Topic 1

Real Business Cycle Theory: Part 1

The Stochastic Neoclassical Growth Model with Variable Labor Supply

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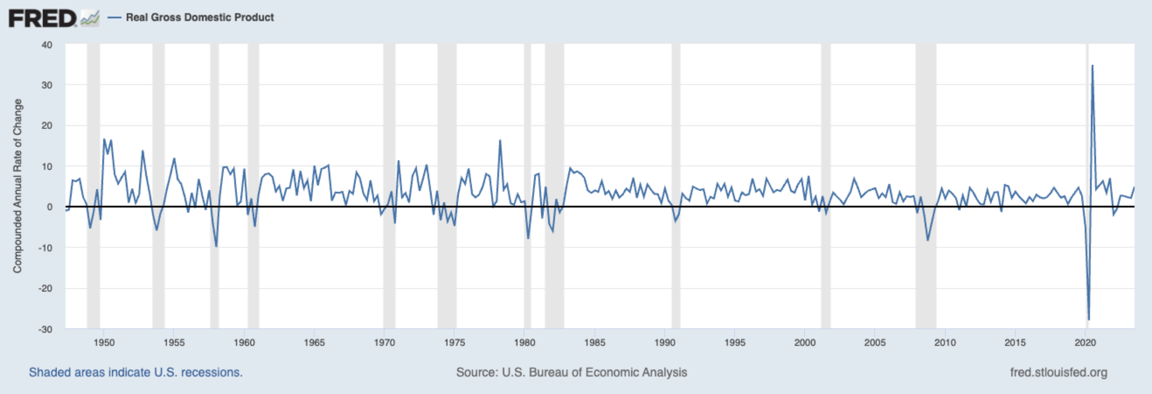
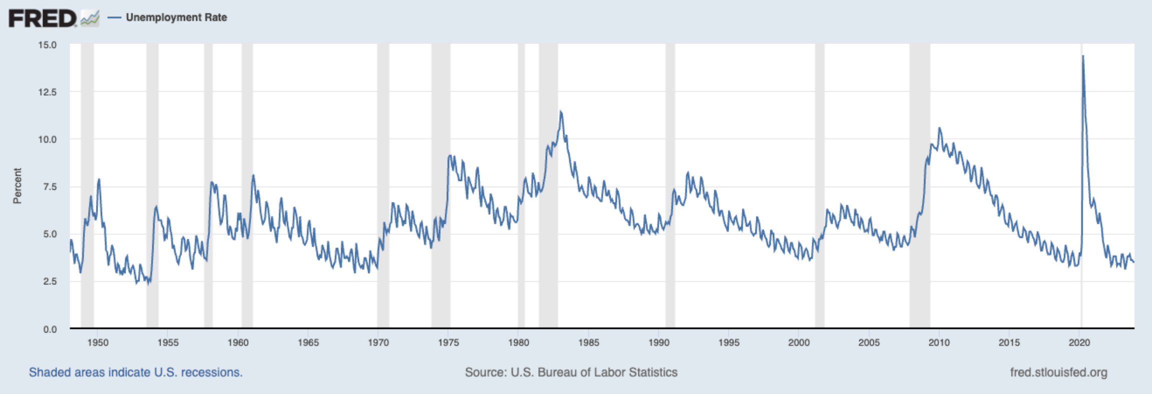
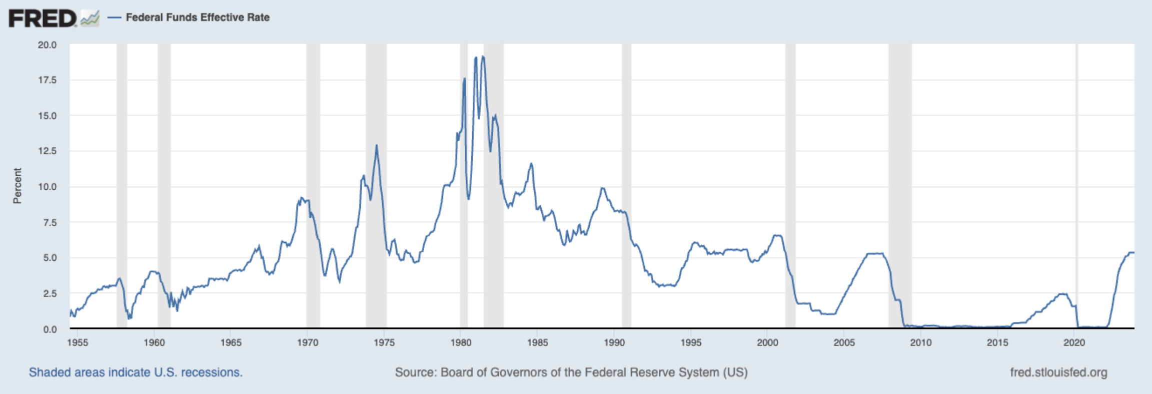

