ECE-GY 6263 Game Theory	Fall 2019
Lecture 10 — Nov 15, 2019	
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1 Overview

In the last lecture we talked about Bayesian games and gave examples on auction problems.

In this lecture we talked about the Bayesian Nash Equilibrium in an auction game, the definition of Revenue Equivalence Theorem and the Revenue-Optimal Mechanism Design.

2 An Auction

In an auction, we assume that:

1. The bidders use the same strategy, which means $\mu_i(\theta_i) = \mu(\theta_i)$.
2. In the first price auction game, the utility function is:

$$u_1(b_1, b_2, ..., b_N, \theta_1) = \begin{cases} \theta_i - b_i & if\ b_1 > max(b_2, \ldots, b_N) \\ 0 & otherwise \end{cases}$$

   using the First Order Function, we can get

$$b_1 = \int_0^{\theta} \frac{\theta f_Y d\theta}{F_Y(\theta_1)} = E(Y|Y \leq \theta_1)$$

$$Y = max(\theta_1, \ldots \theta_N)$$

3. In the second-price auction, the optimal choice for bidder i is to bid the real value. $b_1 = \mu_s(\theta_1) = \theta_1$

Now we assume that μ^* is a Bayesian Nash Equilibrium, the expected utility function for player 1 is:

$$\begin{aligned} &E(u_1(b1, \mu^*(\theta_2), \ldots \mu^*(\theta_N), \theta_1)|\theta_1) \\ &= (\theta_1 - b_1)F_Y(\mu^{*-1}(b_1)) \end{aligned}$$

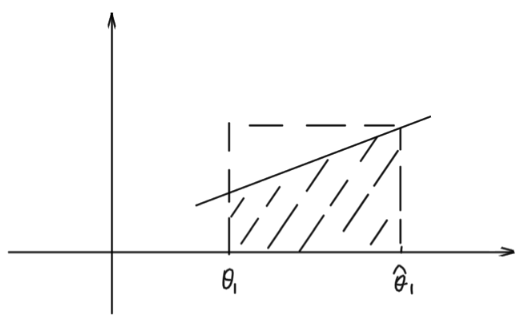
we choose $\hat{b}$ to denote the bid that deviate the BNE $\mu^*(\theta_1)$

$$E(u_1(b1, \mu^*(\theta_2), \ldots \mu^*(\theta_N), \theta_1)|\theta_1) = (\theta_1 - \hat{b_1})F_Y(\mu^{*-1}(\hat{b_1}))$$

we use $\hat{\theta}$ to denote $\mu^{*-1}(\hat{b_1})$ and $\hat{b_1}$ to denote $\mu^*(\hat{\theta_1})$ the former function now can be shown as:

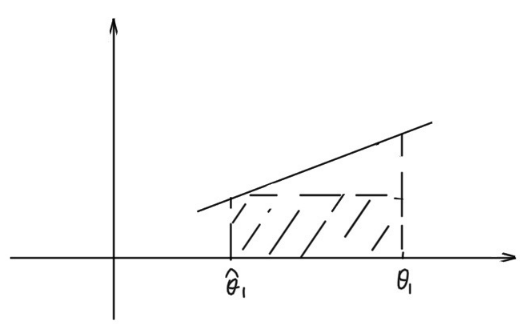
$$\begin{aligned}
&(\theta_1 - \mu^*(\hat{\theta_1}))F_Y(\hat{\theta_1}) \\
&= \theta_1 F_Y(\hat{\theta_1}) - \mu^*\hat{\theta}_1 F_Y(\hat{\theta_1}) \\
&= \theta_1 F_Y(\hat{\theta_1}) - \int_0^{\hat{\theta_1}} \theta f_Y(\theta) d\theta \\
&= \theta_1 F_Y(\hat{\theta_1}) - \theta F_Y(\theta)\Big|_0^{\hat{\theta_1}} + \int_0^{\hat{\theta_1}} F_Y(\theta) d\theta \\
&= \theta_1 F_Y(\hat{\theta_1}) - \hat{\theta_1} F_Y(\hat{\theta_1}) + \int_0^{\hat{\theta_1}} F_Y(\theta) d\theta \\
&= (\theta_1 - \hat{\theta_1}) F_Y(\hat{\theta_1}) + \int_0^{\hat{\theta_1}} F_Y(\theta) d\theta \\
&= (\theta_1 - \hat{\theta_1}) F_Y(\hat{\theta_1}) + \int_0^{\theta_1} F_Y(\theta) d\theta + \int_{\theta_1}^{\hat{\theta_1}} F_Y(\theta) d\theta
\end{aligned}$$

1. $0 \leq \theta_1 \leq \hat{\theta_1}$.



