ECE-GY 6263 Game Theory Fall 2019

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1 Overview

In the last lecture we introduced two-person zero-sum games and non-zero sum games including a Second Price Auction example, computational methods (graphical methods and algorithms) for finding saddle point equilibrium and a paradox (Prisoner's Dilemma).

In this lecture we will look at N-person non-zero sum game.

2 General Model

Definition 1. *For each player i, we can define a finite or infinite set of actions X_i. Player i can play a* ***pure strategy*** *$x_i \in X_i$ or a* ***mixed strategy*** *$p_i \in \Delta_i(X_i)$, which is a probability distribution over actions. In the reality, the mixed strategy can be made with the aid of a random device such as a coin or a die.*

Definition 2. *Denote x_{-i} as all players excluding player i, then we can define* ***utility function*** *as $u_i(x_i, x_{-i}) : \Pi_i X_i \to \mathbb{R}$. The* ***expected utility*** *is denoted as $\tilde{u}_i(p_i, p_{-i}) \equiv \mathbb{E}_{p_i, p-i}\, u_i(x_i, x_{-i})$*

Question: Nash Equilibrium as a solution concept.

- What is Nash Equilibrium
- Why we use Nash Equilibrium?

Definition 3. *(p_i^*, p_{-i}^*) is a* ***Nash Equilibrium*** *in mixed strategies if*

$$\tilde{u}_i(p_i^*, p_{-i}^*) \geq \tilde{u}_i(p_i, p_{-i}^*) \tag{1}$$

for all admissible $p_i \in \Delta(X_i)$ and for all $i \in N$, which is equivalent to

$$p_i^* \in \arg\max_{p_i \in \Delta(X_i)} \tilde{u}_i(p_i, p_{-i}^*) \quad \forall i \tag{2}$$

Note that sometimes $p_i \in \tilde{\Delta}(X_i) \subset \Delta(X_i)$.

Thought Experiment For two-person non-zero sum game, we have

$$p_1^* \in \arg\max_{p_1} \tilde{u}_1(p_1, p_2^*)$$

$$p_2^* \in \arg\max_{p_2} \tilde{u}_2(p_1^*, p_2)$$

Solving p_1^*, p_2^* depend on each other.

3 Analytical Methods for Characteristic Nash Equilibrium

3.1 Best response functions

Definition 4. *Consider the following* ***best response*** *for Player i.*

Given $p_{-i} \in \Pi_{j\neq i}\Delta(X_j)$ (note that $p^*_{-i} \in \Pi_{j\neq i}\Delta(X_j)$ is a point on the space),

$$BR_i(p_{-i}) \equiv \arg\max_{p_i \in \Delta(X_i)} \tilde{u}_i(p_i, p_{-i})$$

Fix p_{-i} and choose p_i, then BR_i is correspondence, a point-to-set mapping.

Remark:

1. $\tilde{u}_i(p_i, p_{-i})$ is continuous in p_i, linear in p_i, and $\Delta_i(X_i)$ is a compact set (according to the property of expectation).
2. Based on Weierstrass's Theorem and property of convexity, we have
   - $BR_i(p_{-i})$ is a convex set.
   - $BR_i(p_{-i})$ is non-empty.
3. Point-to-set mapping. BR_i is "continuous" or not?

   Definition 5. *(Upper semi-continuity) Pick a sequence $p^{(n)}_{-i} \to p_{-i}$ and a sequence $p^{(n)}_i \to BR_i(p^{(n)}_{-i})$ and $p^{(n)}_i \to p_i$ and $p_i \in BR_i(p_{-i})$. If this is true for all p_{-i}, then this mapping is called* ***Upper semi-continuity****. [1]*
4. BR_i is an Upper semi-continuity.
5. The set p_{-i} is a convex compact set.

4 The existence of Nash Equilibrium

For two-person non-zero sum game, we have $p_1 \in BR_1(p_2)$ and $p_2 \in BR_2(p_1)$. For N-person non-zero sum game, consider

$$p \mapsto BR(p)$$

$$p : \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} \qquad BR(p) = \begin{pmatrix} BR_1(p) \\ BR_2(p) \\ \vdots \\ BR_N(p) \end{pmatrix}$$

we have $p \in BR(p)$, solving this is actually a fixed point problem!

Theorem 6. *(Kakutani's Theorem) Let S be a compact and convex subset of $\mathbb{R}^n$, and let f be an upper semi-continuous function which assigns to each $X \in S$ a closed subset of S. Then there exists some $X \in S$ such that $x \in f(x)$.*

Remark: Since $S : \Pi_{i=1}^{N} \Delta(X_i)$ is closed and convex, we can derive the existence of Nash Equilibrium in mixed strategies for finite games.

Thought Experiment: What can go wrong if the game is not finite? e.g. $X_i \equiv [0, 1]$. Use Fixed Point Theorem [1]!