FRED
Real Gross Domestic Product
Compounded Annual Rate of Change
40
30
20
10
0
-10
-20
-30
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
2020
Shaded areas indicate U.S. recessions.
Source: U.S. Bureau of Economic Analysis
fred.stlouisfed.org


FRED
Unemployment Rate
Percent
15.0
12.5
10.0
7.5
5.0
2.5
0.0
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
2020
Shaded areas indicate U.S. recessions.
Source: U.S. Bureau of Labor Statistics
fred.stlouisfed.org


FRED
— Federal Funds Effective Rate
Percent
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
2020
Shaded areas indicate U.S. recessions.
Source: Board of Governors of the Federal Reserve System (US)
fred.stlouisfed.org

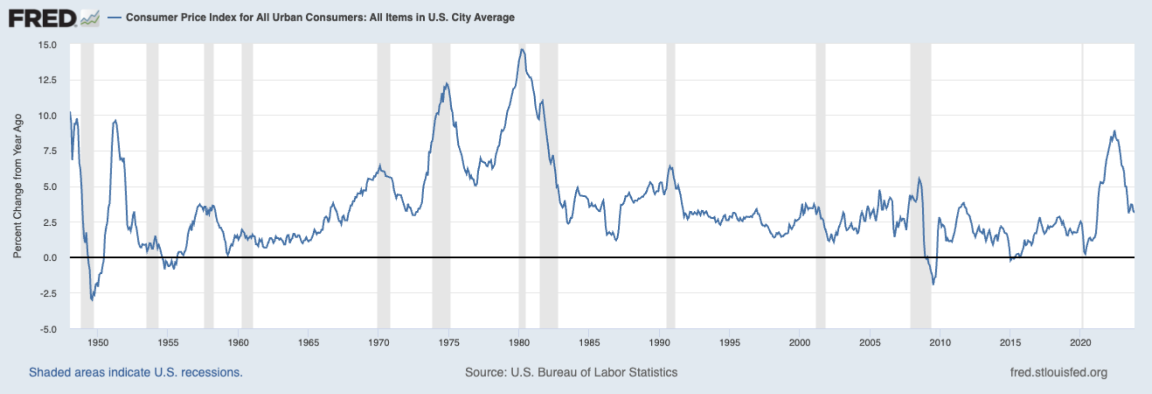
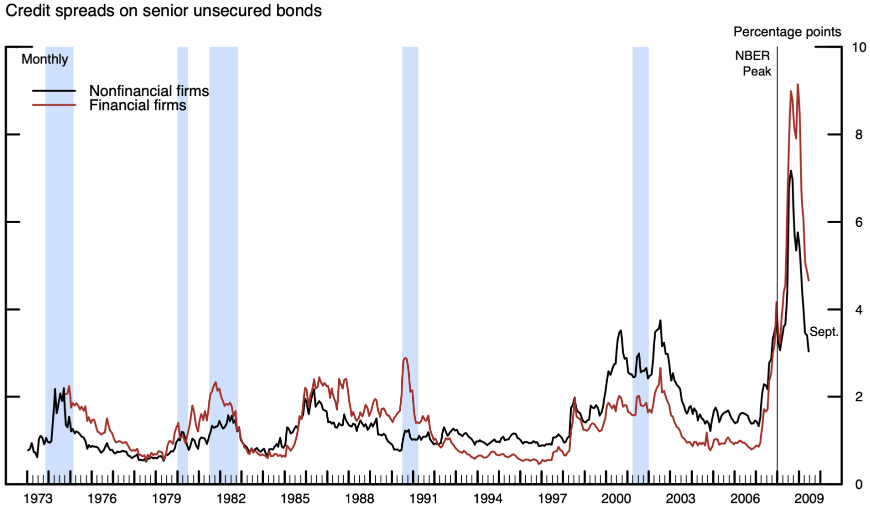

