ECE-GY 6263 Game Theory Fall 2019

Lecture 11 — Novemeber 22, 2019

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1 Overview

In the last lecture we talked about Mechanism Design.

In this lecture we continue to discuss Mechanism Design and then introduce the definition of Dynamic Bayesian Games and Signaling games.

2 Mechanism Design

Remember that the intention of the Bayesian mechanism design is to maximize the expected revenue, we denote the revenue as $m_i(\theta_i)$, θ_i is the valuation of Player i, so the whole problem can be described as the expression below:

$$max_{\pi_i q_i} \sum_{i=1}^{N} E(m_i(\theta_i))$$

Here π_i is the allocation rule and q_i is the payment rule.

2.1 Equivalence between (IC+IR) and three constraints

In the last lecture, we knew that the IC constraint: $\theta_i \alpha_i(\theta_i) - m_i(\theta_i) \geq \theta_i \alpha_i(\hat{\theta}_i) - m_i(\hat{\theta}_i)\ \forall \theta_i, \hat{\theta}_i\ \forall i$ and $\theta_i, \hat{\theta}_i \in \Theta_i$, this IC is equivalent to the following two constaints:

(i) m_i, α_i satisfy:

$$m_i(\theta_i) = m_i(0) + \theta_i \alpha_i(\theta_i) - \int_0^{\theta_i} \alpha_i(\theta) d\theta \quad \forall i$$

Note that :

$$\alpha_i(\theta_i) \equiv E_{\theta_{-i}}(\pi_i(\theta_i, \theta_{-i})|\theta_i)$$

$$m_i(\theta_i) \equiv E_{\theta_{-i}}(\pi i(\theta_i, \theta_{-i}) q_i(\theta_i, \theta_{-i})|\theta_i)$$

(ii) α_i is a non-decreasing function
In the last lecture, we had proved that IC $\Rightarrow$ (i)+(ii), in other words IC is the sufficient condition of (i) and (ii), now we have to prove that (i)+(ii) $\Rightarrow$ IC, which means that IC is also the necessary condition of (i) and (ii).

Proof:

$$m_i(\theta_i) = m_i(0) + \theta_i\alpha_i(\theta_i) - \int_0^{\theta_i} \alpha_i(\theta)d\theta$$

$$\theta_i\alpha_i(\theta_i) - m_i(\theta_i) = \int_0^{\theta_i} \alpha_i(\theta)d\theta - m_i(0) \tag{1}$$

$$\begin{aligned} theta_i\alpha_i(\tilde{\theta}_i) - m_i(\tilde{\theta}_i) &= \tilde{\theta}_i\alpha_i(\tilde{\theta}_i) - \tilde{\theta}_i\alpha_i(\tilde{\theta}_i) + \theta_i\alpha_i(\tilde{\theta}_i) - m_i(\tilde{\theta}_i) \\ &= \int_0^{\tilde{\theta}_i} \alpha_i(\theta)d\theta + (\theta_i - \tilde{\theta}_i)\alpha_i(\tilde{\theta}_i) \end{aligned} \tag{2}$$

Subtract two equations above, we get (1)-(2) : $\int_0^{\theta_i} \alpha_i(\theta)d\theta - \int_0^{\tilde{\theta}_i} \alpha_i(\theta)d\theta - (\theta_i - \tilde{\theta}_i)\alpha_i(\tilde{\theta}_i)$, this result is ≥ 0 if $\theta_i \geq \tilde{\theta}_i$, when $\tilde{\theta}_i \geq \theta_i$, the result is still ≥ 0.

