Topic 1

Real Business Cycle Theory: Part 2

Cyclical Dynamics: Model vs. Data

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Outline

Part 1.
Stochastic growth model with variable labor supply
Planning solution
Decentralized solution
Steady state

Part 2.
Loglinear approximation
Business cycle dynamics: Key properties
Calibration and quantitative performance
Shortcomings
Business cycle accounting: sources of deviations from the data

Review: Model Setup

Representative household:

Consumes C, supplies labor L, and saves capital K which it rents to firms.

Acts competitively - takes the real wage and rental rate on capital as given

Representative firm:

Firm produces output using labor L and capital K

Acts competitively - takes real wage and rental rate on capital as given

Market clearing determines wages, rents and equilibrium quantities

Equivalent to planning solution

Review: Preferences

$$E_t \left[\sum_{i=0}^{\infty} \beta^{t+i}[u(C_{t+i}) - \upsilon(L_{t+i})]\right]$$

with

$$\begin{aligned} u(C) &= \frac{1}{1-\gamma} C^{1-\gamma} \\ &= \log C \text{ iff } \gamma = 1 \end{aligned}$$

$$\upsilon(L) = \frac{1}{1+\varphi} L^{1+\varphi}$$

with $0 < \beta < 1$; $\gamma > 0$; $\varphi > 0$

$\gamma \equiv$ coefficient of relative risk aversion; $\varphi^{-1} \equiv$ Frisch elasticity of labor supply

where $C_t \equiv$ consumption, $L_t \equiv$ labor supply.

Digression on Intensive vs. Extensive Labor Supply

Literal interpretation: Household adjusts L along intensive margin (hours)
However, most hours fluctuations (2/3) are along extensive margin (bodies)
Given complete markets, can re-interpret L as adjustment along extensive margin:
Suppose continuum of measure unity family members who differ according to disutility of work.
Let $j^{\varphi} \equiv$ disutility of work of member j. $L \equiv \#$ of family members working
Given complete consumption insurance within family (each family member consumes the same amount C), family period utility is:

$$\frac{1}{1-\gamma} C^{1-\gamma} - \int_0^L j^{\varphi} dj = \frac{1}{1-\gamma} C^{1-\gamma} - \frac{1}{1+\varphi} L^{1+\varphi}$$

Objective unchanged $\rightarrow$ decision rules unchanged.
While model allows for extensive margin, it ignores search and matching in the labor market and abstracts from incomplete markets

Technology

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t L_t)^{1-\alpha} \\ &= A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha} \end{aligned}$$

where $Y_t \equiv$ output, $A_t^{1-\alpha} \equiv$ total factor productivity, $K_t \equiv$ capital, $L_t \equiv$ labor input. Productivity is labor-augmenting (to ensure a balanced growth path)

Resource Constraint ($\rightarrow$ Law of Motion for Capital):

$$C_t + K_{t+1} = Y_t + (1-\delta) K_t$$

where $0 < \delta < 1$ is the depreciation rate and where TFP obeys

$$\begin{aligned} A_t/\overline{A}_t &= (A_{t-1}/\overline{A}_{t-1})^{\rho} \cdot e^{\epsilon_t} \\ \overline{A}_t/\overline{A}_{t-1} &= G = 1 + g \geq 1 \end{aligned}$$

where $\overline{A}_t =$ trend TFP, $0 \leq \rho < 1$ and ϵ_t is i.i.d. with mean zero.

Behavioral Relations (from Topic 1, Part 1)

Labor market equilibrium:

$$(1-\alpha)A_t(\frac{K_t}{A_t L_t})^{\alpha} = W_t = L_t^{\varphi}/C_t^{-\gamma}$$

Consumption/saving:

$$C_t^{-\gamma} = E_t\{\beta C_{t+1}^{-\gamma} R_{t+1}\}$$

where $R_{t+1} \equiv$ gross return on capital ($r_{t+1} = R_{t+1} - 1 \equiv$ net return):

$$R_{t+1} = \alpha(\frac{K_{t+1}}{A_{t+1}L_{t+1}})^{\alpha-1} + (1-\delta)$$

Transversality condition for household budget constraint ensures non-explosive solution. Note that the behavioral relations come from household and firm decision rules and market clearing (see Part 1).

