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1 Overview

In the last lecture we introduced some basic games, i.e., the Prisoner Dilemma and the Battle of Sex.

In this lecture we formalized finite two-person zero-sum games, introduced pure strategy, mixed strategy, what is nash equilibrium under pure strategy and mixed strategy setting, and how to compute nash equilibrium.

2 Non-cooperative Games

Non-cooperative games have 3 basic elements:

(i) Players:
Q: who are decision makers?
E.g., people, firms, organizations, governments, and so on.

(ii) Actions:
Q1: what can players do?
E.g., enter a bid in an auction, decide when to sell a stock, and so on.
Q2: what is the timing of actions? are actions taken simultaneously or sequentially?
The order of actions is important.

(iii) Payoffs: a function of actions.
Q: what motivates players? What are the payoffs to the various players as a result of the actions?

There exist two different formulations for such games:

(i) **Normal form** (a.k.a matrix form, strategic form) lists what payoffs players get as a function of their actions. This form constitutes a suitable representation of a zero-sum game when each player's information is static in nature, since it suppresses all the dynamic aspects of the decision problem.

Ex: finite n-player normal form game $< N, A, u >$:

Player set: $N = \{1, 2, \ldots, n\}$.

Action set A_i for player i: $a = (a_1, a_2, \ldots, a_n) \in A = A_1 \times A_2 \times \cdots \times A_n$ is an action profile.

Utility function (or payoff function) for player i: $u_i : A \mapsto \mathbb{R}$, $u = (u_1, u_2, \ldots, u_n)$ is a profile of utility.

(ii) **Extensive form (tree)** includes timing of moves. Players move sequentially. This form displays explicitly the evolution of the game and the existing information exchanges between the players.

3 Finite Two-person Zero-sum Games

Definition (Zero-sum game). A zero-sum game is one in which the sum of the individual payoffs for each outcome sum to zero.

Games of pure competition is a class of games where players have exactly opposed interests.

- must be precisely 2 players.
- for all action profile $a \in A$, $u_1(a) + u_2(a) = c$ for some constant c.
  Zero-sum game is a special case when c = 0.
- thus, we only need to store a utility function for one player.

So, **finite two-person zero-sum games** can be defined as:

- Player set: $N = \{1, 2\}$.
- Action set A_i for player i:
  $a = (a_1, a_2) \in A = A_1 \times A_2$, A_1 and A_2 are finite set.
- Utility function (or payoff function) for player i:
  $u_i : A \mapsto \mathbb{R}$
  $u = (u_1, u_2)$ such that $u_1(a) + u_2(a) = 0$ for all $a \in A$

3.1 Matrix Games

The most elementary types of 2-person zero-sum games are **matrix games**. There are two players, P1 (row player) and P2 (column player), and an $m \times n$-dimensional matrix $A = \{a_{i,j}\}$. Each entry $a_{i,j}$ of this matrix A is an outcome of the game, corresponding to P1 pick action i (i_{th} row) and P2 picks action j (j_{th} column). If $a_{i,j}$ is positive, P1 pays $|a_{i,j}|$ toP2; if $a_{i,j}$ is negative, P2 pays $|a_{i,j}|$ to P1. Then, regarded as a rational decision maker, Pl will seek to minimize the outcome of the game, while P2 will seek to maximize it, by independent decisions.

3.1.1 Security Strategy

P1's strategy of playing this game is to secure his losses against any behavior of P2. Hence, P1 picks i^* row which the largest loss is the smallest among all rows, that is the largest entry that is smaller than the largest entry of any other row.

$$\overline{V} \triangleq \max_j a_{i^*,j} \leq \max_j a_{i,j}, i = 1, 2, \ldots, m \tag{1}$$

where $\overline{V}$ is called the *loss ceiling* of P1 (or *upper value of the game*, *security level* of P1), the strategy "row i^*" is called a *security strategy* for P1.

Similarly, P2 will choose to secure his gains against any behavior of P1. Hence, P2 picks column j^* which smallest entry is greater than the smallest entry of any other row.

$$\underline{V} \triangleq \min_i a_{i,j^*} \geq \min_i a_{i,j}, j = 1, 2, \ldots, n \tag{2}$$

where $\underline{V}$ is called the *gain floor* of P2 (or the *lower value of the game*, *security level* of P2). the strategy "column j^*" is called a *security strategy* for P2.

