Strategic Information Transmission in the Employment Relationship*

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Abstract

Formal procedures for dealing with information in organizations may be costly to set up. Informal ones may be more vulnerable to opportunism. We study the tradeoffs by introducing strategic communication a la Crawford and Sobel (1982) into Simon's (1951) model of the employment relationship. A contract specifies the principal's "range of authority" and a fixed wage for the agent. With extreme conflict, optimal contracts minimize the range of authority and preclude communication. With little conflict they maximize the range of authority and induce influential communication. They divide the state space into approximately equal-sized *topics*. Topics are bounded by actions over which the principal has authority and contain approximately equal numbers of cheap-talk actions.

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1 Introduction

Interactions in organizations fall into two categories, formal and informal. Formal rules and organizational structures prescribe rigid procedures on how things have to be done. They are set in advance, hard to change, and controlled by authorities. Informal interactions typically involve information sharing and advice how things should be done. They are spontaneous, easy to adjust, and give discretion to the operator. The formal framework gives the crucial structure to the informal interactions. The informal handling allows to fine-tune the implementation in response to new information. We are interested in the interplay of these two forces and its impact on setting up formal structures.

For concreteness, imagine a principal who anticipates receiving decision-relevant private information and lacks the ability to act on it herself. When hiring an agent to act on her behalf, she has a choice between devising a formal procedure for dealing with the information or handling it informally. Formal procedures may be costly to set up. Informal ones are likely more vulnerable to opportunistic behavior. That raises the question of the optimal mix of formal and informal treatments of anticipated private information in the employment relationship.

We investigate this question in a setting in which formal procedures are contractually arranged and informal ones correspond to cheap talk. We capture the cost of setting up formal procedure by having contracts be incomplete. Cheap talk is implicitly costly because the principal may have an incentive to misrepresent information and the agent may not use the information in the principal's best interest. Relying on contracts ensures greater control over the agent's action, while cheap talk can be more sensitive to the state of the world.

It is common for employment contracts to only partially pin down how the contracting parties deal with uncertain future events. An academic's contract, for example, typically specifies little more than a wage, benefits, a standard teaching load and a research budget. It leaves open specific course assignments, scheduling considerations, possibilities for course reductions, pay for overload teaching, etc. This makes it possible to flexibly respond to new opportunities and challenges. Taking advantage of this flexibility, however, may require sharing private information that has been learned after the contract is signed. This can be a source of inefficiency if there is disagreement about the best use of that information. Contracts will have to balance the costs of a rigid exercise of authority with those arising from imperfect information sharing.

Our environment combines features of Simon's model of the employment relationship and Crawford and Sobel's model of strategic information transmission. A principal, who anticipates privately learning the state of the world, offers a contract to an agent that specifies a fixed wage and a limited number of actions (what Simon calls the principal's "range of authority") that she can ask the agent to perform. Once the principal learns the state of the world, she has a choice between having the agent execute one of the actions specified in the contract or using cheap talk to try to convince the agent to take an action that is not specified in the contract. Arriving at the choice between an action listed in the contract and one induced via cheap talk is a two-stage process: At the ex ante stage, when writing the contract, the principal anticipates for which states of the world she will rely on cheap talk or insisting on a contract provision; at the interim stage, during contract execution, she makes that choice.

We show that every contract creates *topics*: these are subsets of the state space whose boundaries are pinned down by adjacent contract actions. Each topic delimits a domain of the state space that can be the exclusive subject of communication. Near the boundary of a topic, contract actions are taken. Further inside a topic, communication actions are taken, if there is communication in the topic. The larger the size of a topic, the greater is the chance that there is communication in a topic. Within any one topic, communication can be analyzed separately from communication in any other topic. This latter property has the effect of relaxing incentive constraints for communication, and, as we show, in some cases improves it.

There are two dimensions of conflict between the principal and the agent. They disagree over which action to take, i.e., for every state of the world their ideal actions differ. We term this the *action conflict*. An any given state, the size of the action conflict increases in the distance between the ideal actions. The other source of conflict derives from the wage the principal needs to pay the agent. We call this the *compensation conflict*. The less weight the agent places on the wage in their payoff function, the harder it becomes for the principal to use wage payments to compensate the agent for taking actions disliked by the agent. Equivalently, it becomes harder for the principal to compensate the agent the more weight the agent attaches to their (dis)utility from the action, their hedonic utility. Thus the size of the compensation conflict increases with the weight on the agent's hedonic utility.

We find that the sizes of the two dimensions of conflict affect both the complexity of the contract and the degree to which the principal relies on cheap-talk communication. We give a condition for what we call *extreme conflict*. The condition essentially requires that for any given degree of conflict in one dimension there is sufficiently large conflict in the other dimension. With extreme conflict optimal contracts are extremely simple – they specify a single contract action. They also leave no role for cheap talk. In contrast, for any given size of the action conflict, if there is sufficiently *little compensation conflict*, the optimal contract maximizes the number of contract actions. Furthermore, regardless of the size of the compensation conflict, if there is sufficiently *little action conflict* then any optimal contract approximately maximizes the number of contract action used and induces influential communication. Overall, optimal contracts satisfy a *bang-bang property*: contracts are either very simple or approximately maximize the number of contract actions.

For a parameterized version of the model, we completely characterize the optimal menu of contract actions and the role of communication at the optimum. Within each topic specified by the contract, there is maximal communication. The placement of contract actions is in part guided by their impact on communication. Whenever it is feasible to replace a communication action by a contract action, this relaxes incentive constraints for the remaining cheap-talk communication. This encourages locating contract actions in places where this relaxation has the greatest impact. We show that, as a result, in any optimal contract actions are approximately equidistant and topics are of approximately equal size.

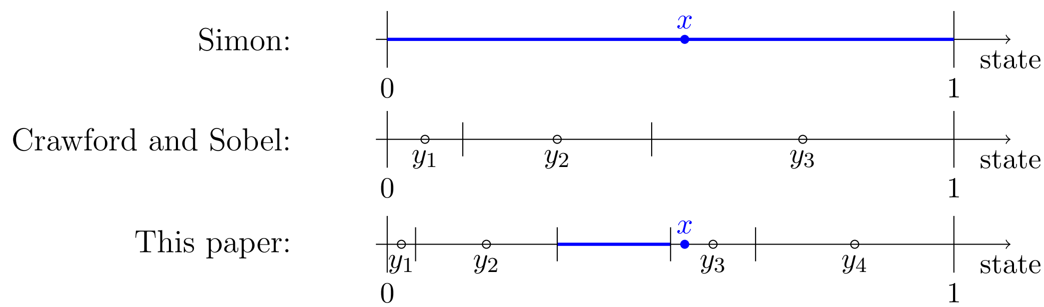

Figure 1: Optimal equilibria in the three different setups for action conflict 0.05 and compensation conflict 0.5.

For a glimpse at some of the key features of optimal equilibria in our environment consider a state that is uniformly distributed on the unit interval, an action space equal to the set of the real numbers, and limit the principal to specifying a single contract action, x. Suppose that both the principal's and the agent's payoffs are given by quadratic loss function, where the agent prefers the action to match the state while the principal prefers an action that exceeds the state by the constant 0.05. Finally, suppose that when offering a contract the principal has to respect the agent's (ex ante) individual rationality constraint (implying a compensation conflict of 0.5).

Figure 1 illustrates the optimal equilibria for an environment with contracts but no cheap talk (in the spirit of Simon, top panel), cheap talk without contracts (Crawford and Sobel, middle panel), and contracts combined with cheap talk (present paper, bottom panel). In the blue region of the state space the principal induces the single contract action x. The contract action satisfies $x > 1/2$, reflecting the principal's desire for higher actions (the agent would prefer $x = 1/2$). The contract relaxes the incentive constraints for communication. As a result, the number of actions y induced by cheap talk increases from three without the contract to four with the contract.

Literature: Simon's employment contracts specify a fixed wage and a "range of authority" (our set of contract actions) for the principal. The fact that the contract is agreed upon before the realization of uncertainty introduces an element of time inconsistency (Strotz (1955)): At the time the contract is agreed upon it needs to reflect the concerns of the agent that are embodied in the agent's individual rationality constraint. Once uncertainty is resolved and the contract is executed, within the parameters of the contract only the principal's preferences matter. We show that in the case of extreme conflict this not only rules out cheap-talk communication but also reduces the "range of authority" to a singleton.

Krishna and Morgan (2008), like us, examine contracting in the environment studied by Crawford and Sobel and, when considering imperfect commitment, impose limits on contractibility. Full revelation, which makes contracts full-detailed complete, is possible but not optimal. In the uniform quadratic environment, optimal contracts induce full revelation for low types and cheap talk with partial pooling for high types; with an extreme bias contracting is of no value. In our case contracts are incomplete by fiat, for small biases contract actions are interspersed with communication actions, and with extreme biases it is optimal to contractually specify a single action.

The literature offers a variety of rationales for why we observe contractual incompleteness, including the writing costs we use to motivate our bound the number of contract actions (Dye (1985), Battigalli and Maggi (2002)). Spier (1992) observes that in the presence of writing (or other transaction) costs, contractual incompleteness may be exacerbated by signaling incentives of the principal to the point where the contract only specifies a fixed wage. Hart and Moore (2008) offer a behavioral justification of employment contracts with a fixed wage: Contracts provide a reference point for feelings of entitlement. The parties are more likely to feel aggrieved when contracts are flexible and in response shade performance. This creates incentives to limit variations of aspects of the contract for which there is extreme conflict, like the wage. Bernheim and Whinston (1998) find that if for some reason contracts have to be incomplete, it may be optimal to increase their incompleteness further. This resonates with our observation for the case of extreme bias: when it is impossible to condition contract actions on the state of the world (which may be unverifiable) it is optimal to only specify a single contract action even though, up to a point, it would be costless to add contract actions.[1]

After we introduce the model in Section 2, we explain the structure of equilibria in Section 3. We illustrate optimal contracts and equilibria in an example in Section 4. Our general results are stated in Section 5, our results for the uniform-quadratic environment in Section 6.

2 Model

A principal (P, she) employs an agent (A, he) to take an action for her. When hiring the agent, the principal does not have all relevant information but anticipates privately learning that information before the agent gets to take the action. The contract between the principal and the agent specifies a fixed wage and a set of possible actions (Simon (1951)). After observing the information, the principal has a choice: she can mandate one of the *contract actions*, or instead, can communicate with the agent by proposing an alternative action (Crawford and Sobel (1982)). The agent has to execute any contract action that he receives; he can freely choose how to respond to proposed *communication actions*.[2]

Principal and agent engage in *contract-writing game* G. At the beginning, the principal offers a contract $X = (\boldsymbol{x}, w)$ to the agent; it consists of a set of contract actions $\boldsymbol{x} =$

[1] If instead, as in Blume, Deimen and Inoue (2022), contracts could coarsely condition on the state of the world it would always be optimal to use the maximal number of actions.

[2] Alternatively and equivalently, in the spirit of Matthews (1989), in the case of communication, we could have the principal provide the agent with information about the state of the world, upon which the agent proposes an action, which the principal either accepts or vetoes by mandating one of the (status quo) contract actions. This would leave the principal always in the role of the final decision maker.

$\{x_1, \ldots, x_K\}$, $x_k \in \mathbb{R}$ with $x_k < x_{k+1}$ and a specification of a fixed wage $w \in \mathbb{R}$ to be paid by the principal to the agent. The set of contract actions may be empty. We assume a bound $\widehat{K}$ on the number of contract actions included in a contract.[3] Each accepted contract induces a *contract subgame* Γ^X. At the beginning of a contract subgame, the principal privately learns the state of the world $t \in [0,1]$ (we sometimes refer to the state as the principal's '*type*'). She then chooses between mandating a contract action $x \in \boldsymbol{x}$ and sending a cheap talk message $m \in M$, where M is an (infinite measurable) space. She does that by sending a message μ from the *generalized message space* $M \cup \boldsymbol{x}$. Upon observing the message μ, the agent takes action $a = x$ if $\mu = x \in \boldsymbol{x}$ and otherwise an action $a = y \in \mathbb{R}$.

If the contract is accepted, state $t \in [0,1]$ is realized, and the agent takes action $a \in \mathbb{R}$, the principal's payoff equals $U^P(a,t,b) - w$ and the agent's payoff equals $\xi U^A(a,t) + (1-\xi)w$. The parameter $b \geq 0$ measures the degree of misalignment of preferences over the action between principal and agent. We refer to the disagreement about the optimal action between principal and agent that is parameterized by b as the *action conflict*. The parameter $\xi \in [0,1)$ indicates the importance the agent attaches to the hedonic utility $U^A(a,t)$ relative to the wage utility w. We call the parameter ξ that captures the difficulty of compensating the agent for an unfavorable action with a higher wage the *compensation conflict*. If the contract offer is rejected, the agent receives a reservation utility $\overline{u}$. We assume that the principal is financially unconstrained and always finds it worthwhile to offer a contract that the agent is willing to accept.

The state is commonly known to be distributed according to a distribution F with continuous density f that is strictly positive on the support $[0,1]$. Using subscripts for derivatives, we assume that $U^A, U^P \in \mathcal{C}^2$, $U^A_{11} < 0$, $U^P_{11} < 0$, $U^A_{12} > 0$, $U^P_{12} > 0$ and there exist $a^*_A(t)$ and $a^*_P(t,b)$ such that $U^A_1(a^*_A(t),t) = 0$, $U^P_1(a^*_P(t,b),t,b) = 0$ and $a^*_P(t,b) > a^*_A(t)$ for all $t \in [0,1]$ and $b > 0$. In the uniform-quadratic environment $U^P(a,t,b) = -(t+b-a)^2$, $U^A(a,t) = -(t-a)^2$, and F is the uniform distribution.

A strategy for the principal in the contract subgame Γ^X is given by $\sigma : [0,1] \to \Delta(M \cup \boldsymbol{x})$, where $\Delta(M \cup \boldsymbol{x})$ denotes the set of probability distributions over $M \cup \boldsymbol{x}$. A strategy for the agent in Γ^X is of the form $\rho : M \cup \boldsymbol{x} \to \mathbb{R}$, with the restriction that for all $x \in \boldsymbol{x}$, $\rho(x) = x$.

For every contract subgame Γ^X, let $E(X)$ denote the set of its Nash equilibrium strategy profiles, with typical element $e^X \in E(X)$. We say that a *contract-equilibrium pair* (X, e^X) *induces an action* $a \in \mathbb{R}$ if there is a set of types $t \in [0,1]$ and a generalized message μ in the support of $\sigma(t)$ for which $\rho(\mu) = a$, such that a is taken with strictly positive probability.