FRED
— Consumer Price Index for All Urban Consumers: All Items in U.S. City Average
Percent Change from Year Ago
15.0
12.5
10.0
7.5
5.0
2.5
0.0
-2.5
-5.0
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
2020
Shaded areas indicate U.S. recessions.
Source: U.S. Bureau of Labor Statistics
fred.stlouisfed.org

Figure 5: Credit Spreads, 1973-2009



Background

- RBC developed as a response to the failure of large macroeconometric models during the late 1960s and early 1970s.
- Objective: Derive a model of fluctuations purely from first principles, where the only exogenous restrictions involve preferences and technology
- Candidate model: Stochastic neoclassical growth model. Among the virtues: A unified theory of the cycle and the trend
- Side-product: "Calibration" introduced as a way to assign values to model parameters. Involves bringing in independent information
- In the end: RBC is a failure as a model of business cycle fluctuations
- But RBC is an important methodological advance. Modern macro models build on RBC by adding "frictions" in order to confront data
- Calibration also controversial, but has had major influence

Outline

Part 1.
Stochastic growth model with variable labor supply
Planning solution
Decentralized solution
Steady state

Part 2.
Loglinear approximation
Business cycle dynamics: Key properties
Calibration and quantitative performance
Shortcomings
Business cycle accounting: sources of deviations from the data

Model Setup

Background Environment:

A stochastic intertemporal general equilibrium with capital and variable labor supply.

Representative household (equivalently: continuum of measure unity identical households).

Markets are competitive, complete and frictionless.
(1st and 2nd welfare theorems apply).

Baseline: no growth (output constant in steady state). Then consider growth.

Model Setup (con't)

Preferences:

$$E_t \left[\sum_{i=0}^{\infty} \beta^{t+i} [u(C_{t+i}) - \upsilon(L_{t+i})] \right]$$

with

$$\begin{aligned} u(C) &= \tfrac{1}{1-\gamma} C^{1-\gamma} \\ &= \log C \text{ iff } \gamma = 1 \end{aligned}$$

$$\upsilon(L) = \tfrac{1}{1+\varphi} L^{1+\varphi}$$

with $0 < \beta < 1$; $\gamma > 0$; $\varphi > 0$

$\gamma \equiv$ coefficient of relative risk aversion; $\varphi^{-1} \equiv$ Frisch elasticity of labor supply

where $C_t \equiv$ consumption, $L_t \equiv$ labor supply.

Model Setup (con't)

Technology:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} = A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}$$

where $Y_t \equiv$ output, $A_t^{1-\alpha} \equiv$ total factor productivity, $K_t \equiv$ capital, $L_t \equiv$ labor input.
Note: Technology labor augmenting to ensure balanced growth path exists.
Resource Constraint ($\rightarrow$ Law of Motion for Capital):

$$C_t + K_{t+1} = Y_t + (1-\delta)\, K_t$$

where $0 < \delta < 1$ is the depreciation rate and TFP obeys:

$$\begin{aligned} A_t/\overline{A}_t &= (A_{t-1}/\overline{A}_{t-1})^{\rho} \cdot e^{\epsilon_t} \\ \overline{A}_t/\overline{A}_{t-1} &= G = 1 + g \geq 1 \end{aligned}$$

where $\overline{A}_t =$ trend TFP, $0 \leq \rho < 1$ and ϵ_t is i.i.d. with mean zero. Baseline: $G = 1$.

Planning Problem

With frictionless markets and no externalities, the planning problem and the decentralized problem yield the same (Pareto efficient) allocation in equilibrium.