Complete Model

Endogenous variables: (Y_t, L_t, C_t, K_{t+1})
Predetermined states: (K_t, A_t)

$$Y_t = A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}$$

$$(1-\alpha)A_t(\frac{K_t}{A_t L_t})^{\alpha} \; = (1-\alpha)\frac{Y_t}{L_t} = \frac{L_t^{\varphi}}{C_t^{-\gamma}}$$

$$1 = E_t\{\beta(\frac{C_{t+1}}{C_t})^{-\gamma}(\alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta)\}$$

$$K_{t+1} = Y_t + (1-\delta)\, K_t - C_t$$

$$A_t/\overline{A} = (A_{t-1}/\overline{A})^{\rho} e^{\epsilon_t}$$

Cyclical driving force: fluctuations in A_t.

Deterministic Steady State

Stationary variables: $\frac{Y}{K}, \frac{K}{AL}, \frac{C}{K}, L$

$\frac{Y}{K}, \frac{K}{AL}, \frac{C}{K}$ determined by production function, consumption/saving relation and resource constraint:

$$\frac{Y}{K} = (\frac{K}{AL})^{\alpha-1}$$
$$1 = \beta(1+g)^{-\gamma}[\alpha\frac{Y}{K} + 1 - \delta]$$
$$\frac{Y}{K} = \frac{C}{K} + \delta + g$$

with $\frac{C'}{C} = 1 + g$ and $\frac{K'}{K} = 1 + g$.
Labor market then determines L :

$$(1-\alpha)\frac{Y}{L} = L^{\varphi}/C^{-\gamma}$$

Note: if $g > 0 \rightarrow \gamma = 1$ required to have L constant along balanced growth path.

Road Ahead

- Loglinear approximation of model around deterministic steady state.
  - For convenience we assume no growth ($g = 0, G = 1$)
- "Calibrate" model parameters.
- Evaluate business cycle dynamics versus quarterly data.

Loglinearization

- Because (i) many macroeconomic series are stationary in growth rates (e.g. GDP); (ii) (for the most part) exhibit relatively small percentage changes and (iii) linear models are easy to work with; we often work with loglinear approximations:

- Consider the following nonlinear equation:

$$g(X_t) = f(Y_t)$$

- Given $Z_t = e^{\log Z_t}$:

$$g(e^{\log X_t}) = f(e^{\log Y_t})$$

- Take a first order expansion around the deterministic steady state

$$g(X) + g'(X)X \cdot d\log X_t \approx f(Y) + f'(Y)Yd\log Y_t$$

where X and Y are steady state values (satisfying $g(X) = f(Y)$)

Loglinearization (con't)

- Given $g(X) = f(Y)$
$$g'(X)X \cdot d\log X_t \approx f'(Y)Yd\log Y_t$$

- Let $z_t \equiv \log(Z_t/Z) = \log(Z_t) - \log(Z)$ for $z_t = x_t, y_t$.

  For small percent changes in Z_t : $\quad z_t \approx d\log Z_t = \frac{dZ_t}{Z}$

- This leads to the following loglinear approximation of $g(X_t) = f(Y_t)$:
$$g'(X)X \cdot x_t = f'(Y)Y \cdot y_t$$

- Via loglinearization, a model that is nonlinear in X_t and Y_t becomes linear in the log-deviations x_t and y_t.

Predetermined versus Forward-Looking Dynamic Variables

- "Dynamic" variable depends on either history or beliefs about the future, or both
  - A variable is history dependent (or predetermined) if it depends on past values of itself and other variables. (e.g., the capital stock).
  - A variable is forward looking if it depends on beliefs about the futue (e.g. consumption, stock prices.)
- Example 1: Backward looking: y_t predetermined. $0 < a < 1$

$$\begin{aligned} y_{t+1} &= a y_t + f_t \\ &= \sum_{i=0}^{T} a^i f_{t-i} + a^{T+1} y_{t-T} = \sum_{i=0}^{\infty} a^i f_{t-i} \end{aligned}$$