Our goal is to characterize subgame-perfect equilibria of the contract-writing game G that are optimal for the principal. If we denote the set of all possible contracts by $\mathfrak{X} = \mathfrak{X}(\widehat{K})$, then a strategy for the principal in the contract-writing game G is $\left(X, \left(\sigma^{X'}\right)_{X' \in \mathfrak{X}}\right)$ and a strategy for the agent is $\left(\rho^{X'}\right)_{X' \in \mathfrak{X}}$. Any contract-equilibrium pair (X, e^X) that the agent anticipates must meet the agent's ex ante participation constraint. Therefore, the principal

[3] This can be motivated by increasing writing costs (Dye (1985)) that prohibit arbitrarily detailed contracts.

– when writing the contract – solves

$$\max_{\substack{X \in \mathfrak{X}(\widehat{K}) \\ e^X \in E(X)}} \mathbb{E}\left[U^P(\rho(\sigma(t)), t, b)\right] - w \quad \text{s.t.} \quad \xi\mathbb{E}\left[U^A(\rho(\sigma(t)), t)\right] + (1-\xi)w \geq \overline{u}.$$

Optimality and our assumption that the principal always finds it worthwhile to satisfy the agent's participation constraint imply that the agent's participation constraint always binds. Therefore, since the wage w is entirely determined by the set of contract actions $\boldsymbol{x}$, we can, and do from hereon, identify any contract X with the set of contract actions (suppressing the wage), and equivalently rewrite the principal's problem as

$$\max_{\substack{X \in \mathfrak{X}(\widehat{K}) \\ e^X \in E(X)}} (1-\xi)\mathbb{E}\left[U^P(\rho(\sigma(t)), t, b)\right] + \xi\mathbb{E}\left[U^A(\rho(\sigma(t)), t)\right]. \tag{1}$$

Thus, ex ante, when writing the contract, the principal maximizes weighted joint surplus. We sometimes refer to the principal in this stage as the *ex-ante principal*. The ex ante principal's payoff function is $U^\xi(a, t, b) := (1-\xi)U^P(a, t, b) + \xi U^A(a, t)$ and her ideal point is $a^*_\xi(t, b)$. By contrast, at the interim stage, when the principal has learned the state and sends a generalized message to the agent, the principal's objective is to maximize her own payoff $U^P(a, t, b)$ from the agent's action, and we sometimes refer to her as the *interim principal*.

3 The structure of equilibria of contract subgames

Fix a contract X. Consider the equilibria of the contract subgame Γ^X. Note that the ideal points of the interim principal and the agent satisfy $a^*_P(t, b) > a^*_A(t)$ for all $t \in [0, 1]$ and $b > 0$. Therefore, independent of X there is a strictly positive lower bound on the distance between communication actions that are induced in any equilibrium.[4] This implies that there is a finite upper bound on the total number of actions (contract actions and communication actions) that can be induced in any equilibrium of the contract-writing game G.

The interim principal's sorting condition $U^P_{12} > 0$ then implies that for each equilibrium in which J actions $a_1 < a_2 < \ldots < a_J$ are induced, there are $J+1$ *critical types* $0 = \theta_0 < \theta_1 < \cdots < \theta_J = 1$, such that all types in (θ_{j-1}, θ_j) strictly prefer to induce action a_j, $j = 1, \ldots, J$. Type θ_j is indifferent between actions a_j and a_{j+1} for $j = 1, \ldots, J-1$. As a result, every equilibrium is essentially (modulo the specification of behavior of types θ_j) equivalent to an equilibrium in which the type set is partitioned into finitely many intervals with endpoints θ_{j-1} and θ_j for the jth interval. Types belonging to the same interval induce the same action, and types belonging to different intervals induce different actions. If the types in (θ_{j-1}, θ_j) induce a communication action, we refer to (θ_{j-1}, θ_j) as a *communication interval*. If instead they induce a contract action, we refer to (θ_{j-1}, θ_j) as a *contract interval*. The agent's sorting condition then implies that if (θ_{j-1}, θ_j) and (θ_j, θ_{j+1}) are both communication intervals, the

[4]This follows immediately from Lemma 1 in Crawford and Sobel (1982) (CS). The fact that, unlike CS, we have contract actions in addition to communication actions does no affect the applicability of their proof.

agent indeed prefers to take a higher action in response to messages sent by types in (θ_j, θ_{j+1}) than in response to messages sent by types in (θ_{j-1}, θ_j), as is required for $a_{j+1} > a_j$ for all $j = 1, \ldots, J-1$. Similarly, if (θ_{j-1}, θ_j) is a contract interval and (θ_j, θ_{j+1}) a communication interval, the agent's sorting condition implies that the agent prefers to take a higher action in response to messages sent by types in (θ_j, θ_{j+1}) than the contract action induced by types in (θ_{j-1}, θ_j) (an analogous observation applies to the case in which the roles of contract and communication intervals are reversed). Hence:

Observation 1 *All equilibria of contract subgames are (essentially) interval partitional and monotonic.*

It is common to refer to the indifference requirement for critical types θ_j, $j = 1, \ldots, J-1$, as those types' *arbitrage condition.* Let $a^*(\theta', \theta'')$ denote the agent's best reply to prior beliefs concentrated on the interval (θ', θ''). Then the arbitrage conditions take the following form:

$$\begin{aligned} &U^P(a_j, \theta_j, b) - U^P(a_{j+1}, \theta_j, b) = 0 \text{ for all } j = 1, \ldots, J-1, \text{ where} \\ &a_j = a^*(\theta_{j-1}, \theta_j) \text{ for all } j \text{ for which } a_j \text{ is a communication action,} \\ &\text{and the remaining actions are contract actions in } X. \end{aligned} \tag{A}$$

Given a contract X, two contract actions $x', x'' \in X$ are *adjacent* if there is no contract action $x \in X$ with $x' < x < x''$. Consider the set of types for which the interim principal's ideal point lies between x' and x'', i.e., $\{t \in [0,1] | x' \leq a_P^*(t, b) \leq x''\}$. Any pair of adjacent contract actions, for which there exist types t' and t'' with $x' = a_P^*(t', b)$ and $x'' = a_P^*(t'', b)$, determines an *inner topic*

$$\mathcal{T}(x', x'') := (x', x'', \{t \in [0,1] | x' \leq a_P^*(t, b) \leq x''\}).$$

Whenever there do *not* exist types t' or t'' with $x' = a_P^*(t', b)$ or $x'' = a_P^*(t'', b)$, we refer to $\mathcal{T}(x', x'')$ as *improper inner topic.* Note that improper inner topics can be empty. In a similar fashion, we define two *outer topics.* The minimal contract action x_1 in X determines the *bottom topic*

$$\mathcal{T}(x_1) := (x_1, \{t \in [0,1] | a_P^*(t, b) \leq x_1\}),$$

which can be empty. The maximal contract action x_K in X determines the *top topic*

$$\mathcal{T}(x_K) := (x_K, \{t \in [0,1] | a_P^*(t, b) \geq x_K\}),$$

which can be empty. A *topic* $\mathcal{T}$ is either an inner, an improper inner, or an outer topic.

Each topic $\mathcal{T}$ induces a game in its own right, with the type distribution restricted to $\mathcal{T}$ and the only contract actions being the ones defining the topic. Refer to that game as a *$\mathcal{T}$-game* and call an equilibrium of that game a *$\mathcal{T}$-equilibrium.* Evidently, every $\mathcal{T}$-equilibrium is itself interval partitional and induces a finite number of actions; except for the actions defining the topic $\mathcal{T}$, these are communication actions.

Let $n(\mathcal{T})$ denote the number of communication actions induced in a $\mathcal{T}$*-equilibrium*. Then, if $\mathcal{T} = \mathcal{T}(x', x'')$ is an inner topic, there are $n(\mathcal{T}) + 1$ critical types $\theta_{\mathcal{T},i}$, $i = 0, \dots, n(\mathcal{T})$. These critical types satisfy

$$U^P\left(x', \theta_{\mathcal{T},0}, b\right) - U^P\left(a^*(\theta_{\mathcal{T},0}, \theta_{\mathcal{T},1}), \theta_{\mathcal{T},0}, b\right) = 0 \tag{2}$$

$$U^P\left(a^*(\theta_{\mathcal{T},i-1}, \theta_{\mathcal{T},i}), \theta_{\mathcal{T},i}, b\right) - U^P\left(a^*(\theta_{\mathcal{T},i}, \theta_{\mathcal{T},i+1}), \theta_{\mathcal{T},i}, b\right) = 0 \text{ for } i = 1, \dots, n(\mathcal{T}) - 1 \tag{3}$$

$$U^P\left(a^*(\theta_{\mathcal{T},n(\mathcal{T})-1}, \theta_{\mathcal{T},n(\mathcal{T})}), \theta_{\mathcal{T},n(\mathcal{T})}, b\right) - U^P\left(x'', \theta_{\mathcal{T},n(\mathcal{T})}, b\right) = 0. \tag{4}$$

Condition (2) ensures that the principal with critical type $\theta_{\mathcal{T},0}$ is indifferent between insisting on the contract action x' and inducing the minimal communication action in topic $\mathcal{T}(x', x'')$. Conditions (3) are the familiar arbitrage conditions for adjacent communication actions, and condition (4) is the requirement that the principal with critical type $\theta_{\mathcal{T},n(\mathcal{T})}$ is indifferent between the maximal communication action in topic $\mathcal{T}(x', x'')$ and insisting on the contract action x''.

Since each topic $\mathcal{T}$ induces a game in its own right, the equilibria of any contract subgame satisfy a separability condition. They are composed of $\mathcal{T}$-equilibria. Essentially, once we fix an equilibrium for each topic, we have an equilibrium for the entire contract subgame.

Observation 2 *Suppose the topics in a contract subgame are* $\mathcal{T}_k$*,* $k = 1, \dots, K + 1$*. Then for any choices of equilibrium outcomes* $\mathcal{O}(\mathcal{T}_k)$ *of the corresponding* $\mathcal{T}_k$*-games, there exists an equilibrium of the entire contract subgame whose outcome agrees with* $\mathcal{O}(\mathcal{T}_k)$ *in each* $\mathcal{T}_k$*-game.*

4 Example

In this section we give examples of optimal contract-equilibrium pairs. We consider contracts with at most two contract actions, $\widehat{K} = 2$, and an agent who gives equal weight to wage and hedonic payoffs, $\xi = 0.5$. See Figure 2 for an illustration. The red dashed lines indicate the boundaries of the topics $\mathcal{T}_1 = [0, x_1 - b]$, $\mathcal{T}_2 = [x_1 - b, x_2 - b]$, and $\mathcal{T}_3 = [x_3 - b, 1]$. All types that are marked blue induce a contract action x_1, or x_2; all remaining types induce a communication action y_i. Notice that each contract action is induced by types from neighboring topics. Critical types θ_j are indifferent between inducing the action below or above.

Consider first low levels of action conflict b. Note that for $b = 0.02$ and $b = 0.05$, the optimal contract actions are approximately equal to $x_1 \approx 0.33$ and $x_2 \approx 0.70$. The distinctive feature of the optimal contract-equilibrium pairs is the induced number of communication actions. While there is one (two) action in each topic for $b = 0.05$ ($b = 0.02$), the number increases to 5 in each topic for $b = 0.005$ and to a total number of 53 communication actions for $b = 0.0005$. By comparison, the numbers $N(b)$ for cheap talk games without contract are $N(0.05) = 3$, $N(0.02) = 5$, $N(0.0005) = 32$. Contracts facilitate information exchange by communication. Note that the lengths of the communication intervals are increasing within a topic but not across topics.

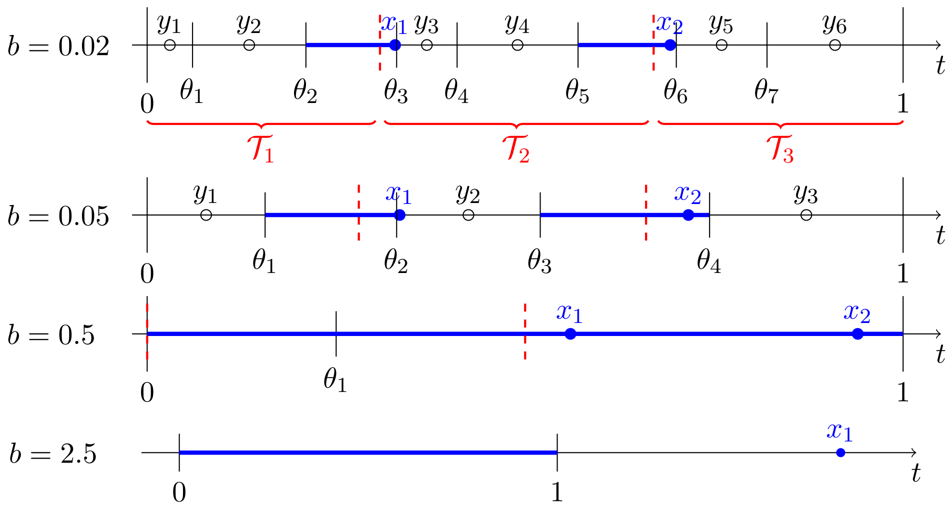

Figure 2: Optimal contract-equilibrium pairs for $\widehat{K} = 2$, $\xi = 0.5$, and different values of b.

Finally, consider larger levels of action conflict. For $b = 0.5$ no communication action is induced, but the state space is split into two segments: types below θ_1 induce action x_1 and types above action x_2. For $b = 2.5$ we have 'extreme conflict': The optimal contract-equilibrium pair is extremely simple: only one contract action $x_1 = 0.5 + (1 - \xi) \cdot b = 1.75$ is induced and there is no role for communication.

5 Optimal contract-equilibrium pairs

In this section, we establish general properties of optimal contract-equilibrium pairs. We find that optimal contracts are extremely simple when there is 'extreme conflict'. Conflict is extreme when given the action conflict there is sufficient compensation conflict, or given the compensation conflict there is sufficient action conflict. If either condition is met, there is substantial disagreement between the ex ante and the interim principal over the optimal action. In this case, the optimal contract has only one contract action and that contract action is implemented with certainty. Thus, there is no communication. By contrast, if there is little conflict, we show that at the optimum, contracts are detailed and there is influential communication.