Combine production function and resource constraints to eliminate $Y_t \rightarrow$
Social planner's sequence problem given initial state (K_t, A_t) :

$$V(K_t, A_t) = \max_{\{C_{t+i}, L_{t+i}, K_{t+1+i}\}_{i\geq 0}} E_t \left[\sum_{i=0}^{\infty} \beta^{t+i} \left(\frac{1}{1-\gamma} C_{t+i}^{1-\gamma} - \frac{1}{1+\varphi} L_{t+i}^{1+\varphi} \right) \right]$$

subject to

$$\begin{aligned} C_t + K_{t+1} &= K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t \\ A_t / \overline{A}_t &= (A_{t-1} / \overline{A}_{t-1})^{\rho} e^{\epsilon_t} \\ K_0 &= K \\ A_0 &= A \end{aligned}$$

Planning Problem: Bellman Equation

$$V(K_t, A_t) = \max_{C_t, L_t, K_{t+1}} \frac{1}{1-\gamma} C_t^{1-\gamma} - \frac{1}{1+\varphi} L_t^{1+\varphi} + \beta E_t\{V(K_{t+1}, A_{t+1})\}$$

subject to

$$C_t + K_{t+1} = K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t$$

The solution yields the policy functions: $C(K_t, A_t), L(K_t, A_t), K_{t+1}(K_t, A_t)$.

Note: (i) Any two policy functions combined with the resource constraint implies the third. (ii) Concavity restrictions on preferences and technology ensure that second order conditions from max. problem are satisfied.

Solution

To solve: (i) use the resource constraint to eliminate C_t in the objective; (ii) optimize w.r.t. (K_{t+1}, L_t); (iii) use the envelope theorem to find $V_1(K_t, A_t)$. $\rightarrow$

$$C_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)\,K_t - K_{t+1}$$

FONC w.r.t. K_{t+1}

$$C_t^{-\gamma} = \beta E_t\{V_1(K_{t+1}, A_{t+1}\}$$

Intuition: Marginal utility cost of foregoing a unit of consumption (LHS) to acquire a unit of capital must equal expected discounted marginal utility benefit from doing so (RHS).

Given $\alpha(\frac{K_t}{A_t L_t})^{\alpha-1} =$ marginal product of capital, then from the envelope theorem:

$$V_1(K_t, A_t) = C_t^{-\gamma}[\alpha(\frac{K_t}{A_t L_t})^{\alpha-1} + 1 - \delta]$$

Solution (con't)

Update one period:

$$V_1(K_{t+1}, A_{t+1}) = C_{t+1}^{-\gamma}[\alpha(\frac{K_{t+1}}{A_{t+1}L_{t+1}})^{\alpha-1} + 1 - \delta]$$

Then combine with the FONC to obtain the first order condition for consumption/saving (known as the "consumption euler condition")

$$C_t^{-\gamma} = E_t\{\beta C_{t+1}^{-\gamma} R_{t+1}\}$$

where $R_{t+1} \equiv$ gross return on capital:

$$R_{t+1} = \alpha(\frac{K_{t+1}}{A_{t+1}L_{t+1}})^{\alpha-1} + (1 - \delta)$$

(Note R_{t+1} is random since A_{t+1} is random.)
Intuition: Equate marginal utility cost of saving (LHS) to expected discounted marginal utility benefit (RHS), where the latter is the gross marginal return on capital weighted by discounted marginal utility of consumption at $t + 1$.

Solution (con't)

First order condition for labor supply:

$$(1-\alpha)A_t\left(\frac{K_t}{A_tL_t}\right)^{\alpha}C_t^{-\gamma} = L_t^{\varphi}$$

Intuition: Equate marginal utility benefit from another unit of labor (LHS: the marginal product of labor weighted by the marginal utility of consumption) with the marginal disutility of supply labor (RHS).

While the consumption/saving decision is intertemporal, the labor supply decision is static (period by period.)