In this case, we assume that $0 \leq \theta_1 \leq \hat{\theta_1}$, then $(\theta_1 - \hat{\theta_1})F_Y(\hat{\theta_1}) + \int_{\theta_1}^{\hat{\theta_1}} F_Y(\theta)d\theta < 0$ so the function $(\theta_1 - \hat{\theta_1})F_Y(\hat{\theta_1}) + \int_0^{\theta_1} F_Y(\theta)d\theta + \int_{\theta_1}^{\hat{\theta_1}} F_Y(\theta)d\theta < \int_0^{\theta_1} F_Y(\theta)d\theta$

2. $0 \leq \hat{\theta_1} \leq \theta_1$.



In this case the rectangle part represents $(\theta_1 - \hat{\theta_1})F_Y(\hat{\theta_1})$. We can see that $(\theta_1 - \hat{\theta_1})F_Y(\hat{\theta_1}) + \int_{\theta_1}^{\hat{\theta_1}} F_Y(\theta)d\theta < 0$ and $(\theta_1 - \hat{\theta_1})F_Y(\hat{\theta_1}) + \int_0^{\theta_1} F_Y(\theta)d\theta + \int_{\theta_1}^{\hat{\theta_1}} F_Y(\theta)d\theta < \int_0^{\theta_1} F_Y(\theta)d\theta$

To maximize $(\theta_1 - \mu^*(\hat{\theta_1}))F_Y(\hat{\theta_1})$, form the above analysis, $\hat{\theta_1} = \theta_1$, so μ^* is BNE.

3 Revenue Equivalence Theorem

Example: From the designer's view. Take Ebay as an example, the expected payment of a bidder is his Probability of winning * Amount of his bid, which could be written as $F_Y(\theta_1)\mu^*(\theta_1)$.

$$
\begin{aligned}
& F_Y(\theta_1)\mu^*(\theta_1) \\
&= F_Y(\theta_1) \int_0^{\theta_1} \frac{\theta f_Y(\theta)d\theta}{F_Y(\theta_1)} \\
&= \int_0^{\theta_1} \theta f_Y(\theta)d\theta \\
\Rightarrow & \int_{\theta_1=0}^{\theta_1=\theta_{max}} [\int_0^{\theta_1} \theta f_Y(\theta)d\theta] p(\theta_1)d\theta_1 \\
&= \int_{\theta_1=0}^{\theta_1=\theta_{max}} \int_0^{\theta_1} \theta f_Y(\theta)p(\theta_1)d\theta d\theta_1 \\
&= \int_{y=0}^{\theta} max[\int_{\theta_1=y}^{\theta_{max}} p(\theta_1)d\theta_1] y f_Y(y)dy \\
&= \int_0^{\theta_{max}} [F_\theta(\theta_m ax) - F_\theta(y)] y f_Y(y)dy \\
&= \int_0^{\theta_{max}} (1 - F_\theta(y)) y f_Y(y)dy
\end{aligned}
$$

The expected revenue under the first price auction to the seller is:

$$
\int_0^{\theta_{max}} yN(1 - F_\theta(y)) f_Y(y)dy
$$

The revenue Equivalence Theorem claims that the seller will gain the same revenue even if the auction strategy is different under the constraints below:

1. The bidder's valuations are independent and identically distributed.
2. They are of the symmetric Bayesian Nash Equilibrium. $\mu_i = \mu$ for all $i \in N$.
3. If $\theta_i = 0$, the expected payment of the bidder is 0.
4. The object goes to the higher bidder.

As we know:

$$u_1 = \begin{cases} \theta_1 - \hat{b_1} & if\ \hat{b_1} > max(\mu(\theta_2), \dots \mu(\theta_N)) \\ 0 & otherwise \end{cases}$$

The expected payoff of the bidder is $E(\theta_1)$ under the condition that $\hat{b_1} > max(\mu(\theta_2), \dots \mu(\theta_N))$. We use $\hat{b_1}$ to denote $\mu(\hat{\theta_1})$ and $\hat{\theta_1}$ to denote $\mu^{-1}(\hat{b_1})$ the expected payoff of the bidder could be shown as $\theta_1 F_Y(\hat{\theta_1}) - a(\hat{\theta_1})$ where a() is the payment rule.

$$\frac{d}{d\hat{\theta_1}}[E(payoff\ to\ player1|\theta_1)] = 0\ when\ \hat{\theta}_1 = \theta_1$$

$$\theta_1 f_Y(\theta_1) - a'(\theta_1) = 0$$

$$\begin{aligned} a(\theta_1) &= \int_0^{\theta_1} \theta f_Y(\theta) d\theta + C \\ &= F_Y(\theta_1) \int_0^{\theta_1} \frac{\theta f_Y(\theta) d\theta}{F_Y(\theta_1)} \\ &= F_Y(\theta_1) E(Y|Y \leq \theta_1) \end{aligned}$$

4 Revenue Optimal Mechanism Design

The auction could be seen as follows:

1. One seller is interested in selling an object.
2. N buyers.
3. Θ_i is the valuation of player i. Θ_i is independent and has support $V_i = [0, \theta_{i,max}] \in R$
4. Mechanism is a set of rules amended by the seller. A bidding strategy μ_i is map from V_i to B_i.
   - Allocation rule. $\Pi_i(b_i, b_{-i})$ is the probability for player i to win, $\Pi_i > 0$ and $\sum_{i=1}^{N} \Pi_i \leq 1$, there is probability that no one wins.
   - Payment rule. If a bidder i wins the object $q_i(b_i, b_{-i})$.