5.1 Extreme conflict

Recall that conflict in our environment has two dimensions: (i) At every state $t \in [0, 1]$, $a^*_P(t, b) > a^*_A(t)$, i.e., the interim principal prefers a higher action than the agent – there is *state conflict.* (ii) For every state, for $\xi > 0$ the ex ante and the interim principal disagree on the optimal action – this is a consequence of there being *compensation conflict*, the difficulty of compensating the agent for the disutility of taking an unfavorable action with a more favorable wage. If there is significant conflict in one dimension while fixing the other, we have *extreme conflict*.

We define extreme conflict with reference to a minimal compact set $\boldsymbol{A}^* \subset \mathbb{R}$, which includes all actions induced in any optimal contract-equilibrium pair irrespective of the compensation conflict (while fixing the environment otherwise). Existence of such a set is shown in the appendix in Lemma A.1.

Definition 1 *There is* ***extreme conflict*** *if*

$$\left[a', a'' \in \boldsymbol{A}^*, a' < a'' \text{ and } U^\xi(a', 1, b) \leq U^\xi(a'', 1, b)\right] \Rightarrow U^P(a', 0, b) < U^P(a'', 0, b).$$

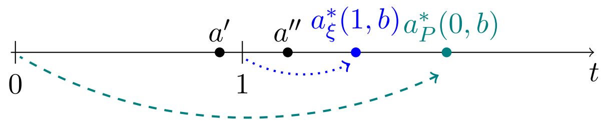

Figure 3: Extreme conflict: the preferred action of the interim principal $a^*_P(0, b)$ for the lowest type is higher than the preferred action of the ex-ante principal $a^*_\xi(1, b)$ for the highest type.

The definition of extreme conflict captures that the interim principal has a much stronger preference for higher actions than the ex ante principal, $a^*_\xi(t, b) < a^*_P(t, b)$. Figure 3 illustrates. In the case of extreme conflict, the lowest type of the interim principal $t = 0$ prefers an action $a^*_P(0, b)$ that is higher than the preferred action $a^*_\xi(1, b)$ of the highest type of ex-ante principal $t = 1$. By concavity of the payoffs this implies that *all* types $t \in [0, 1]$ of the ex-ante principal prefer lower actions than *all* types $t \in [0, 1]$ of the interim principal. Note that the necessary condition for extreme conflict is that $\xi > 0$; for $\xi = 0$ there is never extreme conflict. Finally, considering the uniform-quadratic example with a constant bias b, the condition for having extreme conflict is that $\xi b \geq 1$.

If a contract-equilibrium pair induces more than one action, for any pair of those actions there must be one interim principal type that is indifferent between them. Extreme conflict then implies that for any such pair of actions, the ex ante principal strictly prefers the lower of the two actions. Hence, the ex ante principal would be better off simplifying the contract by having only one contract action, which coincides with the lowest of the actions that are induced in the original contract-equilibrium pair. Further improvements may be possible by picking the single contract action optimally. This is formalized in the following result.

Proposition 1 *With extreme conflict, any contract in an optimal contract-equilibrium pair* (X, e^X) *specifies exactly one contract action. That contract action,* $x^* = \arg\max_a \mathbb{E}_t[U^\xi(a, t, b)]$ *is also the only action that is induced in* e^X.

Proof. Suppose there is extreme conflict. Consider any contract-equilibrium pair (X, e^X) that induces $J > 1$ actions $a_1 < \ldots < a_J$, any of which may be either a contract or a communication action. We will show that (X, e^X) can be improved upon with a contract-equilibrium pair $(X', e^{X'})$ that uses only a single contract action.

Since, by assumption, actions a_1 and a_2 are both induced, there is a type $\theta_1 \in (0,1)$ for which the interim principal is indifferent between those actions, i.e., $U^P(a_1, \theta_1, b) = U^P(a_2, \theta_1, b)$. Therefore, the cross-partial condition for the interim principal implies that $U^P(a_1, 0, b) > U^P(a_2, 0, b)$. By assumption $a_1 < a_2$, and by Lemma A.1 both a_1 and a_2 belong to $\boldsymbol{A}^*$. The last two observations combined with the fact that there is extreme conflict imply that $U^\xi(a_1, 1, b) > U^\xi(a_2, 1, b)$.

Hence, the cross-partial conditions for the interim principal and the agent jointly imply that $U^\xi(a_1, t, b) > U^\xi(a_2, t, b)$ for all $t \in [0,1]$. This and the strict concavity of U^ξ in its first argument for all t imply that $U^\xi(a_1, t, b) > U^\xi(a_j, t, b)$ for all $t \in [0,1]$ and all $j > 1$. It follows that

$$\int_0^1 U^\xi(a_1, t, b) f(t) dt > \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} U^\xi(a_j, t, b) f(t) dt,$$

where types in the intervals (θ_{j-1}, θ_j) induce action a_j, $j = 1, \ldots, J$, $\theta_0 = 0$, and $\theta_J = 1$. Hence, an alternative contract-equilibrium pair $(X', e^{X'})$ in which the set $X' = \{a_1\}$ is a singleton and the equilibrium $e^{X'}$ induces only action a_1 improves on (X, e^X).

Among all contract-equilibrium pairs $(\widetilde{X}, e^{\widetilde{X}})$ in which the contract consists of a single contract action, the one with $\widetilde{X} = \{x^*\}$, where $x^* = \arg\max_a \mathbb{E}_t[U^\xi(a, t, b)]$ is optimal for the ex ante principal. This contract is overall optimal because it strictly dominates a contract without any contract actions, which would induce the single (communication) action

$$y^* = \arg\max_a \mathbb{E}_t[U^A(a, t)] < x^*.$$

□

Under extreme conflict, the ex ante principal cannot leave any choice for the interim principal. Consider the contract with the single ex-ante-optimal contract action. Any additional lower action (which would be preferred by some types of the ex ante principal) will *never* be taken by the interim principal. Any additional higher action (which would be preferred by some types of the ex ante principal) will *always* be taken by the interim principal – which makes it suboptimal to include this option to begin with. The informational advantage of the interim principal is thus of no use for the ex ante principal. Moreover, the extreme disagreement between ex ante principal and interim principal implies that the conflict between interim principal and agent prohibits any communication.

5.2 Little conflict

Optimal contracts under extreme conflict are extremely simple and completely crowd out communication. By contrast, in this section, we show that under some additional assumptions with little conflict optimal contracts are detailed and coexist with rich communication behavior. Say that communication in a contract-equilibrium pair is *influential* if at least two actions are induced by communication. Then, we find that with sufficiently little conflict, every contract-equilibrium pair induces at least $\hat{K}-2$ contract actions and exhibits influential communication.

In this section, we assume that $U^P(a,t,0) = U^A(a,t)$, that $b \geq 0$, and that $U^P_{13}(\cdot) > 0$ everywhere. We thus have $a^*_P(t,b) > a^*_A(t)$ for all $b > 0$ and any increase in b moves the interim principal's preferences away from the agent's. We also require three regularity conditions to hold.

The first of these is condition (M), which is familiar from Crawford and Sobel (1982). For any fixed value of the action conflict b, call a sequence $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_N)$ a *backward solution* if $U^P\left(a^*(\theta_j,\theta_{j+1}),\theta_j,b\right) - U^P\left(a^*(\theta_{j-1},\theta_j),\theta_j,b\right) = 0$, $0 < j < N$ and $\theta_0 > \theta_1$.

We assume that, for a given value of b, if $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ are two backward solutions with $\hat{\theta}_0 = \tilde{\theta}_0$ and $\hat{\theta}_1 > \tilde{\theta}_1$, then $\hat{\theta}_j > \tilde{\theta}_j$ for all $j \geq 2$. (M)

In words, for any two backward solutions, an increase of θ_1 implies an increase of all θ_j, $j \geq 2$.

The second regularity condition is a continuity requirement. Refer to any contract with $K = 0$, i.e., with no contract actions, as a *null contract* (the contract still specifies the wage needed to attract the agent). Every null contract that is accepted turns the contract subgame into a standard *cheap-talk game.* Let $V^A(b)$ denote the agent's maximal equilibrium payoff in the cheap-talk game with bias b.[5] We make the following Convergence Assumption concerning the cheap-talk game:

For every $\varepsilon > 0$ there exists $b_\varepsilon > 0$ such that for all b with $0 < b < b_\varepsilon$ the agent's maximal equilibrium payoff $V^A(b)$ in the cheap-talk game with bias b satisfies

$$|V^A(b) - \mathbb{E}[U^A(a^*_A(t),t)]| < \varepsilon.\text{[6]} \tag{C}$$

Thus as the action conflict converges to zero, the agent's maximal cheap-talk equilibrium payoff converges to the agent's first-best payoff.

Finally, consider any two actions $\underline{x}, \overline{x} \in \mathbb{R}$ with $\underline{x} < \overline{x}$ for which $\overline{x}$ is the principal's ideal point for some type $\bar{t} \in (0,1)$, i.e., $\overline{x} = a^*_P(\bar{t}, b)$. We assume that if $\theta \in (0,1)$ satisfies $U^P(\underline{x},\theta,b) = U^P(\overline{x},\theta,b)$, then

$$\frac{U^A(\overline{x},\theta) - U^A(\underline{x},\theta)}{U^P_2(\overline{x},\theta,b) - U^P_2(\underline{x},\theta,b)} U^P_1(\overline{x},\theta,b) f(\theta) + \int_\theta^{\bar{t}} U^A_1(\overline{x},s) f(s) ds < 0. \tag{N}$$

This condition ensures that the ex ante principal strictly gains from inserting additional contract actions between any two such actions that are not too far part. It is satisfied in the familiar example with quadratic payoffs, constant bias, and a uniform type distribution. It also holds if payoff functions are of the form $U^A(a,t) = V(|t-a|)$, $U^P(a,t,b) = V(|t+b-a|)$, and the type distribution has a non-decreasing density (conditions, which are also sufficient for condition (M) to hold).

[5]Note that this is well defined since for any $b > 0$ the cheap-talk game has essentially only finitely many equilibria and for $b = 0$ there is an equilibrium in which the agent receives his ideal action in every state of the world.

[6]Agastya, Bag and Chakraborty (2015) provide sufficient conditions on primitives for this hold.

Lemma 1 *Suppose that* (X, e^X) *is a contract-equilibrium pair that induces two adjacent contract actions* $\underline{x} < \overline{x}$ *that are not separated by communication. Then, if there exists* $\overline{t} \in (0,1)$ *with* $\overline{x} = a_P^*(\overline{t}, b)$ *and condition (N) is satisfied, one can find* $x' \in (\underline{x}, \overline{x})$*, such that given the contract* $X' = X \cup \{x'\}$*, there is a contract-equilibrium pair* $(X', e^{X'})$ *that the ex ante principal strictly prefers to* (X, e^X)*.*

Proof of Lemma 1. If there exists $t' \in [0,1]$ for which $\underline{x} = a_P^*(t', b)$, let $\underline{t} = t'$ and otherwise let $\underline{t} = 0$. Given any $(\underline{x}, \overline{x})$, define θ_l, θ_h as the types that satisfy $U^P(\underline{x}, \theta_l, b) = U^P(x', \theta_l, b)$ and $U^P(\overline{x}, \theta_h, b) = U^P(x', \theta_h, b)$. These types are well defined since $\underline{x}$ and $\overline{x}$ are induced by assumption.

When adding a contract action $x' \in (\underline{x}, \overline{x})$, payoffs and incentives outside of $[\underline{t}, \overline{t}]$ remain unchanged. Hence, we can limit attention to this interval. The ex ante principal's payoff in $[\underline{t}, \overline{t}]$ is

$$\int_{\underline{t}}^{\theta_l} (1-\xi)U^P(\underline{x}, s, b) + \xi U^A(\underline{x}, s) f(s) ds$$
$$+ \int_{\theta_l}^{\theta_h} (1-\xi)U^P(x', s, b) + \xi U^A(x', s) f(s) ds$$
$$+ \int_{\theta_h}^{\overline{t}} (1-\xi)U^P(\overline{x}, s, b) + \xi U^A(\overline{x}, s) f(s) ds.$$

The derivative with respect to x' equals

$$\left((1-\xi)U^P(\underline{x}, \theta_l, b) + \xi U^A(\underline{x}, \theta_l)\right) f(\theta_l) \frac{d\theta_l}{dx'}$$
$$+ \left((1-\xi)U^P(x', \theta_h, b) + \xi U^A(x', \theta_h)\right) f(\theta_h) \frac{d\theta_h}{dx'} - \left((1-\xi)U^P(x', \theta_l, b) + \xi U^A(x', \theta_l)\right) f(\theta_l) \frac{d\theta_l}{dx'}$$
$$+ \int_{\theta_l}^{\theta_h} \left((1-\xi)U_1^P(x', s, b) + \xi U_1^A(x', s)\right) f(s) ds$$
$$- \left((1-\xi)U^P(\overline{x}, \theta_h, b) + \xi U^A(\overline{x}, \theta_h)\right) f(\theta_h) \frac{d\theta_h}{dx'}.$$

Evaluating this expression at $x' = \overline{x}$ and using the principal's indifference between $\underline{x}$ and $x' = \overline{x}$ at θ_l this simplifies to

$$\xi \left(U^A(\underline{x}, \theta_l) - U^A(\overline{x}, \theta_l)\right) f(\theta_l) \frac{d\theta_l}{dx'}\bigg|_{x'=\overline{x}} + \int_{\theta_l}^{\theta_h} \left((1-\xi)U_1^P(x', s, b) + \xi U_1^A(x', s)\right) f(s) ds \bigg|_{x'=\overline{x}}.$$

Differentiating the interim principal's indifference condition at θ_l, $U^P(\underline{x}, \theta_l, b) \equiv U^P(x', \theta_l, b)$, with respect to x' gives us

$$U_2^P(\underline{x}, \theta_l, b) \frac{d\theta_l}{dx'} - U_1^P(x', \theta_l, b) - U_2^P(x', \theta_l, b) \frac{d\theta_l}{dx'} = 0,$$

which is equivalent to

$$\frac{d\theta_l}{dx'} = \frac{U_1^P(x', \theta_l, b)}{U_2^P(\underline{x}, \theta_l, b) - U_2^P(x', \theta_l, b)}.$$

Evaluation this expression at $x' = \overline{x}$, inserting it into the expression for the derivative of the ex ante principal's payoff in $[\underline{t}, \overline{t}]$, and using the fact that at $x' = \overline{x}$ we have $\theta_l = \theta$ and $\theta_h = \overline{t}$, we obtain

$$\xi(U^A(\underline{x}, \theta) - U^A(\overline{x}, \theta))f(\theta)\frac{U_1^P(\overline{x}, \theta, b)}{U_2^P(\underline{x}, \theta, b) - U_2^P(\overline{x}, \theta, b)} + \int_\theta^{\overline{t}} \left((1-\xi)U_1^P(\overline{x}, s, b) + \xi U_1^A(\overline{x}, s)\right) f(s)ds.$$

By the definition of θ and since and $a_P^*(t, b) > a_A^*(t)$ for all $t \in [0, 1]$, we have $U_2^P(\underline{x}, \theta, b) < 0$, $U_2^P(\overline{x}, \theta, b) > 0$, $U_1^P(\overline{x}, s, b) < 0$ for $s \in [\theta, \overline{t}]$, $U_1^A(\overline{x}, s) < 0$ for $s \in [\theta, \overline{t}]$, and $U^A(\underline{x}, \theta) > U^A(\overline{x}, \theta)$. This implies that the first term in the above sum is positive, while the second is negative. Since the goal is to show that the overall expression is negative, we can take $\xi = 1$ to obtain an upper bound. Condition (N) implies that the expression with $\xi = 1$ is strictly negative and hence that the ex ante principal gains from inserting an additional contract action $x' \in (\underline{x}, \overline{x})$ provided it is sufficiently close to $\overline{x}$. □

We are now in a position to characterize optimal contract-equilibrium pairs when conflict is small in either dimension. With small compensation conflict, optimality requires that the contract is maximally detailed (subject to the writing-cost constraint). With small action conflict, optimal contracts are close to being maximally detailed and accompanied by influential cheap-talk communication. Overall, independent of the size of action conflict and compensation conflict, we get a bang-bang result: optimal contracts are either close to being simple or close to being maximally detailed. Our previous result showed that with extreme conflict optimal contracts are simple. Here we establish that with little conflict they are (close to) maximally detailed.