Necessary and Sufficient conditions for optimality

- First order condition for consumption saving:

$$C_t^{-\gamma} = E_t\{\beta C_{t+1}^{-\gamma}[\alpha(\frac{K_{t+1}}{A_{t+1}L_{t+1}})^{\alpha-1} + (1-\delta)]\}$$

- First order condition for labor supply:

$$(1-\alpha)A_t(\frac{K_t}{A_t L_t})^{\alpha} C_t^{-\gamma} = L_t^{\varphi}$$

- Transversality condition (to rule out unbounded discounted paths for K)

$$\lim_{t\to\infty} \beta^t C_t^{-\gamma} K_{t+1} = 0$$

These equations combined with resource constraint determine C_t, K_{t+1}, L_t.

Necessary and Sufficient conditions for optimality

First order conditions in units of consumption

- Labor supply

$$(1-\alpha)A_t\left(\frac{K_t}{A_tL_t}\right)^{\alpha} = \frac{L_t^{\varphi}}{C_t^{-\gamma}}$$

  - LHS: marginal product of labor (MPL) in units of consumption goods
  - RHS: marginal cost of supply labor in units of consumption goods
    - $\equiv$ "marginal rate of substitution" (MRS) between consumption. and leisure

- Consumption/saving

$$1 = E_t\left\{\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left[\alpha\left(\frac{K_{t+1}}{A_{t+1}L_{t+1}}\right)^{\alpha-1} + (1-\delta)\right]\right\}$$

  - $\beta(\frac{C_{t+1}}{C_t})^{-\gamma} \equiv$ stochastic discount factor (SDF): tells how to value return on assets in units of consumption goods
  - $\rightarrow$ Covariance between SDF and asset return is important (finance 101!)

Complete Model

Endogenous variables: $(Y_t, L_t, C_t, R_{t+1}, K_{t+1})$; Predetermined states: (K_t, A_t)

output:	$Y_t = A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}$
labor:	$(1-\alpha) A_t (\frac{K_t}{A_t L_t})^{\alpha} = \frac{L_t^{\varphi}}{C_t^{-\gamma}}$
consumption/saving:	$1 = E_t\{\beta(\frac{C_{t+1}}{C_t})^{-\gamma} R_{t+1}\}$
gross return on capital:	$R_{t+1} = \alpha(\frac{K_{t+1}}{A_{t+1} L_{t+1}})^{\alpha-1} + 1 - \delta$
resource constraint:	$K_{t+1} = Y_t + (1-\delta) K_t - C_t$
evolution of technology:	$A_t/\overline{A}_t = (A_{t-1}/\overline{A}_t)^{\rho} e^{\epsilon_t}$

Cyclical driving force: Fluctuations in A_t.

Before solving model, we first show equivalence with the decentralized (competitive equilibrium) solution.

Decentralized Solution

- Households:Continuum of measure unity identical households

  Household h consumes $C(h)$, supplies labor $L(h)$, saves capital $K(h)$ which it rents to firms
  Acts competitively - takes real wage W and rental rate on capital Z as given

- Firms:

  Continuum of measure unity firms with identical technologies
  Firm f produces output using labor $L(f)$ and capital $K(f)$
  Acts competitively - takes real wage W and rental rate on capital Z as given

* Market clearing determines W, Z and equilibrium quantities

Household Decision Problem

$\Gamma_t \equiv$ macro state (K_t, A_t);

$$V(K_t(h), \Gamma_t) = \max_{\{C(h), L(h)_t, K(h)_{t+1}\}} E_t \left[\sum_{i=0}^{\infty} \beta^t \left(\frac{1}{1-\gamma} C_t(h)^{1-\gamma} - \frac{1}{1+\varphi} L_t(h)^{1+\varphi} \right) \right]$$

subject to the period budget constraint:

$$C_t(h) = W_t L_t(h) + (Z_t + (1-\delta))K_t(h) - K_{t+1}(h)$$

and a terminal condition on wealth that rules out "Ponzi" schemes:

$$\lim_{\tau\to\infty} \beta^\tau (\tfrac{C_\tau(h)}{C_t(h)})^{-\gamma}[Z_\tau + 1 - \delta)]K_\tau(h) \geq 0$$

(i.e. requires the household to satisfy its' intertemporal budget constraint.)