- Example 2 Forward looking: y_t forward looking. $a > 1$ (and let $d_t = -f_t$)

$$\begin{aligned} y_t &= a^{-1}[E_t\{y_{t+1}\} + d_t] \\ &= \sum_{i=0}^{T} (a^{-1})^i d_{t+i} + (a^{-1})^{T+i} y_{t+1+T} = \sum_{i=0}^{\infty} (a^{-1})^i d_{t+i} \end{aligned}$$

Loglinearization of RBC Model (with no growth)

Let $\widetilde{a}_t = (1-\alpha)a_t$; and $\log(\frac{X_t}{X}) = x_t$

Production function:

$$y_t = \widetilde{a}_t + \alpha k_t + (1-\alpha)\, l_t$$

Labor market equilibrium:

$$\widetilde{a}_t + \alpha(k_t - l_t) = \varphi l_t + \gamma c_t$$

Consumption/Saving:

$$-\gamma c_t = -\gamma E_t \{c_{t+1}\} + E_t \left\{\beta\alpha \frac{Y}{K}(y_{t+1} - k_{t+1})\right\}$$

Law of motion for capital:

$$k_{t+1} = \frac{Y}{K} y_t - \frac{C}{K} c_t + (1-\delta) k_t$$

with $\widetilde{a}_t = \rho \widetilde{a}_{t-1} + \varepsilon_t$. (No constants since model in log deviations from steady state.)

Some Economics: Labor Supply

Rearranging labor market equilibrium (condition):

$$l_t = \varphi^{-1}(\widetilde{a}_t + \alpha(k_t - l_t)) - (\gamma/\varphi)c_t$$

$$= \varphi^{-1}w_t - (\gamma/\varphi)c_t$$

$\varphi^{-1} \equiv$ Frisch labor supply elasticity (percentage response of l_t to one percent change in w_t, holding c_t constant).

Estimates of φ^{-1} depend on whether l_t reflects intensive vs. extensive margin. Low for former (~ 0.5), higher for latter (~ 1.0)

Second term: "wealth" effect on labor supply: $c_t \uparrow \rightarrow l_t \downarrow$. Strength of wealth effect increasing in γ. $\gamma \uparrow \rightarrow$ stronger desire to smooth consumption.

Some Economics: Consumption/Saving

$R_{t+1} = 1 + r_{t+1} = (\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta) \rightarrow R\widehat{r}_{t+1} = .\alpha \frac{Y}{K}(y_{t+1} - k_{t+1}) \rightarrow$

$\widehat{r}_{t+1} = \beta\alpha\frac{Y}{K}(y_{t+1} - k_{t+1})$ (since $R = \beta^{-1}$ and $\widehat{r}_{t+1} = r_{t+1} - r$)

Let $\sigma = \gamma^{-1} \rightarrow$Consumption/saving:

$$c_t = E_t\{c_{t+1}\} - \sigma E_t\left\{\beta\alpha\frac{Y}{K}(y_{t+1} - k_{t+1})\right\} \leftrightarrow$$

$$c_t = E_t\{c_{t+1}\} - \sigma E_t\{\widehat{r}_{t+1}\}$$

Dependence of c_t on $E_t\{c_{t+1}\}$ reflects desire to smooth consumption.
Fluctuation in $E_t\{\widehat{r}_{t+1}\}$ may induce intertemporal substitution of consumption across time:

$\sigma \equiv \frac{1}{\gamma}$ is the intertemporal elasticity of substitution.

$\gamma \uparrow \rightarrow \sigma \downarrow$ since greater desire to smooth consumption.

Solution

To shed some light on the mechanisms that drive output and employment, combine the production function and the labor market equilibrium to obtain:

$$l_t = \frac{1}{\alpha+\varphi}(\widetilde{a}_t + \alpha k_t) - \frac{\gamma}{\alpha+\varphi} c_t$$

$$y_t = \left(1 + \frac{1-\alpha}{\alpha+\varphi}\right)(\widetilde{a}_t + \alpha k_t) - \frac{(1-\alpha)\gamma}{\alpha+\varphi} c_t \rightarrow$$