Theorem: In every matrix game $A = \{a_{i,j}\}$:

(i) the security level of each player is unique,

(ii) there exists at least one security strategy for each player,

(iii) the security level of P1 (the minimizer) never falls below the security level of P2 (the maximizer), i.e.,

$$\min_i a_{i,j^*} = \underline{V} \leq \overline{V} = \max_j a_{i^*,j} \tag{3}$$

where i^* and j^* denote security strategies for Pl and P2, respectively.

Proof: (i) and (ii) are obvious from (1) and (2). To prove (iii), note that $\min_i a_{i,l} \leq a_{k,l} \leq \max_j a_{k,j}$. Let $k = i^*$ and $l = j^*$, then $\underline{V} = \min_i a_{i,j^*} \leq a_{i^*,j^*} \leq \max_j a_{i^*,j} = \overline{V}$. This complete the proof for (iii). And it implies that the outcome of the game will always lie between $\underline{V}$ and $\overline{V}$ if played by rational players.

3.1.2 Saddle Point Equilibrium

Suppose now that P2 is required to announce his choice before P1 makes his choice. In this case, P2 has an advantage over P1. Then, the best play of P1 is to choose one of his security strategy (say row i^*) to security the loss ceiling. Therefore, P2's "optimal" choice (column j^o) would satisfiy

$$a_{i^*,j^o} = \max_j a_{i*,j} = \overline{V} = \min_i \max_j a_{i,j} \tag{4}$$

with "min max" operation designating the order of play in this decision process, which is P1 minimizes his loss ceiling then P2 maximizes his gain. Equation (4) implies that the outcome of the game is equal to the upper value $\overline{V}$, when P2 observes P1's choice. So *minimax strategy* is to minimize one's own maximum loss.

Now, we exchange the order of two players, then P1 has an advantage over P2. Then, the best play of P2 is to choose one of his security strategy (say column j^*) to security the gain floor. Therefore, P1's "optimal" choice (column i^o) would satisfy

$$a_{i^o,j^*} = \max_i a_{i,j^*} = \underline{V} = \max_j \min_i a_{i,j} \tag{5}$$

with "max min" symbol implying that the minimizer acts after the maximizer. Equation (5) indicates that the outcome of the game is equal to the lower value $\underline{V}$, when Pl observes P2's choice. So *maximin strategy* is to maximize one's own minimum gain.

Definition: The pair (i^*, j^*) is a **saddle point equilibrium (SPE)** if $a_{i^*,j} \leq a_{i^*,j^*} \leq a_{i,j^*}$ is satisfied for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Theorem: Let $A = a_{i,j}$ denote an $m \times n$ matrix game with $\underline{V}(A) = \overline{V}(A)$. Then,

(i) A has a saddle point in pure strategy,

(ii) an ordered pair of strategies (i, j) provides a saddle point for A iff i is a security strategy for Pl and j is a security strategy for P2,

(iii) V(A) is uniquely given by $V(A) = \underline{V}(A) = \overline{V}(A)$.

Proof: Let i^* denote a security strategy for Pl and j^* denote a security strategy for P2, which always exist by Thm. 2.1 (ii). Now, since $\underline{V}(A) = \overline{V}(A)$, we have

$$a_{i^*,j} \leq \max_j a_{i^*,j} = \overline{V} = \underline{V} = \min_i a_{i,j^*} \leq a_{i,j^*}$$

$$for\ all\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$$

Let $i = i^*$, $j = j^*$ in this inequality, we get

$$a_{i^*,j^*} = \overline{V} = \underline{V} = a_{i^*,j^*}$$

So, $a_{i^*,j} \leq a_{i^*,j^*} \leq a_{i,j^*}$. This complete (i) and the sufficiency part of (ii). Let row i^*, column j^* be a saddle-point strategy pair. Then,

$$\max_j a_{i^*,j} \leq a_{i,j^*} \leq max_j a_{i,j},\ i = 1, 2, \ldots, m$$

Thus, row i^* is a security strategy for P1. Similarly, we can prove column j^* is a security strategy for P2. This completes the proof of (ii). (iii) is obvious after proving (ii).

So, in a matrix game, the saddle point $a_{i,j}$ is some entry $a_{i,j}$ of the matrix A that has the property that (1) $a_{i,j}$ is the maximum of the i_{th} row and (2) $a_{i,j}$ is the minimum of the j_{th} column. By choosing row i, Player 1 can then keep her loss at most $a_{i,j}$, and Player 2 can win at least $a_{i,j}$ by choosing column j.