Now we can get the proposition:
IC + IR $\equiv$ (i) + (ii) + (iii), here IR: $\theta_i\alpha_i(\theta_i) - m_i(\theta_i) \geq 0 \ \ \forall i \ \ \theta_i \in \Theta_i$, and (iii): $m_i(0) \leq 0$

First we have to prove that IC + IR $\Rightarrow$ (i) + (ii) + (iii), but we only need to show: IR $\Rightarrow$ (iii)
$\theta_i\alpha_i(\theta_i) - m_i(\theta_i) = \int_0^{\theta_i} \alpha_i(\theta)d\theta - m_i(0) \geq 0$, (IR)

When $\theta_i = 0$, we can get: $m_i(0) \leq 0$ which is (iii)

Then prove that (i) + (ii) + (iii) $\Rightarrow$ IC + IR, same as before, we only have to show that: (iii) $\Rightarrow$ IR.
$\theta_i\alpha_i(\theta_i) - m_i(\theta_i) = \int_0^{\theta_i} \alpha_i(\theta)d\theta - m_i(0)$

Because $\int_0^{\theta_i} \alpha_i(\theta)d\theta \geq 0$ and $m_i(0) \leq 0$, we get : $\int_0^{\theta_i} \alpha_i(\theta)d\theta - m_i(0) \geq 0$, which is the IR constraint

Finally we can draw the conclusion that : IC + IR $\equiv$ (i) + (ii) + (iii)

2.2 Objective function

Now, consider the objective function, in other word the expected revenue:

$$\begin{aligned} E(m_i(\theta_i)) &= E(m_i(0) + \theta_i\alpha_i(\theta_i)) - \int_0^{\theta_i} \alpha_i(\theta)d\theta) \\ &= E((m_i(0)) + E(\theta_i\alpha_i(\theta_i))) - E(\int_0^{\theta_i} \alpha_i(\theta)d\theta)) \end{aligned}$$

According to the definition of α_i, we get that:

$$E(\theta_i\alpha_i(\theta_i)) = \int_0^{\theta_{imax}} \theta_i\alpha_i(\theta_i)f_i(\theta_i)d\theta_i$$

$$E(\int_0^{\theta_i} \alpha_i(\theta)d\theta)) = \int_0^{\theta_{imax}} \int_0^{\theta_i} \alpha_i(\theta_i)d\theta f_i(\theta_i)d\theta_i$$
$$= \int_{\theta=0}^{\theta=\theta_{imax}} \int_{\theta_i=0}^{\theta_i=\theta_{imax}} \alpha_i(\theta)f_i(\theta_i)d\theta d\theta_i$$
$$= \int_0^{\theta_{imax}} \alpha_i(\theta)[\int_{\theta_i=0}^{\theta_i=\theta_{imax}} f_i(\theta_i d\theta_i)]$$

The process above is changing the order of integration. Note that: $\int_0^{\theta_{imax}} f_i(\theta_i)d\theta_i = 1 - F_i(\theta)$, F_i is CDF and f_i is PDF.

So, we have the objective function below:

$$E(m_i(\theta_i)) = E(m_i(0)) + \int_0^{\theta_{imax}} \alpha_i(\theta_i)[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}]f_i(\theta_i)d\theta_i$$

And we define that $\varphi_i(\theta_i) = \theta_i - \theta_i - \dfrac{1 - F_i(\theta_i)}{f_i(\theta_i)}$, now we can get:

$$E(m_i(\theta_i)) = E(m_i(0)) + \int_0^{\theta_{imax}} \alpha_i(\theta_i)\varphi_i(\theta_i)f_i(\theta_i)d\theta_i$$

Remember that $\alpha_i = E_{\theta_{-i}}(\pi_i(\theta_i, \theta_{-i})) = \int \pi_i(\theta_i, \theta_{-i})f_{-i}(\theta_{-i})d\theta_{-i}$, so $E(m_i(\theta_i)) = E(m_i(0)) + \int_{\bar{\theta}\in V} \pi_i(\theta_i, \theta_{-i})f_{-i}(\theta_{-i})\varphi_i(\theta_i)f_i(\theta_i)d\bar{\theta}$, we define that: $f(\bar{\theta}) = f_{-i}(\theta_{-i})f_i(\theta_i)$, then we get:

$$E(m_i(\theta_i)) = E(m_i(0)) + \int_{\bar{\theta}\in V} \pi_i(\theta_i, \theta_{-i})\varphi_i(\theta_i)f(\bar{\theta})d\bar{\theta}$$