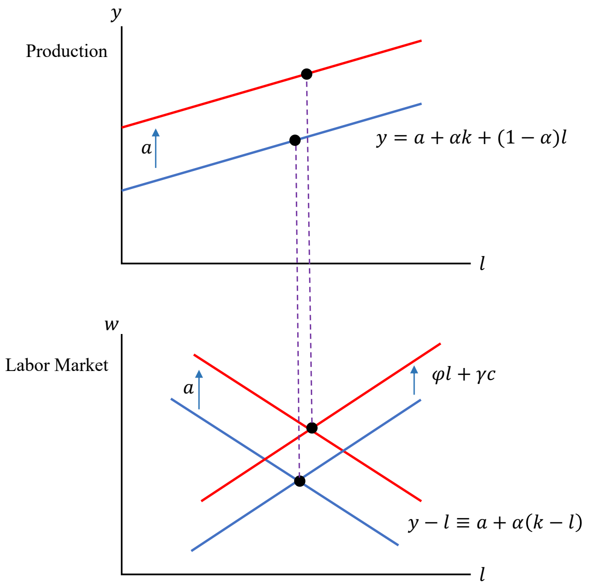
$$y_t = y(\widetilde{a}_t, k_t, c_t)$$

$\widetilde{a}_t + \alpha k_t$ reflects productivity which has both a direct and indirect (through labor demand) effect on y_t.

c_t reflects wealth effect on labor supply.

Three key parameters: α, φ and $\gamma(\equiv \sigma^{-1})$

Effect of Increase in A on Y and L



Solution (con't)

Solving for c_t :

Use the previous relation for y_t to eliminate y_{t+1} in the consumption euler equation and y_t in the resource constraint. Doing so yields the following system of two simultaneous first-order difference equations for c_t and k_{t+1} :

$$c_t = -\sigma E_t \left\{ \beta\alpha \frac{Y}{K} \left(y(\widetilde{a}_{t+1}, k_{t+1}, c_{t+1}) - k_{t+1} \right) \right\} + E_t \left\{ c_{t+1} \right\}$$

$$k_{t+1} = \frac{Y}{KG} y(\widetilde{a}_t, k_t, c_t) - \frac{C}{KG} c_t - \frac{1-\delta}{G} k_t$$

with

$$\widetilde{a}_t = \rho \widetilde{a}_{t-1} + \varepsilon_t$$

with $0 \leq \rho \leq 1$ and where $\widetilde{a}_t$ and k_t are predetermined.

Note that the two first-order difference equations can be combined into a single second-order difference equation in k_t, k_{t+1} and $E_t\{k_{t+2}\}$.

Solution (con't)

Matrix form:

$$\begin{bmatrix} E_t c_{t+1} \\ k_{t+1} \end{bmatrix} = A \begin{bmatrix} c_t \\ k_t \end{bmatrix} + B \cdot \widetilde{a}_t$$

where A is 2x2 and B is 2x1, where k_{t+1} is known at t, and where

$$\widetilde{a}_t = \rho \widetilde{a}_{t-1} + \varepsilon_t$$

- A has two characteristic roots. To ensure stability (convergence to steady state) one is greater than unity (unstable) and the other is less than unity (stable).
- The unstable root is associated with the forward looking variable c_t (consumption) and the stable root is associated with the predetermined state k_t.

Solution (con't)

- Structural system:

$$\begin{bmatrix} E_t c_{t+1} \\ k_{t+1} \end{bmatrix} = A \begin{bmatrix} c_t \\ k_t \end{bmatrix} + B \cdot \widetilde{a}_t$$

$$\widetilde{a}_t = \rho \widetilde{a}_{t-1} + \varepsilon_t$$

- Reduced form policy functions for c_t and k_{t+1} :

$$c_t = \pi_{ca} \widetilde{a}_t + \pi_{ck} k_t$$
$$k_{t+1} = \pi_{ka} \widetilde{a}_t + \pi_{kk} k_t$$

where the π coefficients are functions of the model parameters (in A, B and ρ). and can be obtained by using the method of undetermined coefficients (Campbell, JME 1994).
MofUC: substitute conjectured solutions for c_t, k_{t+1} into system and solve for π coefficients.