Necessary conditions for household optimality

First order condition for consumption/saving:

$$1 = E_t\{\beta(\frac{C_{t+1}(h)}{C_t(h)})^{-\gamma}R_{t+1}\}$$

with

$$R_{t+1} = Z_{t+1} + 1 - \delta$$

First order condition for labor supply:

$$W_t = L_t(h)^{\varphi}/C_t(h)^{-\gamma}$$

W_t and Z_t (and hence R_t) determined in general equilibrium

Note: Identical households of measure unity $\rightarrow C_t(h) = C_t,\ L_t(h) = L_t, K_t(h) = K_t$

Firms

Continuum of measure unity firms
Firms hire labor and rent capital on a period by period basis
No factor adjustment costs $\rightarrow$ factor demand is a static decision
Constant returns and competition $\rightarrow$ zero profits

Firm decision problem:

$$\max_{K_t(f),L_t(f)} \; Y_t(f) - Z_t K_t(f) - W_t L_t(f)$$

subject to:

$$Y_t(f) = A_t^{1-\alpha} K_t(f)^{\alpha} L_t(f)^{1-\alpha}$$

Necessary conditions for firm optimality

First order conditions for capital and labor:

$$\alpha(\frac{K(f)_t}{A_t L_t(f)})^{\alpha-1} = Z_t$$
$$(1-\alpha)A_t(\frac{K_t(f)}{A_t L_t(f)})^{\alpha} = W_t$$

Some implications (from constant returns and factor mobility):

Identical K_t/L_t ratios across firms: from FONCs

$$\frac{K_t(f)}{L_t(f)} = \frac{K_t}{L_t} = \frac{W_t}{Z_t}\frac{\alpha}{1-\alpha}$$

Zero profits: FONCs $\rightarrow Z_t K_t(f) = \alpha Y_t(f)$ and $W_t L_t(f) = (1-\alpha)Y_t(f) \rightarrow$

$$Y_t(f) - Z_t K_t(f) - W_t L_t(f) = 0$$

Individual firm size indeterminate (though size of firm sector pinned down)

Equilibrium

Equilibrium: an allocation (C_t, L_t, Y_t, K_{t+1}) and prices (W_t, Z_t) are a competitive equilibrium iff households and firms are maximizing and all markets clear. Conditions:

output: $Y_t = A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}$

labor market clearing: $(1-\alpha)A_t(\frac{K_t}{A_t L_t})^{\alpha} = W_t = \frac{L_t^{\varphi}}{C_t^{-\gamma}}$

capital rental: $Z_t = \alpha(\frac{K_{t+1}}{A_{t+1}L_{t+1}})^{\alpha-1}$

consumption/saving: $1 = E_t\{\beta(\frac{C_{t+1}}{C_t})^{-\gamma}(Z_{t+1}+1-\delta))\}$

resource constraint: $K_{t+1} = Y_t + (1-\delta)K_t - C_t$

technology: $A_t/\overline{A}_t = (A_{t-1}/\overline{A}_t)^{\rho} e^{\epsilon_t}$

Competitive equilibrium equivalent to planning solution (given frictionless markets and no externalities).

Deterministic Steady State: case of $g = 0$

With $g = 0$, all variables constant in deterministic steady state. Convenient to write system as 4 equations in the following 4 unknowns: $Y/AL, (K/AL), C/AL, L$:
Production function:

$$\frac{Y}{AL} = (\frac{K}{AL})^{\alpha}$$

Consumption/saving

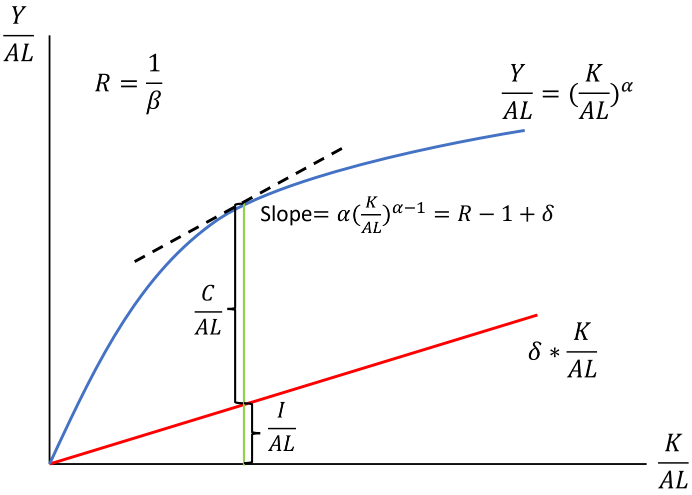
$$1 = \beta R = \beta[\alpha(\frac{K}{AL})^{\alpha-1} + 1 - \delta] \rightarrow \quad \beta^{-1} = R = \alpha(\frac{K}{AL})^{\alpha-1} + 1 - \delta$$

Resource constraint

$$\frac{Y}{AL} = \delta\frac{K}{AL} + \frac{C}{AL}$$

Note: these 3 equations determine $Y/AL, (K/AL), C/AL$
The labor market condition then determines L.

Steady State: $\frac{K}{AL}, \frac{Y}{AL}, \frac{C}{AL}$



Deterministic Steady State: case of $g = 0$

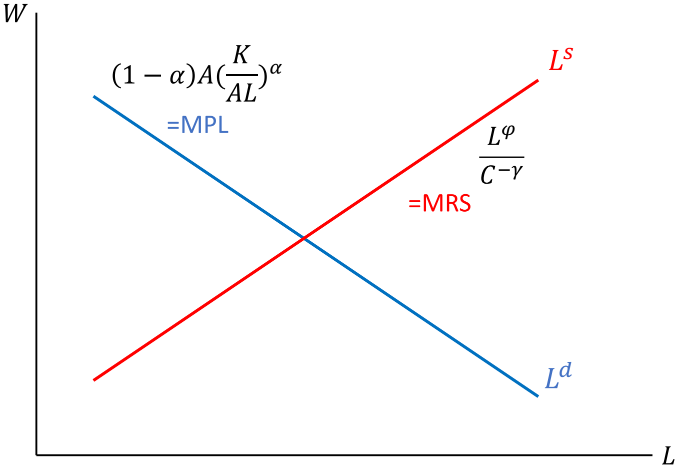
Labor market

$$
\begin{aligned}
(1-\alpha)A\left(\frac{K}{AL}\right)^{\alpha} &= \frac{L^{\varphi}}{C^{-\gamma}} \\
&= \frac{L^{\gamma+\varphi}A^{\gamma}}{(C/AL)^{-\gamma}}
\end{aligned}
$$

LHS: Marginal product of labor: increasing in A and $\frac{K}{AL}$.

RHS: Marginal cost (MRS); increasing in L and in C (wealth effect)

Labor Market Equilibrium



Transition Dynamics

$$(K_t/AL_t) < K/AL \rightarrow \alpha(\frac{K_t}{AL_t})^{\alpha-1} + 1 - \delta = R_t > R = \beta^{-1}$$

$\rightarrow$ increased saving $\rightarrow (\frac{C'}{C} \uparrow) \rightarrow \quad (K_t/AL_t)$ converges to K/AL.

$C_t \downarrow$ due to increased saving $\rightarrow L_t \uparrow \rightarrow Y_t \uparrow$ which speeds convergence.

Deterministic Steady State: Case of Growth

Express as $\frac{Y}{\overline{AL}}, \frac{K}{\overline{AL}}, \frac{C}{\overline{AL}}, L$ (variables that are constant along balanced growth path)
$(\frac{Y}{\overline{AL}}, \frac{K}{\overline{AL}}, \frac{C}{\overline{AL}})$ determined by production function, consumption/saving relation and resource constraint:

$$\frac{Y}{\overline{AL}} = (\frac{K}{\overline{AL}})^{\alpha}$$

$$1 = \beta(\frac{C'}{C})^{-\gamma}[\alpha(\frac{K}{\overline{AL}})^{\alpha-1} + 1 - \delta] = \beta(1+g)^{-\gamma}[\alpha(\frac{K}{\overline{AL}})^{\alpha-1} + 1 - \delta]$$

$$\frac{Y}{\overline{AL}} = (\delta + g)\frac{K}{\overline{AL}} + \frac{C}{\overline{AL}}$$