For N Players, the expected revenue becomes:

$$\sum_{i=1}^{N} E(m_i(\theta_i)) = \sum_{i=1}^{N} E(m_i(0)) + \int_{\bar{\theta}\in V} [\sum_{i=1}^{N} \pi_i(\theta_i, \theta_{-i})\varphi_i(\theta_i)]f(\bar{\theta})d\bar{\theta}$$

2.3 Allocation rules

The rule is that we put positive probability only when $\varphi_i(\theta_i)$ of Player i is non-negative and is the largest, which can be described as : $\pi_i(\theta_i, \theta_{-i}) > 0$ iff $\varphi_i(\theta_i) = max_j \varphi_j(\theta_j)$ and $\varphi_i(\theta_i) \geq 0$. Also, note that: $\sum^i \pi_i \leq 0$

Now, consider the constraints below:
First, remember the (i) constraint in subsection 2.1:

$$m_i(\theta_i) = m_i(0) + \theta_i\alpha_i(\theta_i) - \int_0^{\theta_i} \alpha_i(\theta)d\theta$$

Then, $E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})\pi_i(\theta_i, \theta_{-i})|\theta_i] = m_i(0) + \theta_i E_{\theta_{-i}}[\pi_i(\theta_i, \theta_{-i})|\theta_i] - E[\int_0^{\theta_i} \pi_i(\theta_i, \theta_{-i})|\theta_i]$ (*)

In the case that when $\theta_i = 0$, $q_i(\theta_i, \theta_{-i})\pi_i(\theta_i, \theta_{-i}) = 0 \ \forall i$ (***), we can get: $m_i(0) \equiv E(q_i(0, \theta_{-i})\pi_i(0, \theta_{-i}) = 0$ and $\forall \theta_i, \theta_{-i} : q_i(\theta_i, \theta_{-i})\pi_i(\theta_i, \theta_{-i}) = \theta_i \pi_i(\theta_i, \theta_{-i}) - \int_0^{\theta_i} \pi_i(\theta_i', \theta_{-i}) d\theta_i'$ (**)

Proposition: Assume $\varphi_i(\theta_i)$ is a strictly increasing function (****), under (**) and (***), conditions (i) (ii) (iii) are satisfied.

Because (i) and (iii) are already done, we only have to show (ii) is satisfied, remember (ii) : $\alpha_i(\theta_i) = E_{\theta_{-i}}(\pi_i(\theta_i, \theta_{-i})|\theta_i)$, the probability of winning is non-decreasing.

Consider two conditions below:
1. $\pi_i(\theta_i, \theta_{-i}) > 0$ which means that$\varphi_i(\theta_i)$ is the largest, and when bid $\hat{\theta}_i > \theta_{is}$, we get: $\varphi_i(\hat{\theta}_i) > \varphi_i(\theta_i)$ (Under ****), so α_i is non-decreasing.

2. $\pi_i(\theta_i, \theta_{-i}) = 0$, then $\pi_i(\hat{\theta}_i, \theta_{-i}) \geq \pi_i(\theta_i, \theta_{-i})$ which means that the chances of winning is not decreasing.

Now we can summarize the following allocation rules yielding a revenue-optimal mechanism :
1. (**) + (***) + φ_i is strictly increasing
2. $\pi_i(\theta_i, \theta_{-i}) > 0$ iff $\varphi_i(\theta_i) = max_j \varphi_j(\theta_j)$ and $\varphi_i(\theta_i) \geq 0$

2.4 Two remarks

The total expected revenue is $\sum_i E(m_i(\theta_i))$ and $\sum_i E(m_i(\theta_i)) = \sum_i m_i(0) + E(\sum_i \varphi_i(\theta_i)\pi_i(\theta_i, \theta_{-i}))$. Note that: $\sum_i m_i(0) = 0$, $1 = E(max(\varphi_1(\theta_1), \varphi_2(\theta_2), ..., \varphi_N(\theta_N), 0))$, Remember the allocation rule: $\pi_i(\theta_i, \theta_{-i}) > 0$ iff $\varphi_i(\theta_i) = max_j \varphi_j(\theta_j) \geq 0$

