ECE-GY 6263 Game Theory Fall 2019

Lecture 7 — Oct. 18, 2019

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1 Overview

In the last lecture we covered multistage games and Perfect Subgame Nash Equilibrium.

In this lecture we cover Stackelberg equilibrium in mixed strategies, Multistage Games, and Continuous Kernel Games.

2 Stackelberg Games

Take the following matrix game

$$A = \begin{bmatrix} 1, \frac{1}{2} & 0, 1 \\ 0, 1 & 1, \frac{1}{3} \end{bmatrix}$$

If we say the row player $P1$ is the leader and the column player $P2$ is the follower. With both minimizers $P2$'s best response is as follows. If $P1$ chooses row 1, $P2$ chooses column 1. If $P1$ chooses row 2, $P2$ chooses column 2. This is the pure strategy equilibrium. For a finding a mixed strategy equilibrium.

$$y = (y_1, 1 - y_1), \quad z = (z_1, 1 - z_1)^T$$

with the objective

$$\bar{J}^z(y, z) = y^T B z = (-\frac{7}{6}y_1 + \frac{2}{3})z_1 + \frac{2}{3}y_1 + \frac{1}{3}$$

for z with

$$B = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{3} \end{bmatrix}$$

$$\bar{R}^z = \begin{cases} z_1 = 1 & y_1 > \frac{4}{7} \\ z_1 = 0 & y_1 < \frac{4}{7} \\ z_1 = [0, 1] & y_1 = \frac{4}{7} \end{cases}$$

- If $y_1 > \frac{4}{7}$ then $\bar{J}_1 = y_1$

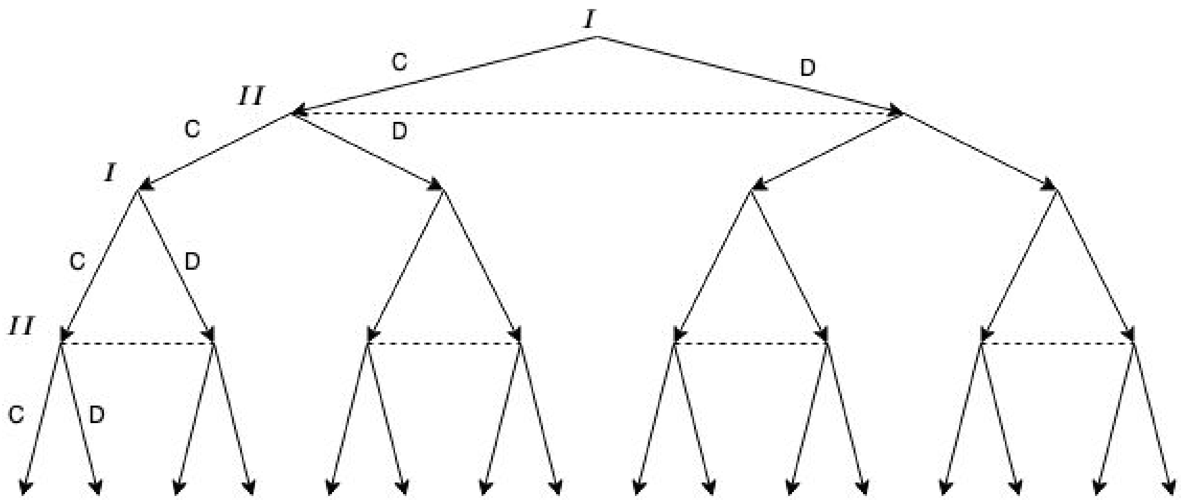

Figure 1: Extensive Form

- If $y_1 < \frac{4}{7}$ then $\bar{J}_1 = 1 - y_1$
- If $y_1 = \frac{4}{7}$ then $\bar{J}_1 = \frac{3}{7} + \frac{1}{7}z_1$ with worst case $\frac{4}{7}$

There is no Stackelberg equilibrium in mixed strategies for this game as there are no point where each strategy is a best response. There is a equilibrium in pure strategies. This highlights the fact that there is always a equilibrium in pure strategies for Stackelberg games but not necessarily in mixed strategies.

3 Multistage

Consider repeated game

$$A = \begin{bmatrix} 3,3 & -1,5 \\ 5,-1 & 0,0 \end{bmatrix}$$

With both players maximizing and the game is played twice and the objective is the summation of the games. The extensive form of this game is shown in figure 1. There are 2 subgames. The probability of a path can be calculated and therefore the expected value given player strategies.

$$\mu_i(t, h_t) = a_i(t)$$
$$a_i(t) \in A_i(t, h_t)$$
$$J_i(\mu_i, \mu_{-i}) = \sum_{t=0}^{T} u_i(t, \mu_i(t, h_t), \mu_{-i}(t, h_t))$$

$$\max_{\mu_i \in \Gamma_i} J_i(\mu_i, \mu_{-i})$$

Definition: A strategy profile $\{\mu_i^*\}_{i=1}^N$ is a Subgame Perfect Nash Equilibrium if for every K, and its associated history $h_k \in H$ the rest of $\{\mu_i^\gamma\}_{i=1}^N$ to the

SPNE can be found using dynamic programming and checked with the One-stage Deviation principle. [1]

Definition: A strategy profile $\{\mu_i^*\}_{i=1}^N$ satisfies One-stage Deviation Condition if no player can gain by deviating at one stage and one history.

