ECE-GY 6263 Game Theory Fall 2019

Lecture 12 — Dec 23rd, 2019

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1 Overview

In the last lecture we discussed the Bayesian Mechanism Design, Incentive Compatible and Individually Rational constraint. In addition, we discussed the allocation rules that yield a revenue optimal mechanism, dynamic game of incomplete information that leads to the Perfect Bayesian Nash Equilibrium, and the signal games as the last topic of the class.

In this lecture we will continue the topic on Bayesian Games with incomplete information, one typical example that this class provide is the multi-states signaling game. We will also cover the estimation control in this class and generalize the signaling game to a multistage N person game and to discuss the existence of Perfect Bayesian Nash Equilibrium.

2 Signaling Game

Assumption 1. *Discrete and finite nature uncertainties labeled as* $T = \{t_1, t_2, \ldots, t_I\}$

Assumption 2. *The prior probability distribution that selected by nature:* $p(t_i), \forall i$, *and* $\sum_{i=1}^{I} p(t_i) = 1$

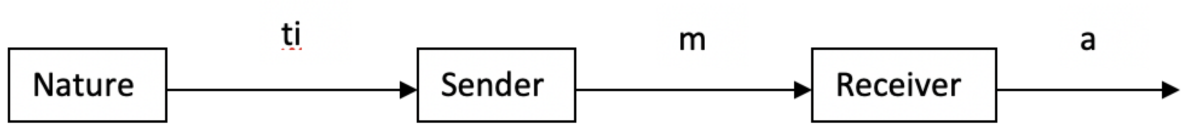

Figure 1: Signaling Game Flowchart

Assumption 3. *The basic flow chart of Signaling Game is shown in Figure 1, where sender knows the type that nature chooses, but receiver cannot distinguish them(receiver can infer the type through sender's action).*

In addition, m *and* a *are sender and receiver's action, with* $m \in \mathcal{M} = \{m_1, m_2, \ldots, m_j\}$, *and* $a \in \mathcal{A} = \{a_1, a_2, \ldots, a_k\}$. *The actions of both sender and receivers are made based on history information with* $m_j = \mu_s(t_i)$, *and* $a_k = \mu_r(m_j)$.

Assumption 4. *Payoffs of sender and receiver:* $U_s(t_i, m_j, a_k), U_r(t_i, m_j, a_k)$

Now, we consider the case of 2-type, 2-message, 2-action Signaling Game as shown in Figure 2.