Solution (con't)

From reduced form policy function for $c_t (= \pi_{ca}\widetilde{a}_t + \pi_{ck} k_t)$, it is possible to solve for the other variables of the model:

$$\begin{aligned} l_t &= \frac{1}{\alpha+\varphi}(\widetilde{a}_t + \alpha k_t) - \frac{\gamma}{\alpha+\varphi} c_t \\ &= \frac{1-\gamma\pi_{ca}}{\alpha+\varphi}\widetilde{a}_t + \frac{\alpha-\gamma\pi_{ck}}{\alpha+\varphi} k_t \end{aligned}$$

$$\begin{aligned} y_t &= \left(1 + \frac{1-\alpha}{\alpha+\varphi}\right)(\widetilde{a}_t + \alpha k_t) - \frac{(1-\alpha)\gamma}{\alpha+\varphi} c_t \\ &= \left(1 + (1-\alpha)\frac{1-\gamma\pi_{ca}}{\alpha+\varphi}\right)\widetilde{a}_t + (\alpha + (1-\alpha)\frac{\alpha-\gamma\pi_{ck}}{\alpha+\varphi}) k_t \end{aligned}$$

$$k_{t+1} = \frac{Y}{KG} y_t - \frac{C}{KG}(\pi_{ca}\widetilde{a}_t + \pi_{ck} k_t) + \frac{1-\delta}{G} k_t$$

Combining the relations for y_t and k_{t+1} $\rightarrow$ policy function for $k_{t+1} (= \pi_{ka}\widetilde{a}_t + \pi_{kk} k_t)$

Solution (con't)

Observe that k_t (i.e. the log-deviation of capital stock from steady state or percent variation of the capital stock) is small over the cycle. Hence we can assume:

$$c_t \approx \pi_{ca}\widetilde{a}_t \rightarrow$$
$$l_t \approx \frac{1-\gamma\pi_{ca}}{\alpha+\varphi}\widetilde{a}_t$$
$$y_t \approx \left(1+(1-\alpha)\frac{1-\gamma\pi_{ca}}{\alpha+\varphi}\right)\widetilde{a}_t$$

Given $I_t = K_{t+1} - (1-\delta)K_t = Y_t - C_t \rightarrow \frac{I}{Y}inv_t = y_t - \frac{C}{Y}c_t \rightarrow$

$$inv_t = \frac{Y}{I}y_t - \frac{C}{I}c_t \rightarrow$$
$$inv_t \approx \frac{Y}{I}[\left(1+(1-\alpha)\frac{1-\sigma\pi_{ca}}{\alpha+\varphi}\right) - \frac{C}{Y}\pi_{ca}]\widetilde{a}_t$$

Key point: inv_t likely more volatile than c_t

- π_{ca} not large due to consumption smoothing (especially if $\widetilde{a}_t$ less persistent, i.e. ρ is low)
- $\frac{Y}{I} > 1$ is large

Calibration

To pick parameter values, use information independent of the business cycle data to be explained: e.g. long run relationships in the data (average growth rate, average labor share of output), parameter estimates from micro studies (labor supply elasticity, etc.).

Productivity not directly observable $\rightarrow$ Detrend the data (e.g. using a Hodrick-Prescott filter). Then recover the Solow residual $\widetilde{a}_t = y_t - \alpha k_t - (1-\alpha) l_t$.

Then use filtered data to estimate the process $\widetilde{a}_t = \rho \widetilde{a}_{t-1} + \epsilon_t$.

Next, generate artificial data by feeding the estimated TFP process into the calibrated model.

From the artificial data, compute a variety of business cycle moments and compare with actual data.

