An Axiomatic Approach to Salience Theory

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Introduction

- I give an axiomatization to the **Salience model** of Bordalo, Gennaioli, and Shleifer (henceforth BGS)
- The decision-theoretic analysis allows understanding:
  - The **minimum relaxation** of Expected Utility (**EU**) needed for Salient thinking;
  - The functional properties implied by the psychological features of **Ordering** (Kahneman 2003) and **Diminishing Sensitivity** (Weber's Law);
  - What is new to previous models;
  - It gives us axioms that can be tested in the lab.

Motivation: Allais Paradox

- The DM is asked to choose between the two **lotteries**

$$L_1^z = \left(2500, \frac{33}{100}; 0, \frac{1}{100}; z, \frac{66}{100}\right), \; L_2^z = \left(2400, \frac{34}{100}; z, \frac{66}{100}\right).$$

- Accordingly to EU theory, the specific value of z **is irrelevant** for the comparison.
- However, we have the following experimental findings

$$L_1^0 \succsim L_2^0 \text{ and } L_2^{2400} \succsim L_1^{2400}.$$

- **Prospect Theory** already explains this phenomenon quite well.

Allais Paradox for Acts

- Instead, suppose that the correlation between the two acts is made explicit:

$L_1^z \backslash L_2^z$	2400	z
0	0.01	0
2500	0.33	0
z	0	0.66

- This version makes clear that L_1^z and L_2^z pay the **common consequence** z **in the same state**.
- Experimental evidence (Conlisk 1989, Birnbaum and Schmidt 2010, BGS 2012):
  - Very **few DMs are reversing** their preferences as z changes.
  - No clear pattern in this reversing.

- The difference between the two versions of the experiment cannot be explained either by PT or by more recent models as Cautious Expected Utility.
- Roughly speaking, Salience explains the phenomenon by assuming that states with an **higher difference in payoffs draws more attention.** Consider an extreme case where the DM only focus on the state where the difference between the two alternatives is higher:
  - for L_1^0 and L_2^0, it is $(2500, 0)$;
  - for $L_2^{2400} \succsim L_1^{2400}$ it is $(0, 2400)$;
  - in the explicitly correlated case, it is always $(0, 2400)$;
  - therefore reversal only in the first case. The same holds for less extreme preferences.
- The primary alternative model stressing the role of correlation is **Regret Theory**.

Preferences sets

- M set of possible prizes.
- $p \in \Delta_s(M \times M)$ is a simple (i.e. with finite support) probability measures over the product space $M \times M$.
- We consider a **preference set** $P \subseteq \Delta_s(M \times M)$, with the following interpretation.
- Let X, Y be two random variables with joint distribution $p \in \Delta_s(M \times M)$.
- I (weakly) prefer to receive the amount specified by X to the amount specified by Y if and only if $p \in P$.

Relaxing EU

- Since it can explain the Allais Paradox, Salience Theory has to relax some of the EU axioms.
- Surprisingly, it is **enough to weaken Transitivity** (maintaining Independence and Continuity) to obtain a Salience Theory representation.

Maintained Axioms

- Given $p \in \Delta_s(M \times M)$, define the conjugate distribution $\bar{p}$ as

$$\bar{p}(x, y) = p(y, x).$$

- The strict preference set $\hat{P}$ is given by those $p \in P$ such that $\bar{p} \notin P$. We consider the following axioms for P.

Completeness If $p \notin P$ then $\bar{p} \in P$.

Independence For all $p \in P$, $q \in \hat{P}$, $\lambda \in (0, 1)$ we have $\lambda p + (1 - \lambda) q \in \hat{P}$.

Archimedean Continuity If $p \in \hat{P}$ and $q \notin P$, there exists α and β in $(0, 1)$ such that

$$\alpha p + (1 - \alpha) q \in \hat{P} \text{ and } \beta p + (1 - \beta) q \notin P.$$

Representation Theorem

- A function $\phi : M \times M \to \mathbb{R}$ is skew-symmetric if $\phi(x, y) = -\phi(y, x)$.

Theorem

P satisfies Completeness, Independence and Archimedean Continuity if and only there exists a skew-symmetric $\phi : M \times M \to \mathbb{R}$ such that

$$p \in P \iff \Phi(p) := \sum_{x,y} p(x, y)\,\phi(x, y) \geq 0.$$

Moreover, ϕ is unique up to a positive linear transformation. In this case, we say that P admits a skew symmetric additive (SSA) representation.