Note that existence of saddle point strategy in pure strategy is not necessary. For the case when the security levels of the players do not coincide no such equilibrium solution can be found within the class of (pure) strategies. Ex. matching point.

3.1.3 Mixed Strategy and Saddle Point Equilibrium

Definition (Mixed strategy): A **mixed strategy** for a player is a probability distribution on the space of his pure strategies.

We denote the strategies of Player 1 and Player 2 as $p \in \triangle_1$ and $p \in \triangle_2$ respectively, where $\triangle_1$ and $\triangle_2$ are some probability distribution sets,
$\triangle_1 = \{p = (p_1, \ldots, p_m)^T : p_i \geq 0,\ for\ i = 1, \ldots, m\ and\ \sum_{i=1}^{m} p_i = 1\}$
$\triangle_2 = \{q = (q_1, \ldots, q_n)^T : q_i \geq 0,\ for\ i = 1, \ldots, m\ and\ \sum_{i=1}^{m} q_i = 1\}$
Note that pure strategy is a special case of mixed strategy. For mixed strategy, we consider expected utility,
$J(p, q) = p^T A q$
Player 1 would like to minimize the average payoff $J(p, q)$, by choosing a probability distribution vector $p \in \triangle_1$ properly, while Player 2 wishes to maximize $J(p, q)$, by choosing a probability distribution vector $q \in \triangle_2$ properly.

Definition: A probability distribution vector $p^* \in \triangle_1$ is called a **mixed security strategy** for P1, in the matrix game A, if the following inequality holds for all $p \in \triangle_1$:

$$\overline{V}_m(A) \triangleq max_{q \in \triangle_2} p^{*T} A q \leq max_{q \in \triangle_2} p^T A q,\ for\ all\ p \in \triangle_1 \tag{6}$$

Here, $\overline{V}_m(A)$ is the **average security level** of P1 (or **the average upper level of the game**). Analogously, a probability distribution vector $q^* \in \triangle_2$ is called a **mixed security strategy** for P2, in the matrix game A, if the following inequality holds for all $1 \in \triangle_2$:

$$\underline{V}_m(A) \triangleq min_{p \in \triangle_1} p A q^* \geq max_{p \in \triangle_1} p^T A q,\ for\ all\ q \in \triangle_2 \tag{7}$$

Here, $\underline{V}_m(A)$ is the **average security level** of P2 (or **the average lower level of the game**).

Definition: A pair of strategies (p^*, q^*) is said to constitute a **saddle point** in mixed strategies, if

$$p^{*T} A q \leq p^{*T} A q^* \leq p^T A q^*,\ for\ all\ p \in \triangle_1,\ q \in \triangle_2 \tag{8}$$

Here $V_m(A) = p^{*T} A q^*$ is known as the **saddle-point value**, or the **value of the game**, in mixed strategies.

Theorem: In mixed strategy matrix game A, the following properties hold:

(i) The average security level of each player is unique.

(ii) There exists at least one mixed security for each player.

(iii) The security level in pure and mixed strategies satisfy the following inequalities:

$$\underline{V}(A) \leq \underline{V}_m(A) \leq \overline{V}_m(A) \leq \overline{V}(A) \tag{9}$$

The middle inequality in (iii) is because the maximizer's security level is lower than the minimizer's security level. The other two inequalities follows from the fact that pure secure strategies of the players are included in their mixed strategy space.

Corollary: In a mixed strategy matrix game, the upper value and the lower value of the game is given, respectively, by

$$\overline{V}_m(A) = \min_p \max_q p^T Aq \tag{10}$$

$$\underline{V}_m(A) = \max_q \min_p p^T Aq \tag{11}$$

Theorem (The Minimax Theorem): In any matrix game A, the average security levels of the palyers in mixed strategies coincides, that is

$$\overline{V}_m(A) = \min_p \max_q p^T Aq = \max_q \min_p p^T Aq = \underline{V}_m(A) \tag{12}$$

This theorem is simply saying that every finite game has a value, and both players have minimax strategies. This theorem states the existence of saddle-point equilibrium, but it does not guarantee the uniqueness of saddle-point equilibrium.

Lemma (Invariance of minimax strategies) If $A = \{a_{i,j}\}$ and $A' = \{a'_{i,j}\}$ are matrices with $a'_{i,j} = ca_{i,j} + b$, where $c > 0$, then the matrix game A has the same minimax strategies for Player 1 and Player 2 as the game with matrix A'. Also, if V denotes the value of the matrix game A, then $V(A') = cV(A) + b$.