2.4.1 Remark 1

Define the following function:

$$r_i(\theta_{-i}) = inf\{Z_i : \varphi_i(Z_i) \geq 0 and \varphi_i(Z_i) \geq max_j \varphi_j(\theta_j)\}$$

$r_i(\theta_{-i})$ yields the "smallest" values of the valuations that the bidder is guaranteed to win

$$\pi_i(\theta_i, \theta_{-i}) = \begin{cases} 1, & \text{if } \theta_i > r_i(\theta_{-i}) \\ 0, & \theta_i < r_i(\theta_{-i}) \end{cases}$$

$$\pi_i(\theta_i, \theta_{-i})q_i(\theta_i, \theta_{-i}) = \theta_i \pi_i(\theta_i, \theta_{-i}) - \int_0^{\theta_i} \pi_i(\theta_i', \theta_{-i}) d\theta_i'$$

Note: $\pi_i(\theta_i, \theta_{-i})q_i(\theta_i, \theta_{-i})$ represents how much to pay for the Player i

If $\theta_i > r_i(\theta_{-i})$

$$\pi_i(\theta_i, \theta_{-i})q_i(\theta_i, \theta_{-i}) = \theta_i - (\theta_i - r_i(\theta_{-i})) = r_i(\theta_{-i})$$

The winner will pay the smallest value that can guarantee his winning.

If $\theta_i < r_i(\theta_{-i})$
$\pi_i(\theta_i, \theta_{-i})q_i(\theta_i, \theta_{-i}) = 0$ which means loss and pay nothing.

2.4.2 Remark 2

When all agents have the same f_i, then

$$\varphi_i \equiv \varphi(\theta_i) = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$$

and because $\varphi(\cdot)$ is a strictly increasing function, we can rewrite the $r_i(\theta_{-i})$ as below:

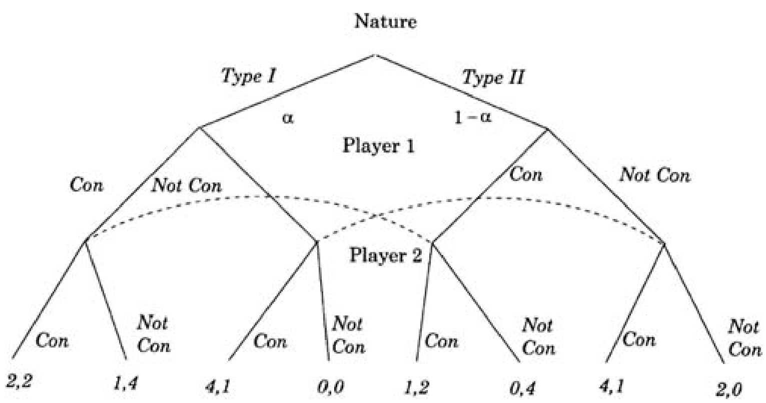
$$\begin{aligned} r_i(\theta_{-i}) &= inf\{Z_i : Z_i \geq \varphi^{-1}(0) and Z_i \geq max_{j \neq i}\theta_j\} \\ &= \{\varphi^{-1}(0), max_{j \neq i}\theta_j\} \end{aligned}$$

Under this condition, the winner pay the second highest bid and $\varphi^{-1}(0)$ is the reserve price.