Proposition 2 *i. For all $\widehat{K}$ and all $b > 0$, there exists $\overline{\xi} > 0$ such that for all $\xi \in (0, \overline{\xi})$, every optimal contract-equilibrium pair induces $\widehat{K}$ contract actions.*

ii. For all $\widehat{K}$, there exists $\overline{b} > 0$ such that for all $b \in (0, \overline{b})$ and all $\xi \in (0, 1)$, every optimal contract-equilibrium pair induces influential communication and at least $\widehat{K} - 2$ contract actions.

iii. In any optimal contract-equilibrium pair, either $K \geq \widehat{K} - 2$ or $K \leq 2$.

Proof of Proposition 2. (i) Fix $b > 0$. To start, let $\xi = 0$. We will show that any contract-equilibrium pair (X, e^X) that induces fewer than $\widehat{K}$ contract actions can be improved upon.

First, consider the case that the maximal action a_n that (X, e^X) induces is a contract action.

If $n = 1$, then either there is an action $a' > a_1$ that the principal prefers to a_1 for a positive measure set of states, or there is such an action $a'' < a_1$. In either case, the principal gains from adding the preferred action.

If $n > 1$ and the next lower action a_{n-1} that is induced by (X, e^X) is also a contract action, we can raise payoffs by adding a contract action x' with $a_{n-1} < x' < a_n$: Both of the actions a_{n-1} and a_n are induced by sets of states (t_{n-2}, t_{n-1}) and $(t_{n-1}, 1)$ that have positive measure. At state t_{n-1} the principal is indifferent between a_{n-1} and a_n. Since $U_{11}^P < 0$, at state t_{n-1} the principal strictly prefers action x' to both a_{n-1} and a_n. Continuity of U^P then implies that there is a positive measure of states at which the principal prefers inducing action x' to inducing either a_{n-1} or a_n.

If $n > 1$, and a_{n-1} is a communication action, consider two possibilities: First, suppose that there is a type $\hat{t}$ who induces action a_n and for whom $a_P^*(\hat{t}, b) > a_n$. Then, if we introduce a new contract action $a_n + \varepsilon$ with $\varepsilon > 0$ sufficiently small, there will be a positive measure of types who strictly prefer inducing $a_n + \varepsilon$ to inducing a_n. All these types induce a_n in (X, e^X). Given the modified contract, there will therefore be an equilibrium in which these types induce action $a_n + \varepsilon$ and all of the remaining types induce the same actions as before.

Second, if $a_P^*(t, b) \leq a_n$ for all types t who induce action a_n in (X, e^X), replace the communication action a_{n-1} by an equivalent contract action and replace the contract action a_n in X by $a_n - \varepsilon$. Type $t = 1$ is one of the types who induces action a_n in (X, e^X). Hence, by assumption $a_P^*(1, b) \leq a_n$, and therefore $\frac{\partial}{\partial a} U^P(1, a_n, b) \leq 0$. Combining this observation with the cross-partial condition, we have

$$\frac{\partial}{\partial a} \int_{t_{n-1}}^{1} U^P(t, a_n, b) f(t) dt = \int_{t_{n-1}}^{1} \frac{\partial}{\partial a} U^P(t, a_n, b) f(t) dt < 0.$$

Thus the principal's expected payoff for types who induce a_n in X strictly increases if we replace a_n in X by $a_n - \varepsilon$, for sufficiently small $\varepsilon > 0$. Denote the replacement contract by X'. Types t for whom $a_P^*(t, b) \leq a_{n-1}$ face the same incentives under X' as they did under X. Types who induce action a_{n-1} under contract X can still do so, or switch to a_n if that is an improvement. Hence, for sufficiently small ε there is a contract-equilibrium pair $(X', e^{X'})$ that the principal ex ante strictly prefers to (X, e^X).

Consider now the case that the maximal action a_n that the contract-equilibrium pair (X, e^X) induces is a communication action. Let a_{j^*} be the minimal action induced by contract-equilibrium pair (X, e^X) such that all actions a_j with $j \geq j^*$ induced by (X, e^X) are communication actions. With condition (M), any two backward solutions $(t_n = 1, t_{n-1}, \ldots)$ and $(t'_n = 1, t'_{n-1}, \ldots)$, with $t'_{n-1} > t_{n-1}$ satisfy $t'_j > t_j$ for all j with $j^* \leq j \leq n-1$. Actions $a(t_{j-1}, t_j)$ and $a(t'_{j-1}, t'_j)$ satisfy $a(t'_{j-1}, t'_j) > a(t_{j-1}, t_j)$ for $j^* \leq j \leq n-1$. Backward solutions $(t'_n = 1, t'_{n-1}, \ldots)$ and the corresponding actions $a(t'_{j-1}, t'_j)$ are continuous in t'_{n-1}. This implies that for $t'_{n-1} > t_{n-1}$ sufficiently close to t_{n-1}, we can find $\varepsilon > 0$ such that type t'_{j^*} is indifferent between actions $a_{j^*} + \varepsilon$ and $a(t_{j^*}, t_{j^*+1})$, and all actions $a(t'_j, t'_{j+1})$, $j = j^*, \ldots, n-1$, are close to the actions $a(t_j, t_{j+1})$, $j = j^*, \ldots, n-1$. Replace the contract X by a contract X' with an additional contract action $a_{j^*} + \varepsilon$. Then the argument we just gave implies that there is a contract-equilibrium pair $(X', e^{X'})$ with the same number of equilibrium actions (i.e., one fewer communication action and one more contract action), actions a_j with $j < j^*$ unchanged, critical types t_j with $j < j^* - 1$ unchanged, and critical types t_j with $j \geq j^* - 1$ replaced by new critical types $t'_j > t_j$. The equilibrium actions a'_j in

$(X', e^{X'})$ for $j \geq j^*$ satisfy $a'_j > a_j$. The interim principal's expected payoff conditional on $t \in (t_{j-1}, t_j)$ from action a'_j would satisfy

$$\int_{t_{j-1}}^{t_j} U^P(t, a'_j, b) f(t) dt > \int_{t_{j-1}}^{t_j} U^P(t, a_j, b) f(t) dt$$

for all $j \geq j^*$ and for $\varepsilon > 0$ sufficiently small. This implies that if we consider only the impact of the raised actions on the interim principal's overall expected payoff, that payoff strictly increases. In addition, fixing the new equilibrium actions a'_j for $j \geq j^*$, the interim principal re-optimizes, which is reflected in the replacement of t_j by t'_j for $j \geq j^* - 1$. This also increases the interim principal's expected payoff.

This establishes that no contract-equilibrium pair (X, e^X) that induces fewer than $\widehat{K}$ contract action can be optimal when $\xi = 0$.

Let $(\hat{X}, e^{\hat{X}})$ be a contract-equilibrium pair that solves the ex ante principal's optimization problem for $\xi = 0$. Notice (i) that with $\xi = 0$ the expected payoff of the ex ante principal equals the expected payoff of the interim principal and (ii) that the existence of the contract-equilibrium pair $(\hat{X}, e^{\hat{X}})$ does not depend on the magnitude of ξ. Let $V^P(K)$ denote the maximal expected payoff of the interim principal from contract-equilibrium pairs with no more than K contract actions. Our observation for $\xi = 0$ implies that $V^P(\widehat{K})$ equals the interim principal's payoff from $(\hat{X}, e^{\hat{X}})$ and that $V^P(\widehat{K}) > V^P(K)$, for all $K < \widehat{K}$. Let $\hat{V}^P = V^P(\widehat{K})$ denote the interim principal's and $\hat{V}^A$ the agent's expected payoffs from the contract-equilibrium pair $(\hat{X}, e^{\hat{X}})$. Since the contract-equilibrium pair $(\hat{X}, e^{\hat{X}})$ is feasible for all ξ, with $\widehat{K}$ contract actions the ex ante principal can achieve a payoff of at least $(1 - \xi)\hat{V}^P + \xi \hat{V}^A$. If instead the ex ante principal used only $K < \widehat{K}$ contract actions, her payoff would be bounded from above by $(1 - \xi)V^P(K) + \xi \times 0$, where we use the fact that 0 is an upper bound on the agent's expected payoff. Evidently, there is exists $\overline{\xi} \in (0, 1)$ such that for all $\xi \in [0, \overline{\xi})$, we have $(1 - \xi)\hat{V}^P + \xi \hat{V}^A > (1 - \xi)V^P(K) + \xi \times 0, \ \forall K < \widehat{K}$.

(ii) We begin by showing that for every $\widehat{K}$, there exists $\overline{b} > 0$ such that for all $b \in (0, \overline{b})$ there is influential communication in every optimal contract-equilibrium pair.

Let $V^P(b)$ denote the principal's maximal payoff over equilibria that maximize the agent's payoff in the cheap-talk game with bias b. Since we are interested in small values of b, we can fix $\hat{b} > 0$ and restrict attention to $b \in [0, \hat{b}]$. All actions $a \in \mathbb{R}$ that are taken in an equilibrium of the cheap-talk game satisfy $a \in [a^*_A(0), a^*_A(1)]$. The set $Z := [a^*_A(0), a^*_A(1)] \times [0, 1] \times [0, \hat{b}]$ is compact. Since U^P is continuous, it is uniformly continuous on Z. Hence, for all $\varepsilon > 0$, there exists $b_1 > 0$ such that for all $(a, t, b) \in Z$ with $b \in [0, b_1)$

$$\left| U^P(a, t, b) - U^A(a, t) \right| < \frac{\varepsilon}{2}$$

and therefore

$$\left| V^P(b) - V^A(b) \right| < \frac{\varepsilon}{2}. \tag{5}$$

Fix $N > \widehat{K}$. Let Φ_N denote the set of all measurable functions $\phi : T \to \mathbb{R}$ that take on no more than N different values in $\mathbb{R}$. Define $V^A_N := \max_{\phi \in \Phi_N} \mathbb{E}[U^A(\phi(t), t)]$ as the maximal agent payoff that can be achieved with no more than N actions. Define $V_N^P(b) := \max_{\phi \in \Phi_N} \mathbb{E}[U^P(\phi(t), t, b)]$ as the maximal principal payoff that can be achieved with no more than N actions when the bias equals b. For any finite set of actions $\tilde{A} \subset \mathbb{R}$, for all but a finite number of types t, we have $\max_{a \in \tilde{A}} U^A(a, t) < U^A(a_A^*(t), t)$. Hence, for every N, $V_N^A < \mathbb{E}[U^A(a_A^*(t), t)]$. Since $\lim_{b \to 0} V_N^P(b) = V_N^A$, we can find $b_2 > 0$ and $V_N^P < \mathbb{E}[U^A(a_A^*(t), t)]$ such that for all $b \in [0, b_2)$ we have $V_N^P(b) < V_N^P < \mathbb{E}[U^A(a_A^*(t), t)]$.

By our convergence assumption, it is the case that for all ε there exists $b_3 > 0$ such that $|V^A(b) - \mathbb{E}[U^A(a_A^*(t), t)]| < \frac{\varepsilon}{2}$ for all $b < b_3$. Combined with (5), this implies that for all ε and $b < \min\{b_1, b_3\}$, we have $|V^P(b) - \mathbb{E}[U^A(a_A^*(t), t)]| < \varepsilon$.

Choose $\varepsilon < \min\{\mathbb{E}[U^A(a_A^*(t), t) - V_N^A, \mathbb{E}[U^A(a_A^*(t), t) - V_N^P]\}$ and $b < \min\{b_1, b_2, b_3\}$. Then $V_N^A < V^A(b)$ and $V_N^P(b) < V^P(b)$ and therefore for $b < \min\{b_1, b_2, b_3\}$, we have $(1-\xi)V_N^P(b) + \xi V_N^A < (1-\xi)V_N^P + \xi V_N^A < (1-\xi)V^P(b) + \xi V^A(b)$. Thus, there exists $\bar{b}$ such that for all $b \in (0, \bar{b})$ the maximal payoff with a null contract exceeds the maximal payoff that can be achieved with no more than N induced actions. Since we assumed that $N > \widehat{K}$ this implies that there exists $\bar{b}$ such that for all $b \in (0, \bar{b})$ the maximal payoff with a null contract exceeds the maximal payoff that can be achieved using no more than one communication action. Hence, for all $b \in (0, \bar{b})$, any optimal contract-equilibrium pair must induce no fewer than two communication actions, and therefore influential communication.