4 Computing SPE

4.1 Graphical Methods

From previous section, a possible way to obtain the mixed saddle-point solution of a matrix game is to determine the mixed security strategies for each player.

Example: A 2×2 matrix game:

P1 \ P2	A	B
A	3	0
B	-1	1

which has no SPE in pure strategy, since $\overline{V} = 1$ and $\underline{V} = 0$. Let $p = (y_1, y_2)' = (y_1, 1 - y_1)'$ and $q = (z_1, z_2)' = (z_1, 1 - z_1)'$ denote P1's and P2's strategies respectively.

Let's consider the average security level for P1 first and fix P2's choice. If P2 chooses A, $J = 3y_1 - (1 - y_1)$. If P2 chooses B, $J = 1 - y_1$. The average outcome of the game is shown in Fig. **??**. The bold line forms the upper envelope. The mixed strategy $(y_1^* = 2/5, y_2^* = 3/5)$ corresponding to the lowest point of the upper envelope is P1's security strategy. By playing strategy of $(y_1^* = 2/5, y_2^* = 3/5)$, P1 can secure his worst loss no greater than 0.6, while P1 might loss more than 0.6 if he choose the other other point.

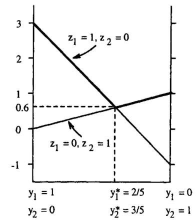

Figure 1: Mixed security strategy of P1 (directly cropped from [BO])

Now, let's consider P2's security strategy and fix P1's choice. If P1 chooses A, $J = 3z_1$. If P1 chooses B, $J = -z_1 + 1 - z_1$. The average outcome of the game is shown in Fig. **??**. The bold line forms the lower envelope. The mixed strategy $(z_1^* = 1/5, z_2^* = 4/5)$ corresponding to the highest point of the lower envelope is P2's security strategy. By playing strategy of $(z_1^* = 1/5, z_2^* = 4/5)$, P2 can secure his gain at least 0.6, while P2 might win less than 0.6 if he choose the other other point.

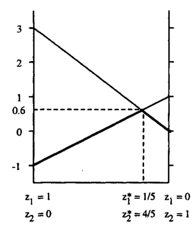

Figure 2: Mixed security strategy of P2 (directly cropped from [BO])

So the SPE for this matrix game is (2/5, 3/5), (1/5, 4/5) and the value of the game is 0.6.

The computational technique discussed above is called the "graphical solution" of matrix games. The graphical method is practical not only for 2×2 matrix games but also for general $2 \times n$ and $m \times 2$ matrix games. Graphical methods become less practical as m or n increases.

4.2 Linear Programming

An alternative to graphical solution when matrix dimensions are high is to convert the matrix game into a linear programming (LP) problem.

Assume a $m \times n$ matrix game $A = \{a_{i,j}\}$, $a_{i,j} > 0$ for all i, j. Then , the average value of the game in mixed strategies is

$$V_m(A) = \min_p \max_q p^T A q = \max_q \min_p p^T A q \tag{13}$$

Here, the first expression $V_m(A) = \min_p \max_q p^T Aq$ can be viewed as first given a $p \in \triangle_1$, then maximizing $p^T Aq$ over $q \in \triangle_2$. So the choice of q depends on p. Therefore, we can rewrite the "min max" expression as

$$\min_p V_1(p) \tag{14}$$

where $V_1(p)$ is a positive scalar valued function of p

$$V_1(p) = \max_q p^T Aq \geq p^T Aq \quad \forall q \in \triangle_2 \tag{15}$$

Pick $q = (1, 0, \ldots, 0)^T$, we can get $V_1(p)1_n \geq A^T p$, where $1_n = (1, \ldots, 1)^T \in \mathbb{R}^n$
Further introducing the notation $\widetilde{p} = p/V_1(p)$ and recalling that the definition of p ($\sum p_i = 1, p_i > 0$). Then, the optimization problem faced by P1 in determining his mixed strategy becomes

$$\begin{aligned} &\min_p V_1(p) \\ s.t.\ A^T\widetilde{p} &\leq 1_n \\ \widetilde{p} &= p/V_1(p) \\ \widetilde{p} &\geq 0 \end{aligned}$$

This is equivalent to the maximization problem

$$\begin{aligned} &\max \widetilde{p}1_m \\ s.t.\ A^T\widetilde{p} &\leq 1_n \\ \widetilde{p} &\geq 0 \end{aligned}$$

which is a standard linear programming problem, which solution gives the mixed security strategy of P1, normalized by the average value of the game. If p^* is P1's mixed saddle-point strategy, then $\widetilde{p}^* = p^*/V_m(A)$ solves the LP problem and the value of the LP problem is given by $V = 1/V_m(A)$.