Theorem 7. *(Brower's Fixed Point Theorem) If S is a compact and convex subset of $\mathbb{R}^n$, f is a continuous function mapping S into itself, then there exists at least one $x \in S$ such that $f(x) = x$.*

Example: If $f : [0, 1] \to [0, 1]$ is a continuous function, then $\exists\, x : \ x = f(x)$.

By Kakutani's argument, we have a point p^* such that $p^* \in BR(p^*)$.

Question: Is p^* a Nash Equilibrium? Yes!

Rough idea: $p_i^* \in BR_i(p_{-i}^*)\ \forall i \Rightarrow (2) \Rightarrow (1)$

Remark: Read the book on John Nash's proof (on Matrix Game)!

- Mixed strategy Nash Equilibrium existence
- Reason to find it
- How to find it
  - Fixed point method
  - Algorithms
  - Learning method

5 Computational method

		y	1-y
		B	S
x	B	(1, 2)	(0, 0)
1-x	S	(0, 0)	(2, 1)

Example: Battle of Sexes problem (B, B) and (S, S) are two pure Nash Equilibriums. Let $x : \mathbb{P}(P_1 \to B)$, $y : \mathbb{P}(P_2 \to B)$, denote the utility functions of two players as $u_i(x, y)$ $i = 1, 2$. Then for P_1, $\tilde{u_1} = xy + 2(1 - x)(1 - y) = x(3y - 2) + (2 - 2y)$, his best response strategy is

$$BR_1(y) = \begin{cases} 0, & y < \frac{2}{3} \\ 1, & y > \frac{2}{3} \\ [0, 1], & y = \frac{2}{3} \end{cases}$$

Note this is a point-to-set mapping which is upper semi-continuity.

Similarly for P_2 we have $\tilde{u_2} = 2xy + (1-x)(1-y) = y(3x-1) + (1-x)$, his best response strategy is

$$BR_2(x) = \begin{cases} [0,1], x = \frac{1}{3} \\ 0, x < \frac{1}{3} \\ 1, x > \frac{1}{3} \end{cases}$$

x, y should satisfy $y \in BR_2(x)$ and $x \in BR_1(x)$. We can solve this using graphical method.

Thought Experiment A: What if the utility functions are non-linear?

Thought Experiment B: Using Indifference Principle for inner solutions!

Thought Experiment C: Perturbations, Equilibrium Selection and Refinements of Nash Equilibrium. Related to "trembling hand", whenever there are errors/mistakes, it eventually will back to saddle points.

Best-response Dynamics: $P_i^{(n+1)} \in BR_i(P_{-i}^{(n)})$ at round n, if it goes to steady state then $p^* \in BR(p^*)$.

6 Fictitious-Play Learning Algorithm

Consider 2-player problem:

	L	R
U	(3, 3)	(0, 0)
D	(4, 0)	(1, 1)

(1) The 2 players choose $x_i \in X_i$ at time $t = 1, 2, \cdots$ where $X_1 = \{U, D\}$, $X_2 = \{L, R\}$.

(2) Define $\eta_i^t : S_{-i} \to N$ as the number of times player i observed the action S_{-i} played before time t.

- $\eta_1^0 = (3, 0)$
- $\eta_2^0 = (1, 2.5)$

(3) Players form a prediction on other players' strategies

$$\mu_i^t(x_i) = \frac{\eta_i^t(x_i)}{\sum\limits_{x_i' \in X_i} \eta_i^t(x_i')}$$

- $\mu_1^0 = (1, 0)$
- $\mu_2^0 = (\frac{1}{3.5}, \frac{1}{2.5})$

(4) Player i: $x_i \in \arg\max\limits_{x_i \in X_i} \mathbb{E}_{\mu_i^t}(x_i, x_{-i})$ at time t

Example:

- Round 1: $\mu_1^0 = (1,0)$, $\mu_2^0 = (\frac{1}{3.5}, \frac{1}{2.5})$, P1: D, P2: L
- Round 2: $\eta_1^1 = (4,0)$, $\mu_1^1 = (1,0)$, $\mu_2^1 = (\frac{1}{4.5}, \frac{3.5}{4.5})$, P1: D, P2: R
- Round 3: converges to $\mu_1^t \to p_2^*$, $\mu_2^t \to p_1^*$ (equilibrium)

Example: IBM robots play rock-and-scissors

Thought Experiment:

1. Fictitious play can be viewed as a interpretation as a learning process approaches to equilibrium. Outcome of rational learning should not change over times or over place. Prediction should be stationary.
2. Using arbitration or recommendation for players to avoid "Prisoners' Dilemma".

References

[1] Basar, Tamer, and Geert Jan Olsder. *Dynamic noncooperative game theory. Vol. 23. Siam, 1999.*