Now, in order to reach a contradiction, suppose that there is influential communication in an optimal contract-equilibrium pair but the number K of contract actions that are induced satisfies $K < \widehat{K} - 2$. Then either there are two adjacent communication actions or there is a contract action (not necessarily directly) between two communication actions. In case there are two adjacent communication actions, we can replace both with contract actions x' and x'', where $x' < x''$. Let $\theta' \in (0, 1)$ be the type for which the interim principal is indifferent between x' and x''. Then $a_P^*(\theta', b) < x''$ and $a_P^*(1, b) > x''$. Therefore, there exists $\bar{t} \in (\theta', 1)$ for which $a_P^*(\bar{t}, b) = x''$. Since after replacement we have $K < \widehat{K}$, by Lemma 1 we can strictly improve on the contract-equilibrium pair (X, e^X) by introducing an additional contract action, contrary to our assumption that (X, e^X) was optimal. Next, consider the case in which there is a contract action (not necessarily directly) between two communication actions. Then we can find a pair of a contract action x and a communication action y that are adjacent and satisfy $x < y$. Let $\theta' \in (0, 1)$ be the type for which the principal is indifferent between x and y. Then, using the fact that y is a communication action, $a_P^*(\theta', b) < y$ and $a_P^*(1, b) > y$. Therefore, there exists $\bar{t} \in (\theta', 1)$ for which $a_P^*(\bar{t}, b) = y$. Hence, letting $\underline{x} = x$ and replacing the communication action y with the contract action $\overline{x} = y$, we can satisfy the requirement in condition (N) that there exists $\bar{t} \in [0, 1]$ for which $a_P^*(\bar{t}, b) = \overline{x}$. Since after replacement we have $K < \widehat{K}$, by Lemma 1 we can strictly improve on the contract-equilibrium pair (X, e^X) by introducing an additional contract action, contrary to our assumption that (X, e^X) was optimal.

(iii) To derive a contradiction, suppose that there is an optimal contract-equilibrium pair (X, e^X) with $2 < K < \widehat{K} - 2$. Since $K > 2$, three contract actions are induced. Hence, we can choose three adjacent actions that are induced, at least one of which is a contract action. From $K < \widehat{K} - 2$ it follows that if any of the three chosen actions are communication actions, we can replace them by contract actions, without changing payoffs. The fact that one of the three chosen actions was a contract action to begin with implies that after the replacement, we continue to have $K < \widehat{K}$. After the replacement, three adjacent contract actions x', x'', and x''' with $x' < x'' < x'''$ are induced. Thus by assumption, there are types $\theta', \theta'' \in (0,1)$ such that at θ' (θ'') the principal is indifferent between x' and x'' (x'' and x'''). Therefore, there exists $\bar{t} \in (\theta', \theta'')$ for which $a^*_P(\bar{t}, b) = x''$. Hence, letting $\underline{x} = x'$ and $\overline{x} = x''$, we can satisfy the requirement in condition (N) that there exists $\bar{t} \in [0,1]$ for which $a^*_P(\bar{t}, b) = \overline{x}$. Since after replacement we have $K < \widehat{K}$, by Lemma 1 we can strictly improve on the contract-equilibrium pair (X, e^X) by introducing an additional contract action, contrary to our assumption that (X, e^X) was optimal. □

6 Uniform quadratic environment

In this section we characterize optimal contract-equilibrium pairs in the uniform-quadratic environment when there is significant interplay of contracting and communication. For this, we assume that the bias is small enough for optimality to require the use of communication actions (Proposition 2), in addition to contract actions. Our main characterization below then shows that topics are of similar size: the number of induced communication actions within each topic typically differs by one but at most by four.

For any contract $X = \{x_1, x_2, \ldots, x_K\}$, we use the following notation: $\mathcal{T}_1 \coloneqq \mathcal{T}(x_1)$, $\mathcal{T}_k \coloneqq \mathcal{T}(x_{k-1}, x_k)$, $k = 2, \ldots, K$, and $\mathcal{T}_{K+1} \coloneqq \mathcal{T}(x_K)$. Notice that in this environment the type set that is associated with any inner topic is of the form $[x_{k-1} - b, x_k - b]$. We suppose $\widehat{K} \geq 5$. Thus by Proposition 2 at least 3 contract actions will be used in any optimal contract-equilibrium pair. Having three contract actions implies by Lemma B.2 in the appendix that the lowest and highest contract actions satisfy $x_1 \geq -b$ and $x_K \leq 1 + b$. Hence for topic $\mathcal{T}_{K+1}$ we have the associated type set $[x_K - b, 1]$; for topic $\mathcal{T}_1$ we have $[0, x_1 - b]$ if $x_1 \geq b$, otherwise, $\mathcal{T}_1$ is empty and we have $[0, x_2 - b]$ for the improper inner topic $\mathcal{T}_2$.

Definition 2 *An* ***n-step*** $\boldsymbol{\mathcal{T}}$***-equilibrium*** *is a $\mathcal{T}$-equilibrium that induces $n \in \mathbb{N}_0$ communication actions.*

For any topic $\mathcal{T}$, we denote the maximal number n for which there is an n-step $\mathcal{T}$-equilibrium by $N(\mathcal{T})$. Moreover, for any topic $\mathcal{T}_k$ with an n_k-step $\mathcal{T}_k$-equilibrium, we denote the corresponding communication actions by $y_{k,i}$, $i = 1, \ldots, n_k$. For each of those communication actions, there is a minimal type $\theta_{k,i-1}$ and a maximal type $\theta_{k,i}$ willing to induce that action (these might equal 0 or 1). We refer to $(\theta_{k,i-1}, \theta_{k,i})$ as the ith *communication interval* in $\mathcal{T}_k$. This is the set of types who strictly prefer to induce action $y_{k,i}$. Figure 4 illustrates.

Notice that, unlike in the leading CS example, here the boundary conditions are endogenous and belong to the interior of a topic. For example for an inner topic, types in $(x_{k-1} - b, \theta_{k,0})$ induce the contract action x_{k-1}; types in $(\theta_{k,n_k}, x_k - b)$ induce the contract action x_k; and, the remaining types induce the communication actions $y_{k,i} = \frac{\theta_{k,i} + \theta_{k,i-1}}{2}$, $i = 1, \ldots, n_k$.

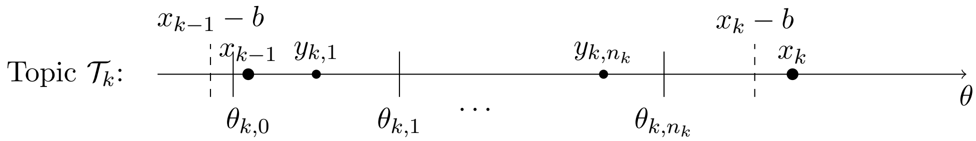

Figure 4: Types in topic $\mathcal{T}_k$ induce contract actions x_k and x_{k-1}, and communication actions $y_{k,i}$, $i = 1, \dots, n_k$.

We can now state our result that shows that topics are of similar size. Note that for $\widehat{K} \geq 5$ most topics are inner topics.

Proposition 3 *For every optimal contract-equilibrium pair (X, e^X) and all inner topics $\mathcal{T}$ and $\mathcal{T}'$ generated by the contract X,*

$$|N(\mathcal{T}) - N(\mathcal{T}')| \leq 1,$$

and for all topics $|N(\mathcal{T}) - N(\mathcal{T}')| \leq 4$.

The result illustrates the interplay of the formal structure of the contract and the informal interaction through communication. The contract is used to structure communication and it sets guardrails for what communication can be used for. The different topics in which communication takes place are of roughly equal size. The intuition for this equality stems from the well-known fact that in the uniform quadratic environment communication intervals increase in size the higher the types – within a topic. This implies that communication becomes less informative if a topic gets larger. On the other hand, more communication actions are better than fewer and there is no interaction between topics. Therefore it is optimal to equalize the number of communication actions across all topics.

The proof of the proposition proceeds through a sequence of steps, summarized in lemmas stated in the appendix. We begin by showing how the number of possible communication actions in each topic is constrained by the size of that topic (Lemma B.4). We then express the ex ante principal's payoff in each topic as a function of the number of communication actions in that topic (Lemma B.5). In Lemma B.6 we show that it is optimal to maximize this number in each topic.

As a consequence, when considering increasing the size of one topic at the expense of another, the principal faces a tradeoff: communication opportunities shift from the shrinking to the growing topic. To satisfy communication incentives, it is necessary in large topics to have large communication intervals, which limits the efficacy of communication. Therefore to benefit most from communication, it is preferable to try to equalize the size of topics. This is formalized in our key result in Lemma B.8, where we show that the maximal numbers of communication actions in two neighboring inner topics can at most differ by one. To extend our result of similar sizes of neighboring inner topics to all inner topics, we show that switching two neighboring inner topics, say $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ along with the corresponding respective $\mathcal{T}_k$- and $\mathcal{T}_{k+1}$- equilibrium behavior, preserves incentive compatibility and leaves the principal's ex ante payoff unchanged (Lemma B.7). Figure 5 illustrates.

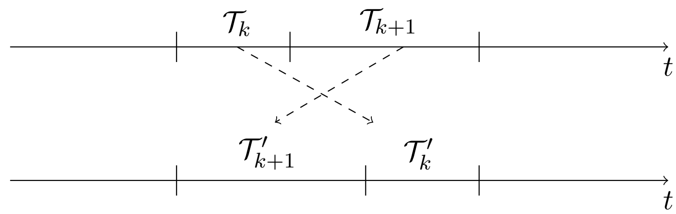

Figure 5: Topics and translated topics.

To complete the picture, in Lemma B.8 we also establish bounds on the differences in the number of communication actions in adjacent topics that need not be inner topics. Specifically, if $\mathcal{T}_1$ is nonempty, this difference between $\mathcal{T}_1$ and $\mathcal{T}_2$ is at most one; if $\mathcal{T}_1$ is empty, $\mathcal{T}_2$ has at most one communication action, implying that $\mathcal{T}_3$ to $\mathcal{T}_K$ have at most two; and the difference between the numbers of communication actions in $\mathcal{T}_K$ and $\mathcal{T}_{K+1}$ is at most two. Topics that are not inner topics cannot be switched with other topics. Hence, adding the numbers of possible differences of communication actions across all topics, we get that for any arbitrary two topics they can differ by at most four.

Finally, when there influential communication in $\mathcal{T}_2$, then $\mathcal{T}_1$ is non-empty (Lemma B.9). As a consequence, with sufficiently little conflict – and therefore a large number of communication actions – the maximal difference across all topics in the proposition reduces to $|N(\mathcal{T}) - N(\mathcal{T}')| \leq 3$.

A Appendix

A set $\boldsymbol{A} = [a_\ell, a_h] \subset \mathbb{R}$ is a *set of candidate actions* if for every $\xi \in [0,1)$ and every optimal contract-equilibrium pair (X, e^X), all actions $a \in \mathbb{R}$ that are induced by that contract-equilibrium pair satisfy $a \in \boldsymbol{A}$. The following observation shows that independent of the compensation conflict we can restrict attention to a compact subset of the action space.

Lemma A.1 *There exists a set of candidate actions* $\boldsymbol{A}$.

Let $\mathcal{A} := \{\boldsymbol{A} \subset \mathbb{R} | \boldsymbol{A} \text{ is a set of candidate actions}\}$ and $\boldsymbol{A}^* := \bigcap_{\boldsymbol{A} \in \mathcal{A}} \boldsymbol{A}$. It is easily seen that $\boldsymbol{A}^*$ is a nonempty closed interval and the smallest set of candidate actions. In the sequel we will refer to $\boldsymbol{A}^*$ as the *set of relevant actions*. The following observation is immediate.

Observation A.1 *For every* $\xi \in [0,1)$ *and every optimal contract-equilibrium pair, all actions* $a \in \mathbb{R}$ *that are induced by that contract-equilibrium pair satisfy* $a \in \boldsymbol{A}^*$.

Proof of Lemma A.1. Strict concavity and the cross-partial condition imply that the interim principal strictly prefers the higher (lower) of any two actions less than $a_A^*(0)$ (greater than $a_P^*(1,b)$). Therefore, any contract-equilibrium pair that does not induce at least one action $a \in [a_A^*(0), a_P^*(1,b)]$ induces at most two actions. Moreover, these two actions $a' < a''$ satisfy $a' < a_A^*(0) < a_P^*(1,b) < a''$.

Any contract-equilibrium pair that does not induce at least one action $a \in [a_A^*(0), a_P^*(1,b)]$ while inducing two actions $a' < a''$ (satisfying $a' < a_A^*(0) < a_P^*(1,b) < a''$) can be improved upon: Let $\hat{t}$ be the type who satisfies $U^P(a', \hat{t}, b) = U^P(a'', \hat{t}, b)$. Evidently, we can find $\underline{a}$ and $\overline{a}$ such that $a' < \underline{a} < a_A^*(0) < a_P^*(1,b) < \overline{a} < a''$ and $U^P(\underline{a}, \hat{t}, b) = U^P(\overline{a}, \hat{t}, b)$. Given the actions $\underline{a}$ and $\overline{a}$, the interval of types for which the interim principal chooses the lower action remains the same. For all types in that interval both interim principal and agent, and therefore the ex ante principal, prefer the action $\underline{a}$ to the action a'. Similarly, the interval of types for which the interim principal chooses the higher action remains the same, and for all types in that interval both interim principal and agent, and therefore the ex ante principal, prefer the action $\overline{a}$ to the action a''.

Combining these observations, it follows that every optimal contract-equilibrium pair induces at least one action $a \in [a_A^*(0), a_P^*(1,b)]$.

Let

$$\underline{u}^P := \min_{\substack{a \in \left[a_A^*(0), a_P^*(1,b)\right] \\ t \in [0,1]}} U^P(a, t, b).$$

Since any optimal contract-equilibrium pair induces at least one action in the interval $[a_A^*(0), a_P^*(1,b)]$, $\underline{u}^P$ is a lower bound on the interim principal's utility, regardless of the type. It is well-defined since U^P is continuous and $[a_A^*(0), a_P^*(1,b)] \times [0,1]$ is compact. Strict concavity of U^P in its first argument implies that there exists $a_h > a_P^*(1,b)$ such that $\underline{u}^P > U^P(a, 1, b)$ for all $a > a_h$.

Similarly, there exists $a_\ell < a^*_A(0)$ such that $U^P(a, 0, b) < \underline{u}^P$ for all $a < a_\ell$.