For "max min" expression, similarly, we can rewrite as

$$\min_q V_2(q) \tag{16}$$

where $V_2(q)$ is a positive scalar valued function of q

$$V_2(q) = \min_p p^T Aq \leq p^T Aq \quad \forall p \in \triangle_1 \tag{17}$$

By introducing $\widetilde{q} = q/V_2(q)$, we can rewrite the optimization problem as

$$\begin{aligned} &\min \widetilde{q}1_n \\ s.t.\ A^T\widetilde{q} &\geq 1_m \\ \widetilde{q} &\geq 0 \end{aligned}$$

In conclusion, we show that for every positive matrix game, there exist two dual LP problems whose solutions yields the saddle-point solution of the game in mixed strategy. Then, we can use efficient linear programming algorithms, e.g., simplex methods and interior methods, to solve high-dimensional matrix games. From Lemma (Invariance of minimax strategies) we discussed in last section, the positivity here is not restrictive.

Lemma Let A and B be 2 $m \times n$ matrices related to each other by $A = B + c1_m 1_n^T$ for some constant c. Then,

(i) every mixed saddle-point strategy for matrix A also constitute to mixed saddle-point strategy solution for matrix B,

(ii) $V_m(A) = V_m(B) + c$

4.3 Learning

So far, the graphical method and linear programming method are both offline methods. We can find saddle-point equilibrium by online methods as well, e.g. Fictitious play.

5 N-person Nonzero-sum Games

A general finite N-person nonzero-sum game in normal form can be defined as:

1. N players denoted by P1, P2, ..., PN.
2. Each player P_i has finite pure strategies X_i, with the element of X_i denoted as x_i. For mixed strategy, the strategy profile $P = (p_i, p_{-i})$ constitutes of mixed strategy $p_i \in \triangle_i(x_i)$ for each player.
3. The outcome of the game is a vector of payoffs of all players, $a^i_{x_i,x_{-i}} = u_i(x_i, x_{-i})$. For mixed strategies, the outcome of the game is a vector of payoffs of all players, $a^i_{p_i,p_i} = u_i(p_i, p_{-i})$
4. Players make their own decision independently.

Definition An N-tuple of strategies $n_1^*, \ldots, n_N^*$, with $n_i^* \in M_i$, $i \in N$, is said to constitute a noncooperative (Nash) equilibrium solution for an N-person nonzero-sum static finite game in normal form, as formulated above, if the following inequality are satisfied for all $x_i \in X_i$, $i \in N$:

$$a^{i*} \triangleq a^i_{x_i^*,x_{-i}^*} \leq a^i_{x_i,x_{-i}^*}$$

In mixed strategy, a strategy profile (p_i^*, p_{-i}^*) is a Nash equilibrium if the following inequality are satisfied for all $p_i \in \triangle(x_i)$, $i \in N$:

$$a^{i*} \triangleq u^i(p_i^*, p_{-i}^*) \leq u^i(p_i, p_{-i}^*)$$

Theorem Every N-person static finite game in normal form admits a Nash equilibrium solution in mixed strategies.

Here's the same equation but that can be referenced:

$$a^2 + b^2 + c^2 \tag{18}$$

Equation **??** has been known for a long time. Sometimes you might want to align equations:

$$\begin{aligned} (x+y)^2 - (x-y)^2 &= (x^2 + 2xy + y^2) - (x^2 - 2xy - y^2) \\ &= 4xy \end{aligned}$$

or align them with numbers to be referenced:

$$(x+y)^2 - (x-y)^2 = (x^2 + 2xy + y^2) - (x^2 - 2xy - y^2) \tag{19}$$
$$= 4xy \tag{20}$$

or where only some of them can be easily referenced:

$$\begin{aligned} (x+y)^2 - (x-y)^2 &= (x^2 + 2xy + y^2) - (x^2 - 2xy - y^2) \\ &= 4xy \qquad (21) \end{aligned}$$

6 Reference

- T. Başar and G.J. Olsder, Dynamic Noncooperative Game Theory, 2nd edition, Classics in Applied Mathematics, SIAM, Philadelphia, 1999
- D. Fudenberg and J. Tirole, Game Theory, MIT Press, 1991.
- Thomas S. Ferguson, Game Theory, Second Edition, 2014 (available online)