Example: Repeated Prisoners Dilemma

$$A = \begin{bmatrix} 3,3 & -1,5 \\ 5,-1 & 0,0 \end{bmatrix}$$

Played infinite times

$$J_i^{(k)} = \sum_{t=k}^{\infty} \mu_i(t, \mu_i(t, h_t), \mu_{-i}(t, h_t)) = \sum_{t=k}^{\infty} \alpha^k u_i(t, \mu_i, \mu_{-i})$$

with

$$\alpha \in (0, 1)$$

Consider a trigger strategy, Play C if D does not appear in history and Play D if D appears in history.

$$J_i^{(k)} = \sum_{t=k}^{\infty} 3\alpha^t = 3\frac{\alpha^k}{1-\alpha}$$

$$J_i^{(k)} = \sum_{t=k}^{k'-1} +5\alpha^{k'} + \sum_{t=k'}^{\infty} 0$$

One-stage deviation condition

$$\sum_{t=k}^{k'-1} +5\alpha^{k'} \leq \sum_{t=k}^{\infty} 3\alpha^t$$

$$5\alpha^{k'} \leq \sum_{t=k'}^{\infty} 3\alpha^t$$

$$5\alpha^{k'} \leq 3\frac{\alpha^{k'}}{1-\alpha}$$

this is satisfied for

$$\frac{2}{5} \leq \alpha \leq 1$$

4 Constraints

There may be constraints on player actions

Thought Experiment: Player 1 and 2 have actions 0,1 and 2 with the constraint that the sum of there actions must be less than or equal to 2.

5 Continuous Games

Before the action space was finite $u^i = 0, 1$ but we will now consider infinite action spaces $u^i = [0, 1]$ or continuous action spaces. In addition there may be constraints. Unlike finite games a equilibrium is not guaranteed.

Example: An infinite game without an equilibrium is a single player game with objective e^{-x} and constraint $x > 0$ with the player being a minimizer. The optimal value of 0 cannot be achieved. Instead though we can get within ϵ of the optimal value

Definition: For a given $\epsilon \geq 0$ an N-tuple $(u_\epsilon^{1*}, \dots, u_\epsilon^{n*})$ with $u_\epsilon^i \in U, i \in N$, is called a (pure) ϵ-Nash equilibrium solution for an N-person games if

$$J^i(u_\epsilon^{1*}, \dots, u_\epsilon^{n*}) \leq \inf_{u_i}(u_\epsilon^{1*}, \dots, u^i, \dots, u_\epsilon^{n*}), \forall i \in N$$

with $\epsilon = 0$ being the Nash equilibrium.

Example: Cournot Duopoly with 2 firms. This problem represents companies problem of how much product to produce. Two companies wish to maximize their profit with a simple model of supply and demand. Each company produces a quantity q_1, q_2 That the price and ultimately their profit is determined by.

$$U_i(q_1, q_2) = q_i(a - q_1 - q_2) - cq_i$$

where a is a given constant related to the supply demand function that sets the price and c is a given constant representing the cost to produce one unit of the product. This function of utility is a concave quadratic. The best response for a player can be found by applying first order conditions for optimality.

$$\frac{\partial U_1}{\partial q_1} = 0 = a - c - q_2 - 2q_1$$

$$q_1 = \frac{a-c-q_2}{2}$$
$$q_2 = \frac{a-c-q_1}{2}$$

therefore the Nash Equilibrium is

$$q_1^* = q_2^* = \frac{a-c}{3}$$

Now consider a monopoly where there is only one play who decides the quantity. The new objective function is

$$U(q) = q(a-q) - cq$$

with maximum

$$q = \frac{a-c}{2}$$

The total quantity in the monopoly is less than the case where there are two firms. The profit is greater as well

$$\frac{(a-c)^2}{4}$$

with the profit for two firms being

$$\frac{(a-c)^2}{9}$$

This can also be turned into a repeated game where the firms maximize total profit. We can investigate if a trigger strategy is an equilibrium again. The two strategies are to play as a monopoly or to play the one shot optimal value.

$$\begin{bmatrix} \frac{a-c}{4}, \frac{a-c}{4} & \frac{a-c}{4}, * \\ *, \frac{a-c}{4} & \frac{a-c}{3}, \frac{a-c}{3} \end{bmatrix}$$

Suppose P1 deviates at t

$$J^D = \sum_{k=t'}^{t-1} \frac{(a-c)^2}{8} \alpha^k + \sum_{k=t'+1}^{\infty} \alpha^k \frac{(a-c)^2}{9} + q_1(t)$$

$$\delta J = J^D - J' = q_1(t)[\frac{3}{4}(a-c) - q_1(t)]\alpha^t - \frac{(a-c)}{8}\alpha^t + \sum_{k=t+1}^{\infty} \alpha^k (a-c)^2 (\frac{1}{9} - \frac{1}{8}) \leq 0$$

$$\alpha \geq \frac{9}{17}$$

[2]

References

[1] Drew Fudenberg and Jean Tirole. *Game Theory*. Cambridge, MA: MIT Press, 1991.

[2] Michael Maschler, Eilon Solan, and Shmuel Zamir. *Game Theory*. Cambridge University Press, 2013.