- Recall that under EU

$$\begin{aligned} p \in P \quad & \Leftrightarrow p_1 \succsim^P p_2 \\ & \Leftrightarrow \textstyle\sum_x p_1(x)\,u(x) \geq \sum_y p_2(y)\,u(y) \\ & \Leftrightarrow \textstyle\sum_{x,y} p_1(x)\,p_2(y)\,(u(x) - u(y)) \geq 0. \end{aligned}$$

BGS Decision criterion

- We revisit the specific criterion proposed by BGS.
- To identify the Salient pairs of payoff, they use the concept of Salience function.
- $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is a Salience function if satisfies:

1. **Ordering**: If $x' \leq y'$, $x \leq y$ and $[x', y'] \subseteq [x, y]$, then $\sigma(x, y) \geq \sigma(x', y')$;
2. **Diminishing sensitivity**: if $k, x, y \in \mathbb{R}_+$, then $\sigma(x + k, y + k) \leq \sigma(x, y)$;
3. **Symmetry**: $\sigma(x, y) = \sigma(y, x)$

- Their main example is

$$\sigma(x, y) = \frac{|x - y|}{|x + y + 1|}$$

BGS Decision criterion

- Under BGS Theory of Choice, the DM adopts the following δ-σ decision criterion:

$$p \in P \Leftrightarrow \sum_{(x,y)} (x-y)\, \delta^{\frac{1}{\sigma(x,y)}} p(x,y) \geq 0.$$

- The main idea is that **states with higher Salience are overweighted** using a distortion $\sigma(x,y)$. EU$\Rightarrow \delta = 1$.
- **Proposition 1** The Salience Theory model of BGS satisfies Completeness, Independence, and Archimedean Continuity.
- It is easy to see that the former can be embedded in the latter: let

$$\phi(x,y) = (x-y)\, \delta^{\frac{1}{\sigma(x,y)}}.$$

Salience Properties

- So far we have focused on the **weakenings of EU** necessary for a Salience Theory Representation.
- Now, we try to understand the **additional restrictions** implied by Salient Thinking.
- In particular, we define **Ordering** and **Diminishing Sensitivity** axiomatically, and we characterize them as properties of ϕ.
- From now on focus on monetary consequences, $M = \mathbb{R}$.

Axiomatization: Ordering

Definition

We say that P satisfies **Ordering** if for every

$$x_H \geq x_L \geq y_H \geq y_L$$

we have that

$$p = \left((x_H, y_L), \frac{1}{4}; (x_L, y_H), \frac{1}{4}; (y_L, x_L), \frac{1}{4}; (y_H, x_H), \frac{1}{4} \right) \in P.$$

The axiom captures the idea that, the best way to make the first component more desirable is to have an event with **extremely high difference** between outcomes (x_H, y_L).

Proposition 2 If P admits a δ-σ representation, it P satisfies Ordering if and only if σ satisfies Ordering.

- A **strict version of Ordering is not compatible with EU**.

Axiomatization: Diminishing Sensitivity

Definition

We say that P satisfies **Diminishing Sensitivity** if for every $x \geq y \geq 0$, and $k \in \mathbb{R}_+$

$$p = \left((x, y), \frac{1}{2}; (y + k, x + k), \frac{1}{2} \right) \in P.$$

- **Proposition 3** If P admits a δ-σ representation with linear utility, P satisfies Diminishing Sensitivity if and only if σ satisfies Diminishing Sensitivity.

Proposition 4 Let P admit an EU Representation. Then P satisfies Diminishing Sensitivity if and only if it satisfies **risk-aversion** in the gain domain.

- Therefore Diminishing Sensitivity is a generalization of Risk Aversion to Non-Transitive Preferences. What makes Salience Theory incompatible with EU is **Ordering.**

The interplay between Cognitive Sciences and Decision Theory

- The axiomatic approach takes the **revealed choice** of the DM as the **only observable**.
- Under this approach, there is **little distinction** between Salience and Regret Theory.
- However, they are extremely different in terms of **neurologic** underpinning and **welfare** implications.
- Supplementing choice data with neural evidence is essential.