Labor market then determines L:

$$(1-\alpha)\frac{Y}{L} = L^{\varphi}/C^{-\gamma} \rightarrow$$

$$(1-\alpha)A(\frac{K}{AL})^{\alpha} = \frac{L^{\gamma+\varphi}A^{\gamma}}{(C/AL)^{-\gamma}}$$

Balanced Growth Path

Now suppose there is positive trend growth in TFP:

$$\overline{A}_t/\overline{A}_{t-1} = G = 1 + g > 1$$

The steady state now corresponds to a balanced growth path where the quantities Y, C, K grow at the gross growth rate G, while L is constant (given that population is assumed to be constant).

Note R is now increasing in g: $R = \beta^{-1}(1+g)^{\sigma}$

In general allowing for trend TFP growth (and also population growth) leads to only minor changes in both the steady state and cyclical dynamics.

Allowing for growth does place restrictions on preferences, however.

Balanced Growth Path (con't)

Labor market equilibrium:

$$(1-\alpha)\frac{Y}{L} \cdot C^{-\gamma} = L^{\varphi}$$
$$MPL \cdot MUC = MDUL$$

To have a balanced growth path with constant labor, need $MPL \cdot MUC$ constant

Rewrite labor market equilibrium:

$$(1-\alpha)\frac{Y}{C} \cdot C^{1-\gamma} = L^{1+\varphi}$$

Given $\frac{Y}{C}$ constant in a balanced growth path, L constant requires $\gamma = 1$: $\rightarrow$

$$(1-\alpha)\frac{Y}{C} = L^{1+\varphi}$$

$\gamma = 1 \rightarrow$

$$u(C) - \nu(L) = \log C - \tfrac{1}{1+\varphi} L^{1+\varphi}$$

With logarithmic preferences, along a balanced growth path *MUC* declines at a rate that exactly offsets the increase in *MPL* to keep the product constant:
$MPL \cdot MUC = (1-\alpha)\frac{Y}{L}\frac{1}{C} = (1-\alpha)\frac{Y}{C}\frac{1}{L}$, where $\frac{Y}{C}$ is constant along a BGP.
Intuitively, with log preferences, in steady state wealth effect on labor supply (from $\frac{1}{C}$) exactly offsets substitution effect (from increasing $W = (1-\alpha)\frac{Y}{L}$).

With $\gamma > 1$, L will decline as output grows due to the wealth effect on labor supply (captured by $C^{-\gamma}$)

$$(1-\alpha)\frac{Y}{C} \cdot C^{1-\gamma} = (1-\alpha) Y \cdot C^{-\gamma} = L^{1+\varphi}$$

Preferences with $\gamma \neq 1$ that permit balanced growth path

$$\begin{aligned} U(C,L) &= \frac{1}{1-\gamma}[C^{\eta}(1-L)^{1-\eta}]^{1-\gamma} \\ &= \log[C^{\eta}(1-L)^{1-\eta}] \quad \text{if} \quad \gamma = 1 \end{aligned}$$

with $0 < \eta < 1$ and $\gamma > 0$

$$U_1(C,L) = \eta C^{\eta-1}(1-L)^{1-\eta}\tfrac{1}{1-\gamma}[C^{\eta}(1-L)^{1-\eta}]^{-\gamma}$$
$$U_2(C,L) = (1-\eta)C^{\eta}(1-L)^{-\eta}\tfrac{1}{1-\gamma}[C^{\eta}(1-L)^{1-\eta}]^{-\gamma}$$

labor market equilibrium:

$$(1-\alpha)\tfrac{Y}{L}U_1(C,L) = U_2(C,L) \rightarrow$$
$$(1-\alpha)\tfrac{Y}{C} = \tfrac{1-\eta}{\eta}\tfrac{L}{1-L}$$

$\rightarrow L$ constant along a balanced growth path

Road Ahead

- Loglinear approximation of model around deterministic steady state.
- "Calibrate" model parameters.
- Evaluate business cycle dynamics versus data.