Each player submit a bid $b_i \in B_i$ which could be different from V_i. The Mechanism Design Problem is to maximize the revenue.

Definition (Revelation Principle): The seller can achieve its maximum revenue by choosing $B_i = V_i$ and by imposing the condition that each bidder's best response to the mechanism is to truthfully reveal its type.

$$\alpha_i(\theta_i) = E_{\theta_{-i}}(\pi_i(\theta_i, \theta_{-i})|\theta_i)$$
$$m_i(\theta_i) = E_{\theta_{-i}}(q_i(\theta_i, \theta_{-i})\pi_i(\theta_i, \theta_{-i})|\theta_i)$$

The player i's payoff is

$$\theta_i \alpha_i(\theta_i) - m_i(\theta_i)$$

The incentive compatibility is

$$\theta_i\alpha_i(\theta_i) - m_i(\theta_i) \geq \theta_i\alpha_i(\hat{\theta}_i) - m_i(\hat{\theta}_i)\ \forall\theta_i, \hat{\theta}_i \in V_i$$
$$\theta_i\alpha_i(\theta_i) - m_i(\theta_i) \geq 0$$

RM-Mechanism Design Problem is to $maximize \sum_{i=1}^{N} E(m_i(\theta_i))$

Prop: (IC) is equivalent to the following constraints:

i. m_i, α_i satisfy $m_i(\theta_i) = m_i(\theta) + \theta_i\alpha_i(\theta_i) - \int_o^{\theta_i} \alpha_i(\theta)d\theta\ \forall i.$

ii. α_i is a non-decreasing function.

Proof: (IC) $\Rightarrow$ i+ ii

$$u_i(\tilde{\theta}_i) = \theta_i\alpha(\tilde{\theta}_i) - m_i(\tilde{\theta}_i)$$
$$\frac{d}{d\tilde{\theta}_i}\theta_i\alpha(\tilde{\theta}_i) - m_i(\tilde{\theta}_i) = 0\ when\ \tilde{\theta}_i = \theta_i$$
$$\theta_i\alpha_i'(\theta_i) - m_i'(\theta_i) = 0$$
$$m_i'(\theta_i) = \theta_i\alpha_i'(\theta_i)$$

$$\begin{aligned} m_i(\theta_i) &= m_i(0) + \int_0^{z_i} z\alpha_i'(z)dz \\ &= m_i(0) + [\, z\alpha_i(z)\Big|_0^{\theta_1} - \int_0^{\theta_i} \alpha_i(z)dz] \\ &= m_i(0) + \theta_i\alpha_i(\theta_i) - \int_0^{\theta_i} \alpha_i(z)dz \end{aligned}$$

To show α_i is a non-decreasing function $\tilde{\theta}_i, \hat{\theta}_i\forall V_i$

i $\tilde{\theta}_i\alpha_i - m_i(\tilde{\theta}_i) \geq \tilde{\theta}_i\alpha_i(\hat{\theta}_i - m_i(\hat{\theta}_i))$.

ii $\hat{\theta}_i\alpha_i - m_i(\hat{\theta}_i) \geq \hat{\theta}_i\alpha_i(\tilde{\theta}_i - m_i(\tilde{\theta}_i))$.

i+ ii:

$$(\tilde{\theta}_i - \hat{\theta}_i)\alpha_i(\tilde{\theta}_i) + (\hat{\theta}_i - \tilde{\theta}_i)\alpha_i(\hat{\theta}_i) \geq 0$$

$$(\tilde{\theta}_i - \hat{\theta}_i)(\alpha_i(\tilde{\theta}_i) - \alpha_i(\hat{\theta}_i)) \geq 0$$

$$If\ \tilde{\theta}_i \geq \hat{\theta}_i, \quad \alpha_i(\tilde{\theta}_i) \geq \alpha_i(\theta_i)$$
$$If\ \tilde{\theta}_i \leq \hat{\theta}_i, \quad \alpha_i(\tilde{\theta}_i) \leq \alpha_i(\theta_i)$$

References

[1] Gibbons R. A primer in game theory[J]. pp. 154-167, 1992.