Combining these two observations with the cross-partial condition that holds for U^P implies that the interim principal would never induce an action that is outside the interval $[a_\ell, a_h]$. □

B Uniform quadratic case

Lemma B.1 *Any contract-equilibrium pair (X, e^X) in Γ^X can induce at most one contract action x' with $x' \leq b$ and at most one contract action x'' with $x'' \geq 1 + b$.*

Proof of Lemma B.1. Consider any contract with $1 + b \leq x_{K-1} < x_K$. Then, since all types θ of the interim principal have ideal points $\theta + b \leq 1 + b$ and their payoff functions are strictly concave in the action taken, they all strictly prefer x_{K-1} over x_K. Hence, the interim principal would never induce x_K. An analogous argument establishes the first part of the statement. □

Lemma B.2 *For every optimal contract-equilibrium pair (X, e^X) in Γ^X that induces $K \geq 3$ contract actions, the lowest of those actions, x_1, satisfies $x_1 \geq -b$, and the highest of those actions, x_K, satisfies $x_K \leq 1 + b$.*

Proof of Lemma B.2. Bound on x_1:

Consider a contract-equilibrium pair (X, e^X) in Γ^X in which $K \geq 3$ contract actions are induced and $x_1 < -b$. We will show that there exists a contract-equilibrium pair $(X', e^{X'})$ in $\Gamma^{X'}$ that raises both the interim principal's and the agent's expected payoffs, and therefore the ex ante principal's payoff.

Let ℓ denote the length of the interval of types who induce contract action x_1 in equilibrium e^X of Γ^X. Since x_1 is induced by assumption, we know that $\ell > 0$. Since none of the types $\theta \geq x_2 - b$ induce action x_1, it follows that $x_2 \geq b + \ell$.

Let X' be the contract obtained from X by replacing x_1 with $x'_1 = x_2 - \ell$, and otherwise leaving the contract unchanged. Since $K \geq 3$, it follows from Lemma B.1 that $x_2 - b < 1$.

We have $x'_1 - b = x_2 - \ell - b \geq (b + \ell) - \ell - b = 0$. Hence $[x'_1 - b, x_2 - b] \subset [0, 1]$.

Evidently, there exists an equilibrium $e^{X'}$ in $\Gamma^{X'}$ in which (1) all types $\theta \geq x_2 - b$ induce the same actions as they did in the equilibrium e^X of Γ^X and (2) for every type θ with $\ell < \theta < x_2 - b$ who induced action a in the equilibrium e^X of Γ^X, type $\theta - \ell$ induces action $a - \ell$ in the equilibrium $e^{X'}$ of $\Gamma^{X'}$.

In the equilibrium $e^{X'}$ in $\Gamma^{X'}$ types in $[x'_1 - b, x'_1 - b + \ell/2)$ induce action x'_1 and types in $(x'_1 - b + \ell/2, x_2 - b]$ induce action x_2. Hence, the agent's expected payoff from types in the interval $[x'_1 - b, x_2 - b]$ in the equilibrium $e^{X'}$ in $\Gamma^{X'}$ equals

$$-\int_0^{\ell/2} (s - b)^2 ds - \int_{\ell/2}^{\ell} (s - (\ell + b))^2 ds.$$

In contrast, the agent's expected payoff from types in the interval $[0,\ell]$ in the equilibrium e^X in Γ^X equals

$$-\int_0^\ell (s-x_1)^2 ds.$$

Hence, the agent gains from replacing equilibrium e^X in Γ^X with equilibrium $e^{X'}$ in $\Gamma^{X'}$ if

$$-\int_0^\ell (s-x_1)^2 ds < -\int_0^{\ell/2} (s-b)^2 ds - \int_{\ell/2}^\ell (s-(\ell+b))^2 ds.$$

This inequality is equivalent to

$$\ell x_1 - x_1^2 < -b^2 + \frac{1}{4}\ell^2.$$

Since $\ell > 0$ and $x_1 < 0$ by assumption, a sufficient condition for the latter inequality to be satisfied is that

$$-x_1^2 < -b^2.$$

Since $x_1 < 0$ by assumption, the latter inequality holds as long as $x_1 < -b$. Hence, if $x_1 < -b$, replacing the contract-equilibrium pair (X, e^X) with $(X', e^{X'})$ raises the agent's payoff.

It remains to show that the interim principal also gains from replacing the pair (X, e^X) with the pair $(X', e^{X'})$. For this purpose, it suffices to show that every type $x_1' - b + l \in [x_1' - b, x_2 - b]$ of the interim principal has a higher expected payoff in the equilibrium $e^{X'}$ of $\Gamma^{X'}$ than does type $l \in [0,\ell]$ in the equilibrium e^X of Γ^X. The interim principal's payoff when her type is $l \in [0,\ell]$ in the equilibrium e^X of Γ^X equals $-(l+b-x_1)^2$. Her payoff when her type is $x_1' - b + l \in [x_1' - b, x_2 - b]$ in the equilibrium $e^{X'}$ of $\Gamma^{X'}$ equals at least $-(x_1' - b + l + b - x_1')^2 = -l^2$. Since we assumed that $x_1 < -b$, it follows that $-(l+b-x_1)^2 < -l^2$. Thus for every type $x_1' - b + l \in [x_1' - b, x_2 - b]$, the interim principal has a higher expected payoff in the equilibrium $e^{X'}$ of $\Gamma^{X'}$ than does type $l \in [0,\ell]$ in the equilibrium e^X of Γ^X.

Since both the agent and the interim principal gain from replacing the pair (X, e^X) with the pair $(X', e^{X'})$, we obtain the desired contradiction to the assumption that (X, e^X) is optimal for the ex ante principal.

An analogous argument establishes the bound on x_K. □

Lemma B.3 *For any inner and improper inner topic $\mathcal{T}_k$, in any n_k-step $\mathcal{T}_k$-equilibrium, the critical types $\theta_{k,i}$ $i = 0, \ldots, n_k$ satisfy the difference equation*

$$\theta_{k,i} - \theta_{k,i-1} = \frac{\theta_{k,n_k} - \theta_{k,0}}{n_k} - 2b\left(n_k - 2i + 1\right), \ \text{for } i = 1, \ldots, n_k, \tag{6}$$

with boundary conditions

$$\theta_{k,0} = \frac{(2n_k+1)x_{k-1} + x_k - 2b(1+n_k)^2}{2(1+n_k)}, \ \theta_{k,n_k} = \frac{x_{k-1} + (2n_k+1)x_k - 2b(1+n_k)^2}{2(1+n_k)}. \tag{7}$$

For outer topic $\mathcal{T}_{K+1}$, the boundary conditions are of the form

$$\theta_{K+1,0} = \frac{2n_{K+1}x_K + 1 - 2bn_{K+1}(n_{K+1}+1)}{2n_{K+1}+1}, \quad \theta_{K+1,n_{K+1}} = 1. \tag{8}$$

If the outer topic $\mathcal{T}_1$ is nonempty, then for this topic the boundary conditions are of the form

$$\theta_{1,0} = 0, \quad \theta_{1,n_1} = \frac{2x_1n_1 - 2bn_1(n_1+1)}{2n_1+1}. \tag{9}$$

Proof of Lemma B.3. By Lemma B.1 and Lemma B.2, we have that $x_2 \geq b$ and $x_K \leq 1+b$. Consider topics $\mathcal{T}_k$ for $k = 2, \ldots, K$.

For the remainder of the proof, we suppress the index k for the critical types and for the number of steps in topic $\mathcal{T}_k$, writing θ_i for $\theta_{k,i}$ as well as n for n_k. The interim principal's arbitrage conditions for critical types θ_i with $0 < i < n$ are

$$\theta_i + b - \frac{\theta_{i-1}+\theta_i}{2} = \frac{\theta_{i+1}+\theta_i}{2} - \theta_i - b$$

and hence, for these critical types we have

$$\theta_{i+1} - \theta_i = \theta_i - \theta_{i-1} + 4b \quad \text{for } 0 < i < n \tag{10}$$

as usual. The arbitrage conditions for the two remaining critical types, θ_0 and θ_n, are

$$\theta_0 + b - x_{k-1} = \frac{1}{2}(\theta_1 + \theta_0) - \theta_0 - b \quad \text{and} \tag{11}$$

$$\theta_n + b - \frac{1}{2}(\theta_n + \theta_{n-1}) = x_k - \theta_n - b. \tag{12}$$

Fixing θ_0 and θ_1, iterating the expression in equation (10), and summing the resulting interval lengths $\theta_{i'} - \theta_{i'-1}$ between θ_0 and θ_i gives us

$$\theta_i - \theta_0 = i(\theta_1 - \theta_0) + 4b\frac{1}{2}i(i-1) \text{ for } i = 1, \ldots, n, \tag{13}$$

which implies that

$$\theta_i - \theta_{i-1} = \theta_1 - \theta_0 + 4b(i-1) \tag{14}$$

and

$$\theta_1 - \theta_0 = \frac{\theta_n - \theta_0}{n} - 2b(n-1). \tag{15}$$

Hence,

$$\theta_i - \theta_{i-1} = \frac{1}{n}(\theta_n - \theta_0) - 2b(n - 2i + 1), \tag{16}$$

which establishes (6) in the statement of the Lemma. Equations (13) and (15) imply

$$\theta_i = \frac{i}{n}(\theta_n - \theta_0) + \theta_0 - 2bi(n-i). \tag{17}$$

Using (17) to substitute for θ_1 in (11) we obtain

$$\theta_n = (2n+1)\theta_0 + 2bn(n+1) - 2nx_{k-1}. \tag{18}$$

Using (17) to substitute for θ_{n-1} in (12), we obtain

$$\theta_0 = (2n+1)\theta_n + 2bn(n+1) - 2nx_k. \tag{19}$$

Solving the system of equations (18) and (19) gives us θ_0 and θ_n.

Finally, with $n = 0$, the arbitrage condition becomes $\theta_0 + b - x_{k-1} = x_k - \theta_0 - b$, which is equivalent to (7).

The results for $k = 1$ and $k = K + 1$ can be proven analogously. □

Lemma B.4 *For every inner and improper inner topic $\mathcal{T}_k$, in any n_k-step $\mathcal{T}_k$-equilibrium, $x_k - x_{k-1} > 2bn_k(n_k+1)$. In any n_{K+1}-step $\mathcal{T}_{K+1}$-equilibrium, $1 - (x_K - b) > 2bn_{K+1}^2$. In any n_1-step $\mathcal{T}_1$-equilibrium, $x_1 > 2bn_1^2$.*

Proof of Lemma B.4. In the proof, we suppress the index k for the topic on the critical types, thus writing θ_i for what would otherwise be $\theta_{k,i}$ as well as n for n_k.

If there is a nontrivial interval of types in topic $\mathcal{T}_k$ who induce a communication action, then this action must be strictly greater than x_{k-1}. Therefore, for communication in topic $\mathcal{T}_k$, we must have $\theta_0 > x_{k-1} - b$, from the arbitrage condition for θ_0. In addition, rearranging the arbitrage condition, we have $\theta_1 - \theta_0 = 2\theta_0 + 4b - 2x_{k-1}$. The right-hand side of this equation is strictly increasing in θ_0 and, since $\theta_0 > x_{k-1} - b$, bounded from below by $2(x_{k-1} - b) + 4b - 2x_{k-1} = 2b$. To summarize, we need to respect the constraints

$$\theta_0 > x_{k-1} - b, \text{ and} \tag{20}$$

$$\theta_1 - \theta_0 > 2b. \tag{21}$$

Using (21), equation (6) in Lemma B.3 for $i = 1$ implies that we need to satisfy the condition

$$\frac{\theta_n - \theta_0}{n} - 2b(n-1) > 2b. \tag{22}$$

The minimal length $\theta_1 - \theta_0$ of the first step is greater than $2b$ (by (21)). Each of the $(n-1)$ additional steps adds $4b$ to the length of the previous step, according to equation (6) in Lemma B.3. Therefore, it follows that the length of the nth step, $\theta_n - \theta_{n-1}$, is greater than $2b + (n-1)4b$. Using the arbitrage condition ($\theta_n + b - \frac{1}{2}(\theta_n + \theta_{n-1}) = x_k - \theta_n - b$) for θ_n, this implies that $\theta_n + b < x_k - [b + 2(n-1)b + b]$. Hence,

$$\theta_n < x_k - 2nb - b. \tag{23}$$

Combine (20), (22), and (23) to obtain $\frac{x_k - 2nb - b - (x_{k-1} - b)}{n} - 2b(n-1) > 2b$, which is equivalent to

$$x_k - x_{k-1} > 2bn(n+1). \tag{24}$$

The results for $k = 1$ and $k = K + 1$ can be proven analogously. $\square$

Denote the ex ante principal's payoff from an n-step $\mathcal{T}$-equilibrium by $\Pi^\xi(n, \mathcal{T})$.

Lemma B.5 *For every inner topic $\mathcal{T}_k$, in an n_k-step $\mathcal{T}_k$-equilibrium, the principal's* ex ante *payoff in topic $\mathcal{T}_k$ is given by*

$$\Pi^\xi(n_k, \mathcal{T}_k) = \frac{(x_{k-1} - x_k)((x_{k-1} - x_k)^2 + 4b^2(1 + n_k)^2(3 + 2n_k + n_k^2 - 3(1 - \xi)))}{12(1 + n_k)^2}. \tag{25}$$

In an n_{K+1}-step $\mathcal{T}_{K+1}$-equilibrium, the principal's ex ante *payoff in topic $\mathcal{T}_{k+1}$ is given by*

$$\Pi^\xi(n_{K+1}, \mathcal{T}_{K+1}) = \frac{(-1 + x_K)^3 - 3b(-1 + x_K)^2(1 - \xi) + b^2(-1 + x_K)\left(4n_{K+1}(1 + n_{K+1})\left(1 + n_{K+1} + n_{K+1}^2\right) + 3(1 - \xi)\right)}{3(1 + 2n_{K+1})^2}$$

$$+ \frac{b^3\left(-1 + 4n_{K+1}\left(-1 + n_{K+1}\left(n_{K+1}(2 + n_{K+1}) - 3(1 + n_{K+1})^2(1 - \xi)\right)\right)\right)}{3(1 + 2n_{K+1})^2}.$$

In an n_1-step $\mathcal{T}_1$-equilibrium, the principal's ex ante *payoff in topic $\mathcal{T}_1$ is given by*

$$\begin{aligned}\Pi^\xi(n_1, \mathcal{T}_1) =& \frac{1}{3(1 + 2n_1)^2}(-x_1^3 + b^2x_1(4n_1(1 + n_1)(-2 + (n_1 - 2)n_1) - 3(1 - \xi)) + \\ & 3bx_1^2(1 - \xi) + b^3(1 + 4n_1(2 + n_1(3 - n_1 - 2n_1^2(2 + n_1) + 3(1 + n_1)^2(1 - \xi))))).\end{aligned}$$

Suppose $\mathcal{T}_1$ is empty, then the principal's ex ante *payoff in an n_2-step $\mathcal{T}_2$-equilibrium in improper inner topic $\mathcal{T}_2$ is given by*

$$\begin{aligned}\Pi^\xi(n_2, \mathcal{T}_2) =& -\frac{1}{12(1 + n_2)^2}[-4b^3(1 + n_2)^2 + (3 + 4n_2(2 + n_2))x_1^3 + 3x_1^2x_2 - 3x_1x_2^2 + x_2^3 \\ & - 12b(1 + n_2)^2x_1^2(1 - \xi) - 4b^2(1 + n_2)^2((3 + n_2(2 + n_2))(x_1 - x_2) + 3(-2x_1 + x_2)(1 - \xi))].\end{aligned}$$

Proof of Lemma B.5. In the proof, we suppress the index k for the topic on the critical types, thus writing θ_i for what would otherwise be $\theta_{k,i}$ and n for n_k.