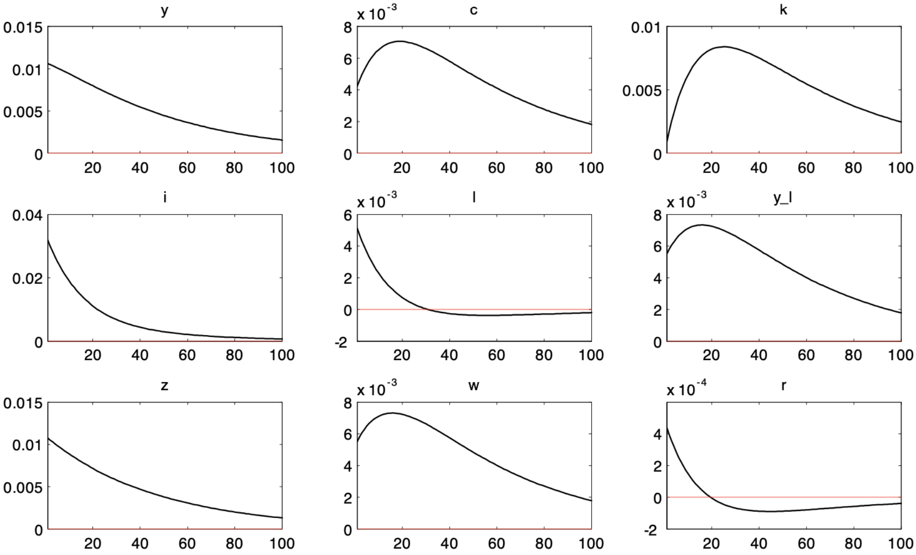
Parameter Choices (example from King and Rebelo)

Six "economic" parameters $(\beta, \gamma = \sigma^{-1}, \varphi, \alpha, \delta, g, \rho, \sigma_a^2)$

$\beta = 0.9375$ annually (0.984 quarterly) to match average return on capital.
$g = 0.016$ annually (0.004 quarterly) to match average growth in output per capita.
$\alpha = 0.33$ to match capital share.
$\delta = 0.10$ (0.025 quarterly) to match capital depreciation rate.
$\varphi^{-1} = 1$ to match evidence on Frisch elasticity of labor supply (extensive margin).
$\gamma = 1$ log utility to ensure a balanced growth path exists

Two additional parameters governing productivity process (ρ, σ_a^2)
Obtained from estimates of filtered Solow residuals

Note: calibration does not allow for parameter uncertainty making it hard to assess how confident one can be in model performance.


y
c
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x 10^-3
x 10^-4
0.015
0.01
0.005
0
0.04
0.02
8
6
4
2
-2
20
40
60
80
100

See Tables 1 and 3 in the Handbook chapter of King and Rebelo. Here only some key moments are reported

	std% data	std% model	correlation with output data	correlation with output model
output	1.81	1.39	1	1
consumption	1.35	0.61	0.88	0.94
investment	5.30	4.09	0.80	0.99
hours	1.79	0.67	0.88	0.97
labor productivity	1.02	0.75	0.55	0.98

Properties

- A reasonably calibrated model with a_t as the sole driving force can generate a standard deviation of output equal to seventy percent of actual output fluctuations (for postwar data pre-1984).(Note $a_t = (1-\alpha)z_t$ in the previous figure).
- The model can produce about half the volatility of hours.
- Investment is more volatile than consumption (as in the data).
- Fluctuations are Pareto efficient - no scope for policy.

Properties (con't)

- Consumption smoothing:

  Iterating the consumption/saving relation forward and imposing a terminal condition:

$$c_t = \sum_{i=0}^{\infty} -\sigma E_t \left\{ \widehat{r}_{t+1+i} \right\}$$

  with

$$\widehat{r}_{t+1} = \alpha \frac{Y}{K} \left(y_{t+1} - k_{t+1} \right)$$

- As long as long-term interest rates are not too variable, consumption behavior will be smooth.

Shortcomings

1. There is no internal propagation of shocks: y_t is driven only by a_t.

2. Unlikely that high frequency variation in the Solow residual reflects true movements in TFP. Total factor productivity a_t is not observed directly but measured as a residual. Suppose $Y_t = A_t^{1-\alpha} \left(U_t^K K_t \right)^{\alpha} \left(U_t^N L_t \right)^{1-\alpha}$. The production function includes unmeasured variations of factor utilization. The loglinearized Solow residual is then $a_t^o = a_t + \alpha u_t^K + (1-\alpha) u_t^N$. Thus utilization may be driving the high frequency variation in the Solow residual.

3. The productivity/hours correlation at the high frequency has shifted from positive to negative, post-1984 (due mainly to "jobless" recoveries).

4. Under a reasonable calibration the model cannot account for the magnitude of employment fluctuations.

5. Monetary/financial frictions are absent from this model.