Judy Garland, Liza Minnelli⟩, ⟨Catelyn Stark, Sansa Stark⟩, ⟨Eileen Prince, Severus Snape⟩, ...}

- Write this relation in predicate notation.

- Write the relation in (31) in predicate notation.

2.3. Functions

- *Functions* are special relations. This means that every function is a relation, but not every relation is also a function.
- In a function, there can't be two pairs that agree in their first element but disagree in their second element. That is, if $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$ are both pairs of the function and $x_1 = x_2$, then also $y_1 = y_2$.
- The relation in (31) is a function, because every positive integer has exactly one positive integer following it.
- We can also write this function in the following way.

(34) $f(x) = x + 1$

In-class Exercise 6

- Is the relation from In-Class Exercise 5 also a function? Come up with two more examples, one of a function and one of a relation that is not a function.

What you need to know

Key notions: set, member/element, singleton set, empty set, cardinality, subset, proper subset, superset, proper superset, intersection, union, subtraction, difference, complement set, complement, ordered pair, relation, domain, range, Cartesian product, function

Answers to the following questions:
- What can be an element of a set? Of an ordered pair?
- What is the main difference between sets and ordered pairs?
- What property must a relation have in order to be a function?

Skills:
- Use predicate notation to describe sets.