For $n = 0$, the *ex ante* principal's payoff in $[x_{k-1} - b, x_k - b]$, is given by

$$-\int_{x_{k-1}-b}^{\theta_0} (1 - \xi)\left((s + b) - x_{k-1}\right)^2 + \xi\left(s - x_{k-1}\right)^2 ds - \int_{\theta_0}^{x_k - b} (1 - \xi)\left((s + b) - x_k\right)^2 + \xi\left(s - x_k\right)^2 ds.$$

This reduces to

$$-(1 - \xi)\frac{1}{3}\left((\theta_0 - x_{k-1} + b)^3 + (x_k - b - \theta_0)^3\right) - \xi\frac{1}{3}\left((\theta_0 - x_{k-1})^3 + (x_k - \theta_0)^3\right)$$

$$= \frac{(x_{k-1} - x_k)((x_{k-1} - x_k)^2 + 12b^2\xi)}{12}.$$

For $n \geq 1$, the *ex ante* principal's payoff from an n-step equilibrium in $[x_{k-1} - b, x_k - b]$, is given by

$$
\begin{aligned}
&-\int_{x_{k-1}-b}^{\theta_0} (1-\xi)\left((s+b) - x_{k-1}\right)^2 + \xi\left(s - x_{k-1}\right)^2 ds \\
&-\sum_{i=1}^{n}\int_{\theta_{i-1}}^{\theta_i} (1-\xi)\left((s+b) - \frac{\theta_{i-1}+\theta_i}{2}\right)^2 + \xi\left(s - \frac{\theta_{i-1}+\theta_i}{2}\right)^2 ds \\
&-\int_{\theta_n}^{x_k-b} (1-\xi)\left((s+b) - x_k\right)^2 + \xi\left(s - x_k\right)^2 ds.
\end{aligned}
$$

The interim principal's payoff over the range $[\theta_0, \theta_n]$, in which she induces communication actions rather than contract actions in $\mathcal{T}_k$, equals

$$
-\sum_{i=1}^{n}\int_{\theta_{i-1}}^{\theta_i}\left((s+b) - \frac{\theta_{i-1}+\theta_i}{2}\right)^2 = -\frac{1}{12}\sum_{i=1}^{n}\left(\theta_i - \theta_{i-1}\right)^3 - \left(\theta_n - \theta_0\right)b^2.
$$

Analogously, the agent's payoff over that range is

$$
-\sum_{i=1}^{n}\int_{\theta_{i-1}}^{\theta_i}\left(\frac{\theta_{i-1}+\theta_i}{2} - s\right)^2 = -\frac{1}{12}\sum_{i=1}^{n}\left(\theta_i - \theta_{i-1}\right)^3.
$$

Using (6) in Lemma B.3 and noting that

$$
\sum_{i=1}^{n}\left(\frac{\theta_n - \theta_0}{n} + 2b(2i - n - 1)\right)^3 = \frac{(\theta_n - \theta_0)^3}{n^2} + 4b^2(\theta_n - \theta_0)(n+1)(n-1),
$$

the *ex ante* principal's payoff in $[x_{k-1} - b, x_k - b]$ reduces to

$$
\begin{aligned}
&-(1-\xi)\frac{1}{3}\left(\theta_0 - x_{k-1} + b\right)^3 - \xi\frac{1}{3}\left(\left(\theta_0 - x_{k-1}\right)^3 + b^3\right) \\
&-\frac{1}{12}\frac{(\theta_n - \theta_0)^3}{n^2} - \frac{1}{3}b^2(\theta_n - \theta_0)(n^2 - 1) - (1-\xi)(\theta_n - \theta_0)b^2 \\
&-(1-\xi)\frac{1}{3}\left(x_k - b - \theta_n\right)^3 - \xi\frac{1}{3}\left(\left(x_k - \theta_n\right)^3 - b^3\right).
\end{aligned}
$$

We can now insert the values of θ_n and θ_0 given in equation (7) from Lemma B.3. Simplifying, we obtain

$$
\Pi^{\xi}(n_k, \mathcal{T}_k) = \frac{(x_{k-1} - x_k)((x_{k-1} - x_k)^2 + 4b^2(1+n)^2(3 + 2n + n^2 - 3(1-\xi)))}{12(1+n)^2}.
$$

The results for $k = 1, 2$ and $k = K + 1$ can be proven analogously. $\square$

Lemma B.6 *For all k, the $N(\mathcal{T}_k)$-step equilibria maximize the ex ante principal's payoff among the equilibria in topic $\mathcal{T}_k$.*

Proof of Lemma B.6. Consider $k = 3, \ldots, K$. We compare the principal's payoff derived in Lemma B.5 for n_k and $n_k - 1$ steps. The payoff difference in topic $\mathcal{T}_k$ is equivalent to

$$\Pi^{\xi}(n_k, \mathcal{T}_k) - \Pi^{\xi}(n_k - 1, \mathcal{T}_k) = \frac{(1 + 2n_k)(x_{k-1} - x_k)}{12n_k^2(1 + n_k)^2}(4b^2n_k^2(1 + n_k)^2 - (x_k - x_{k-1})^2).$$

Lemma B.4 implies that the expression is strictly positive.

The results for $k = K + 1$ and $k = 1, 2$ can be proven analogously. $\square$

We say that the contract $X' = \{x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_K\}$ *switches the topics* $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ of contract $X = \{x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_K\}$ if $x'_k = x_{k+1} - (x_k - x_{k-1})$.

Lemma B.7 *Let X' switch the inner topics $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ of contract X. Then, for any equilibrium e^X in Γ^X with n_κ communication actions in $\mathcal{T}_\kappa$, $\kappa = 1, \ldots, K + 1$, there exists a payoff-equivalent equilibrium $e^{X'}$ in $\Gamma^{X'}$ with communication actions $y'_{\kappa i} = y_{\kappa i}$, $i = 1, \ldots, n_\kappa$ for all $\kappa \in \{1, \ldots, k - 1, k + 2, \ldots, K + 1\}$ with $n_\kappa \neq 0$, $y'_{k+1,i} = y_{k,i} + (x_{k+1} - x_k)$ for $i = 1, \ldots, n_k$ if $n_k \neq 0$, and $y'_{k,i} = y_{k+1,i} - (x_k - x_{k-1})$ for $i = 1, \ldots, n_{k+1}$ if $n_{k+1} \neq 0$.*

Proof of Lemma B.7. Denote the set of communication actions in the equilibrium e^X by Y and in the postulated equilibrium $e^{X'}$ by Y'. Without loss of generality, let the equilibrium e^X be in pure strategies. For any two actions a and $a' > a$, let $\theta(a, a')$ denote the type who is indifferent between those two actions. Since $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ are inner topics, $\theta(a, a')$ is well defined for all $a, a' \in \mathcal{T}_k \cup \mathcal{T}_{k+1} = \mathcal{T}'_{k+1} \cup \mathcal{T}'_k$.

For all actions $a \in X \cup Y$, let $T(a)$ be the set of types $t \in [0, 1]$ who strictly prefer inducing that action to inducing any other action in $X \cup Y$. Similarly, for all actions $a \in X' \cup Y'$, let $T'(a)$ be the set of types $t \in [0, 1]$ who strictly prefer inducing action a to inducing any other action in $X' \cup Y'$. We begin by identifying for each action $a \in X' \cup Y'$ the set of types $T'(a)$. There are four ranges of types to consider.

Types $t \leq x_{k-1} - b$: Since these types have ideal points that are no larger than x_{k-1}, they strictly prefer action x_{k-1} to any higher action in $X' \cup Y'$. The set of actions in $X' \cup Y'$ that are less than or equal to x_{k-1} is the same as in $X \cup Y$. Therefore all of the types below $t \leq x_{k-1} - b$ have an incentive to induce the same actions given $X' \cup Y'$ that they prefer to induce given $X \cup Y$.

Types $t \in (x_{k-1} - b, x'_k - b)$: The distance between the actions x_{k-1} and $y'_{k,1}$ is the same as that between x_k and $y_{k+1,1}$: $y'_{k,1} - x_{k-1} = y_{k+1,1} - (x_k - x_{k-1}) - x_{k-1} = y_{k+1,1} - x_k$. Therefore, $\theta(x_{k-1}, y'_{k,1}) = \theta(x_k, y_{k+1,1}) - (x_k - x_{k-1})$. Similarly, $\theta(y'_{k,i}, y'_{k,i+1}) = \theta(y_{k+1,i}, y_{k+1,i+1}) - (x_k - x_{k-1})$ for $i = 1, \ldots, n_{k+1} - 1$ and $\theta(y'_{k,n_{k+1}}, x'_k) = \theta(y_{k+1,n_{k+1}}, x_{k+1}) - (x_k - x_{k-1})$. This implies that types in the interval $(x_{k-1} - b, \theta(x_{k-1}, y'_{k,1}))$ strictly prefer to induce action x_{k-1}, types in the interval $(\theta(x_{k-1}, y'_{k,1}), \theta(y'_{k,1}, y'_{k,2}))$ strictly prefer to induce action $y'_{k,1}$, types in the interval $(\theta(y'_{k,i}, y'_{k,i+1}), \theta(y'_{k,i+1}, y'_{k,i+2}))$ strictly prefer to induce action $y'_{k,i+1}$, for

$i = 1, \ldots, n_{k+1} - 2$, types in the interval $(\theta(y'_{k,n_{k+1}-1}, y'_{k,n_{k+1}}), \theta(y'_{k,n_{k+1}}, x'_k))$ strictly prefer to induce action $y'_{k,n_{k+1}}$, and types in the interval $(\theta(y'_{k,n_{k+1}}, x'_k), x'_k - b)$ strictly prefer to induce action x'_k.

Types $t \in (x'_k - b, x_{k+1} - b)$: The distance between the actions x'_k and $y'_{k+1,1}$ is the same as that between x_{k-1} and $y_{k,1}$: $y'_{k+1,1} - x'_k = y_{k,1} + (x_{k+1} - x_k) - (x_{k+1} - (x_k - x_{k-1})) = y_{k,1} - x_{k-1}$. Therefore, $\theta(x'_k, y'_{k+1,1}) = \theta(x_{k-1}, y_{k,1}) + (x_{k+1} - x_k)$. Similarly, $\theta(y'_{k+1,i}, y'_{k+1,i+1}) = \theta(y_{k,i}, y_{k,i+1}) + (x_{k+1} - x_k)$ for $i = 1, \ldots, n_k - 1$ and $\theta(y'_{k+1,n_k}, x_{k+1}) = \theta(y_{k,n_k}, x_k) + (x_{k+1} - x_k)$. This implies that types in the interval $(x'_k - b, \theta(x'_k, y'_{k+1,1}))$ strictly prefer to induce action x'_k, types in the interval $(\theta(x'_k, y'_{k+1,1}), \theta(y'_{k+1,1}, y'_{k+1,2}))$ strictly prefer to induce action $y'_{k+1,1}$, types in the interval $(\theta(y'_{k+1,i}, y'_{k+1,i+1}), \theta(y'_{k+1,i+1}, y'_{k+1,i+2}))$ strictly prefer to induce action $y'_{k+1,i+1}$, for $i = 1, \ldots, n_k - 2$, types in the interval $(\theta(y'_{k+1,n_k-1}, y'_{k+1,n_k}), \theta(y'_{k+1,n_k}, x'_k))$, strictly prefer to induce action y'_{k+1,n_k}, and types in the interval $(\theta(y'_{k+1,n_k}, x_{k+1}), x_{k+1} - b)$, strictly prefer to induce action x_{k+1}.

Types $t \geq x_{k+1} - b$: Since these types have ideal points that are no less than x_{k+1}, they strictly prefer action x_{k+1} to any lower action in $X' \cup Y'$. The set of actions in $X' \cup Y'$ that are greater than or equal to x_{k+1} is the same as in $X \cup Y$. Therefore all of the types above $x_{k+1} - b$ have an incentive to induce the same actions given $X' \cup Y'$ that they prefer to induce given $X \cup Y$.

Taking action $y \in Y'$ is optimal for the agent given prior beliefs concentrated on $T'(y)$. This follows since, as we have seen, (a) for actions $y \in Y \cap Y'$ the set $T'(y)$ is the set of types who induce y in e^X and (b) for each action $y \in Y' \setminus Y$ the set $T'(y)$ and the action y are translated by the same amount, guaranteeing that y remains the midpoint of $T'(y)$. To specify the interim principal's strategy in $e^{X'}$ (up to a set of types of measure zero) let the behavior of types $t \leq x_{k-1} - b$ and $t \geq x_{k+1} - b$ remain unchanged from e^X when we replace the contract X by X'. Have types in the interval $\big(\theta(y'_{k,i-1}, y'_{k,i}), \theta(y'_{k,i}, y'_{k,i+1})\big)$ send the same message that is sent by types in the interval $(\theta(y_{k+1,i-1}, y_{k+1,i}),\ \theta(y_{k+1,i}, y_{k+1,i+1}))$ in e^X. And, let types in the interval $\big(\theta(y'_{k+1,i-1}, y'_{k+1,i}), \theta(y'_{k+1,i}, y'_{k+1,i+1})\big)$ send the same message that is sent by types in the interval $(\theta(y_{k,i-1}, y_{k,i}), \theta(y_{k,i}, y_{k,i+1}))$ in e^X. To complete the description of $e^{X'}$, have the agent's strategy prescribe to take action y in response to the message sent by types in $T'(y)$ according to the specified interim principal's strategy for all $y \in Y'$, and, for some arbitrary action $\hat{y} \in Y'$, to take that action after all other messages.