3 Signaling Games

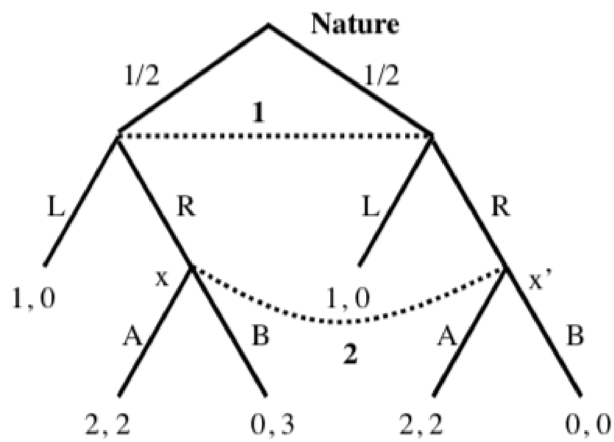
3.1 Dynamic games of incomplete information

Consider the following game tree



Player 1 knows the nature, but Player 2 does not know. The equilibrium of this kind of game is called Perfect Bayesian Nash Equilibrium.

Example: Consider the following game:



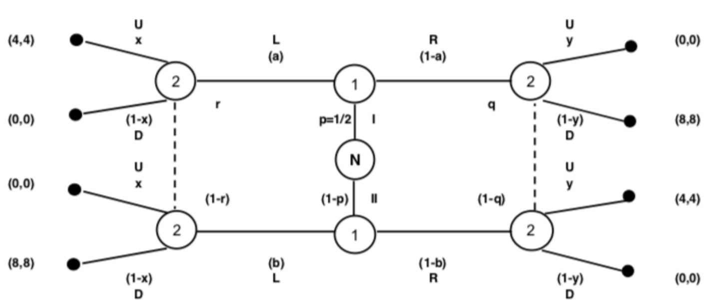
In this game, (L, B) is a subgame perfect equilibrium.

As in the previous game, (L, B) is an SPE in this example because there are no subgames. Note, however, that there are in fact beliefs for which B is an optimal choice for player 2. If player 2 places probability at least $2/3$ on being at x given that he is at $h = x, x'$, then B is an optimal choice. However, these beliefs seem quite unreasonable: if player 1 chooses R, then player 2 should place equal probability on being at either x or x'.

3.2 Signaling games

1. Nature draws a type t_i for the sender of feasible types $T = [t_i, ..., t_I]$ according to $p(t_i)$, $p(t_i) > 0$, $\sum_{i=1}^{I} p(t_i) = 1$
2.Sender observes t_i, and then chooses a message m_j from a set of feasible message $M = \{m_1, ..., m_J\}$, $m_i = \mu(t_j)$
3.Receiver observes message m_j, chooses an action a_u from action space $A = \{a_1, ..., a_k\}$, $a_u = \nu_\Delta(m_j)$
4. $U_S(t_i, a_u)$, $U_R(t_i, a_u)$

Example: Consider the following game:



This is a typical signaling game that follows the classical "beer and quiche" structure. The game starts with a "decision" by Nature, which determines whether player 1 is of type I or II with a probability $p = Pr(I)$ (in this example we assume $p = \frac{1}{2}$). Player 1, after learning his type, must decide whether to play L or R. Since player 1 knows his type, the action will be decided conditioning on it.

After player 1 moves, player 2 is able to see the action taken by player 1 but not the type of player 1. Hence, conditional on the action observed she has to decide whether to play U or D.

Hence, it's possible to define strategies for each player (π_i) as

• For player 1, a mapping from types to actions
$\pi_1 : I \to aL + (1-a)R$ and $II \to bL + (1-b)R$
• For player 2, a mapping from player 1's actions to her own actions
$\pi_2 : L \to xU + (1-x)D$ and $R \to yU + (1-y)D$

In this game there is no subgame since we can not find any single node where a game completely separated from the rest of the tree starts. This is why we cannot use the concept of Subgame perfection in this type of games. However it is necessary to identify subforms, which are trees separated from the whole game that starts from an information set instead than from a single node. In this case, there are two subforms, the one that start after player 2 sees player 1 played L and the other after observing R.

References

[1] R. Gibbons, *Game Theory for Applied Economics*, Princeton University Press, 1992.

[2] D. Fudenburg and J.Tirole, *Game Theory*, MIT Press, 1991.