Next, we show that the equilibrium $e^{X'}$ in $\Gamma^{X'}$ is payoff equivalent to the equilibrium e^X in Γ^X. Note first that each type $t \in [0,1] \setminus (\mathcal{T}_k \cup \mathcal{T}_{k+1})$ induces the same action in e^X as in $e^{X'}$. Hence for all these types, payoffs do no vary with the two equilibria. Next consider types in $\mathcal{T}_k \cup \mathcal{T}_{k+1}$. For each $a \in X \cup Y$ with $x_{k-1} \leq a \leq x_{k+1}$, define $P(a) := T(a) \cap (\mathcal{T}_k \cup \mathcal{T}_{k+1})$ and let $\mathcal{P}$ be the collection off all these sets. (Note that $P(a) = T(a)$ for $a \neq x_{k-1}, x_{k+1}$.) Similarly, for each $a \in X' \cup Y'$ with $x_{k-1} \leq a \leq x_{k+1}$, define $P'(a) := T'(a) \cap (\mathcal{T}_k \cup \mathcal{T}_{k+1})$ and let $\mathcal{P}'$ be the collection off all these sets. By construction there is a bijection from $\mathcal{P}$ to $\mathcal{P}'$ with the following properties: (a) for each $P(a) \in \mathcal{P}$ the image $P'(a')$ under this bijection satisfies that $P'(a')$ is the Minkowski sum of $P(a)$ and $\{d\}$ and $a' = a + d$ for some $d \in \mathbb{R}$; and, (b) types in $P(a)$ induce action a in equilibrium e^X and types in $P(a')$ induce action a' in equilibrium $e^{X'}$. This implies that for almost all types in $\mathcal{T}_k \cup \mathcal{T}_{k+1}$, payoffs are the same in equilibrium $e^{X'}$ as they are in e^X. Combining the observations about types $t \in [0,1] \setminus (\mathcal{T}_k \cup \mathcal{T}_{k+1})$ and types $t \in \mathcal{T}_k \cup \mathcal{T}_{k+1}$, it follows that expected equilibrium payoffs are the same in equilibrium e^X of Γ^X and equilibrium $e^{X'}$ of $\Gamma^{X'}$. □

Lemma B.8 *For every optimal contract-equilibrium pair (X, e^X) and all inner topics $\mathcal{T}$ and $\mathcal{T}'$ generated by the contract X,*

$$|N(\mathcal{T}) - N(\mathcal{T}')| \leq 1.$$

If $\mathcal{T}_1$ is non-empty, $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy

$$|N(\mathcal{T}_2) - N(\mathcal{T}_1)| \leq 1.$$

If $\mathcal{T}_1$ is empty, $\mathcal{T}_3$ and $\mathcal{T}_2$ satisfy

$$|N(\mathcal{T}_2) - N(\mathcal{T}_3)| \leq 1.$$

Moreover, $\mathcal{T}_K$ and $\mathcal{T}_{K+1}$ satisfy

$$|N(\mathcal{T}_{K+1}) - N(\mathcal{T}_K)| \leq 2.$$

Proof. We first establish the following claim.

Claim 1 *Suppose that $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ are inner topics that satisfy $N(\mathcal{T}_{k+1}) > N(\mathcal{T}_k)+1$. Then the derivative of $\Pi^\xi(N(\mathcal{T}_k), \mathcal{T}_k) + \Pi^\xi(N(\mathcal{T}_{k+1}), \mathcal{T}_{k+1})$ with respect to x_k is strictly positive. And, if $N(\mathcal{T}_{k+1}) + 1 < N(\mathcal{T}_k)$ the derivative is strictly negative.*

By Lemma B.5, the sum of the payoffs in $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ is given by

$$\frac{1}{12(n_k+1)^2}(x_{k-1} - x_k)\left((x_{k-1} - x_k)^2 + 4b^2(1+n_k)^2\left(3 + 2n_k + n_k^2 - 3(1-\xi)\right)\right)$$
$$+\frac{1}{12(n_{k+1}+1)^2}(x_k - x_{k+1})\left((x_k - x_{k+1})^2 + 4b^2(1+n_{k+1})^2\left(3 + 2(n_{k+1}) + n_{k+1}^2 - 3(1-\xi)\right)\right).$$

The derivative of this sum with respect to x_k equals

$$\frac{1}{12}\left(-4b^2 n_k(n_k+2) + 4b^2 n_{k+1}(n_{k+1}+2) - \frac{3(x_{k-1}-x_k)^2}{(1+n_k)^2} + \frac{3(x_k - x_{k+1})^2}{(1+n_{k+1})^2}\right).$$

By Lemma B.4, the assumption that n_k and n_{k+1} attain their maximal feasible values implies

$$\begin{aligned} 2b(n_{k+1}+1)(n_{k+1}+2) \geq\, & x_{k+1} - x_k > 2bn_{k+1}(n_{k+1}+1) \\ 2b(n_k+1)(n_k+2) \geq\, & x_k - x_{k-1} > 2bn_k(n_k+1). \end{aligned}$$

We use these bounds to show that the derivative is positive under the stated condition. To derive a lower bound on the derivative, replace $(x_{k+1}-x_k)$ by its lower bound $2bn_{k+1}(n_{k+1}+1)$ (as it enters positively) and replace $(x_k - x_{k-1})$ by its upper bound $2b(n_k+1)(n_k+2)$ (as it enters negatively) in the derivative. Replacing yields

$$
\begin{aligned}
&\frac{1}{12}\left(-4b^2 n_k(n_k+2) + 4b^2 n_{k+1}(n_{k+1}+2)\right) \\
&\quad - \frac{1}{12}\left(\frac{3(2b(n_k+1)(n_k+2))^2}{(1+n_k)^2} + \frac{3(2bn_{k+1}(n_{k+1}+1))^2}{(1+n_{k+1})^2}\right) \\
=& -\frac{b^2}{3}\left(n_k(n_k+2) - n_{k+1}(n_{k+1}+2) + 3(n_k+2)^2 + 3n_{k+1}^2\right) \\
=& \frac{2}{3}b^2(2n_{k+1} - 2n_k - 3)(2 + n_k + n_{k+1}),
\end{aligned}
$$

The derivative is thus strictly positive for $n_{k+1} > n_k + \frac{3}{2}$. Since n_{k+1} and n_k are integers, this holds if $n_{k+1} > n_k + 1$.

Similarly, to show that the derivative is negative under the stated condition, we derive an upper bound on the derivative. Replacing $(x_{k+1}-x_k)$ by its upper bound $2b(n_{k+1}+1)(n_{k+1}+2)$ (as this enters positively) and replacing $(x_k - x_{k-1})$ by its lower bound $2bn_k(n_k+1)$ (as this enters negatively) in the derivative yields

$$
\begin{aligned}
&\frac{1}{12}\left(-4b^2 n_k(n_k+2) + 4b^2 n_{k+1}(n_{k+1}+2)\right) \\
&\quad - \frac{1}{12}\left(\frac{3(2bn_k(n_k+1))^2}{(1+n_k)^2} + \frac{3(2b(n_{k+1}+1)(n_{k+1}+2))^2}{(1+n_{k+1})^2}\right) \\
=& -\frac{2}{3}b^2(2n_k - 2n_{k+1} - 3)(2 + n_k + n_{k+1}).
\end{aligned}
$$

The derivative is thus strictly negative for $n_{k+1} + \frac{3}{2} < n_k$. Since n_{k+1} and n_k are integers, this holds if $n_{k+1} + 1 < n_k$. This establishes the claim (analogous claims hold for $k = 1, 2$ and $k = K+1$).

In order to derive a contradiction to the statement of the lemma, suppose that for some k and some $s \geq 2$, $\mathcal{T}_k$ and $\mathcal{T}_{k+s}$ are inner topics with $|N(\mathcal{T}_k) - N(\mathcal{T}_{k+s})| > 1$ for some optimal contract-equilibrium pair (X, e^X). Then, by Lemma B.7, there exists a payoff-equivalent contract-equilibrium pair $(X', e^{X'})$ for which the contract X' switches the topics $\mathcal{T}_{k+s-1}$ and $\mathcal{T}_{k+s}$ of the contract X. By, if necessary, repeatedly applying this argument we can conclude that there exists a payoff equivalent contract-equilibrium pair $(\widetilde{X}, e^{\widetilde{X}})$ with adjacent inner topics $\widetilde{\mathcal{T}}_k$ and $\widetilde{\mathcal{T}}_{k+1}$ such that $|N(\widetilde{\mathcal{T}}_k) - N(\widetilde{\mathcal{T}}_{k+1})| > 1$. This, however, is ruled out by Claim 1 for optimal contract-equilibrium pairs. □

Lemma B.9 *For every optimal contract-equilibrium pair (X, e^X) with $n_2 > 1$, $x_1 \geq b$.*

Proof. We establish the claim by showing that the derivative of $\Pi^{\xi}(n_2, \mathcal{T}_2)$ with respect to x_1 is strictly positive for $n_2 > 1$, if $x_1 < b$.

Assume that $x_1 < b$. The derivative of $\Pi^{\xi}(n_2, \mathcal{T}_2)$ with respect to x_1 is given by

$$\frac{1}{(12(1+n)^2)}(-3(3+8n+4n^2)x_1^2-6x_1x_2+3x_2^2+4b^2(1+n)^2(3+2n+n^2-6(1-\xi))+24b(1+n)^2x_1(1-\xi)).$$

We use the following bounds to show that the derivative is strictly positive under the stated condition.

$$\begin{aligned} -b \leq x_1 < b \quad &\text{(by Lemma B.2 and by assumption)} \\ -b + 2bn(n+1) < x_2 < b + 2b(n+1)(n+2) \quad &\text{(by Lemmas B.2 and B.4)} \\ 0 \leq \ (1-\xi) \ \leq 1 \quad &\text{(by assumption)} \end{aligned}$$

A lower bound on the derivative is given by

$$\begin{aligned} &\frac{1}{(12(1+n)^2)}[-3(3+8n+4n^2)b^2 - 6b(b+2b(n+1)(n+2)) + 3(-b+2bn(n+1))^2 \\ &\qquad + 4b^2(1+n)^2(3+2n+n^2-6\cdot 1) + 24b(1+n)^2(-b)\cdot 1] \\ &= \frac{1}{(12(1+n)^2)}[8b^2(1+n)^2(2n^2+n-9)]. \end{aligned}$$

This is strictly positive for $n > 1$. □

Proof of Proposition 3. Suppose that (for arbitrary topics) $|N(\mathcal{T}) - N(\mathcal{T}')| > 4$. Then, by Lemma B.8, $\mathcal{T}$ and $\mathcal{T}'$ cannot both be inner topics. By Lemma B.8, we have $|N(\mathcal{T}_{K+1}) - N(\mathcal{T}_K)| \leq 2$. Combining this fact with our observation that $\left|N(\tilde{\mathcal{T}}) - N(\tilde{\mathcal{T}}')\right| \leq 1$ for any two inner topics and repeatedly applying Lemma B.7 rules out that one of the two topics $\mathcal{T}$ and $\mathcal{T}'$ is an inner topic and the other is $\mathcal{T}_{K+1}$.

Suppose that $x_1 \geq b$. Then all topics $\mathcal{T}_k$ with $k = 2, \ldots, K$ are inner topics. Furthermore, by Lemma B.8 we have that $|N(\mathcal{T}_2) - N(\mathcal{T}_1)| \leq 1$. This rules out the topics $\mathcal{T}$ and $\mathcal{T}'$ are $\mathcal{T}_1$ and $\mathcal{T}_2$. Combining the fact $|N(\mathcal{T}_2) - N(\mathcal{T}_1)| \leq 1$ with the observation that for all inner topics $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}}'$ we have $\left|N(\tilde{\mathcal{T}}) - N(\tilde{\mathcal{T}}')\right| \leq 1$ and repeatedly applying Lemma B.7, implies that $|N(\mathcal{T}) - N(\mathcal{T}')| \leq 1$ whenever one of the topics $\mathcal{T}$ and $\mathcal{T}'$ is an inner topic and the other is $\mathcal{T}_1$. A second implication, using the fact that $|N(\mathcal{T}_{K+1}) - N(\mathcal{T}_K)| \leq 2$, is that for all topics $\mathcal{T}$ and $\mathcal{T}'$, we have $|N(\mathcal{T}) - N(\mathcal{T}')| \leq 3$.

Suppose that $x_1 < b$. Then all topics $\mathcal{T}_k$ with $k = 3, \ldots, K$ are inner topics.

We can divide this case into two subcases: Either $\mathcal{T}_2$ has two or more communication actions, or it has either no or only one communication action.

If there are two or more communication actions in $\mathcal{T}_2$, then by Lemma B.8 we get a contradiction to the assumption defining this subcase, that $x_1 < b$. This takes us back to the case with $x_1 \geq b$ for which we established that for all topics $\mathcal{T}$ and $\mathcal{T}'$, we have $|N(\mathcal{T}) - N(\mathcal{T}')| \leq 3$.

If there are fewer than two communication actions in $\mathcal{T}_2$, then, remembering that $x_1 < b$ the topic $\mathcal{T}_1$ is empty, we have $|N(\mathcal{T}_1) - N(\mathcal{T}_2)| \leq 1$. Furthermore, by Lemma B.8, we have that $|N(\mathcal{T}_2) - N(\mathcal{T}_3)| \leq 1$; $|N(\mathcal{T}_{K+1}) - N(\mathcal{T}_K)| \leq 2$; and we have $\left|N(\tilde{\mathcal{T}}) - N(\tilde{\mathcal{T}}')\right| \leq 1$ for any two inner topics $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}}'$; and, using Lemma B.7 we can repeated exchange any to adjacent inner topics. The combination of these facts implies that for all topics $\mathcal{T}$ and $\mathcal{T}'$, we have $|N(\mathcal{T}) - N(\mathcal{T}')| \leq 4$. $\square$

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