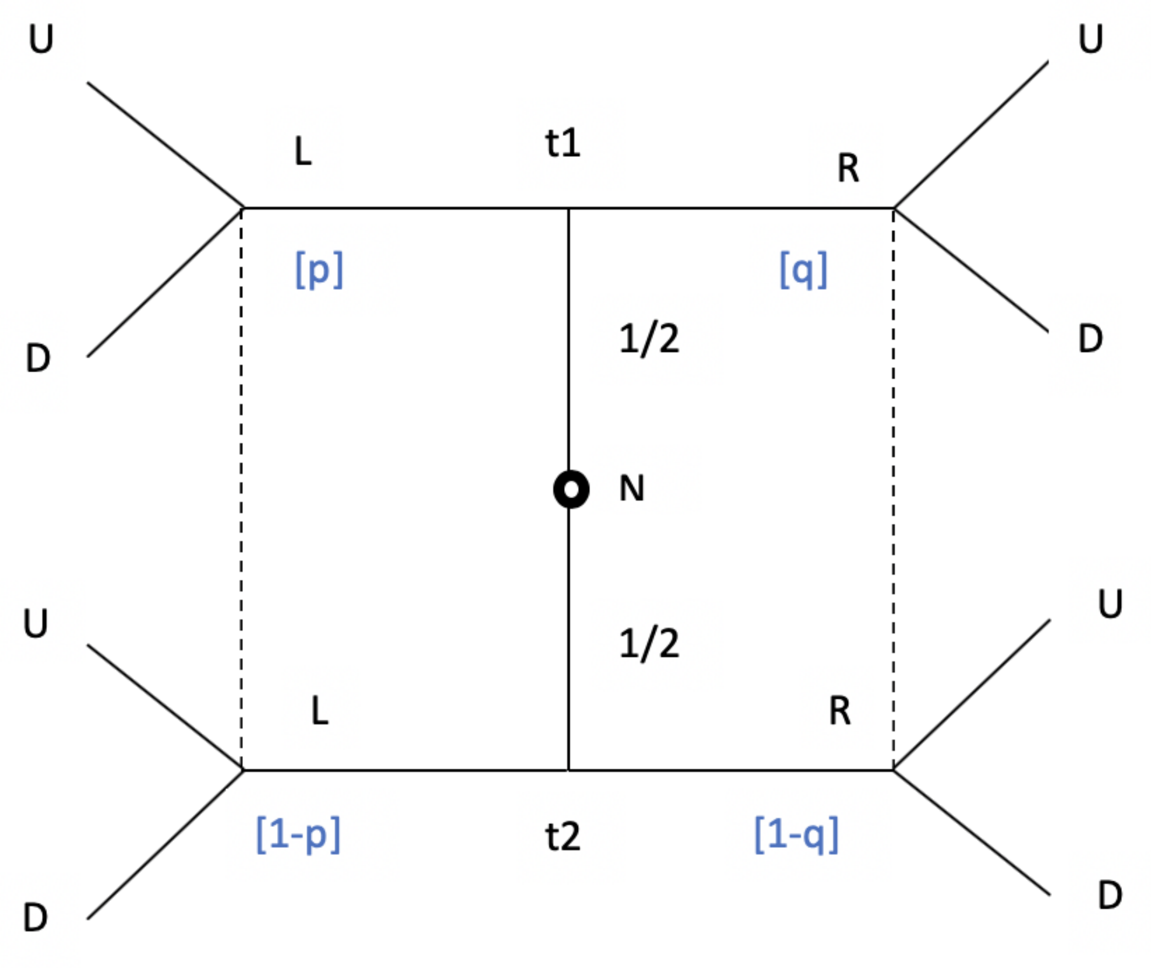

Figure 2: 2-type, 2-msg, 2-action Signaling Game

According to the above payoff graph, there are two different strategies for the sender:

- Pooling Strategy: such as choosing (L, L) or (R,R), the same strategy, for both t_1 and t_2.
- Separating Strategysuch as choosing (L,R) or (R,L) for t_1 and t_2.
- Mixture: there is not mixture strategy in this case, but an example of mixture strategy is shown as below:

$$m_j = \begin{cases} L \ if \ t_i = t_1 \\ L \ if \ t_i = t_2 \\ R \ if \ t_i = t_3 \end{cases}$$

For the receiver, it is easier for them if sender has separating strategy.

3 Perfect Bayesian Nash Equilibrium

Recall that the Nash Equilibrium we learnt before does not acquire the trembling hand property, but Perfect Bayesian Nash Equilibrium(PBNE) has.

We continue using the payoff chart shown in Figure 2, and we introduce the beliefs of the receiver: (p, 1-p), (q,1-q), where 0¡p,q¡1, representing the probability of each state that the receiver believes, given the sender's action.

Given both players maximizers, we have the following sequential rationality:

1. Receiver: $\max_{a_k} \sum_{t_i \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k)$, where $a_k = a^*(m_j)$, and $\mu(t_i|m_j)$ represents the conditional probability that nature actually chooses t_i when the sender acts m_j.
2. Sender: $\max_{m_j} U_S(t_i, m_j, a_k) = \max_{m_j} U_S(t_i, m_j, a^*(m_j))$, where $m_j = m^*(t_i) = \mu_s(t_i)$
3. By Bayesian's Rule: $\mu(t_i|m_j) = \dfrac{p(t_i)\sigma(m_j|t_i)}{\sum_{t_i \in T} p(t'_i)\sigma(m_j|t'_i)}$, where $p(t_i)$ labels the probability of each nature state, and $\sigma(m_j|t_i)$ represents the conditional probability that the sender choose m_j at nature state t_i.

Given such sequential rationality distribution, it is shown that the receiver's belief should be consistent with sender's action.

Example A signaling game with specific payoffs is shown as below.

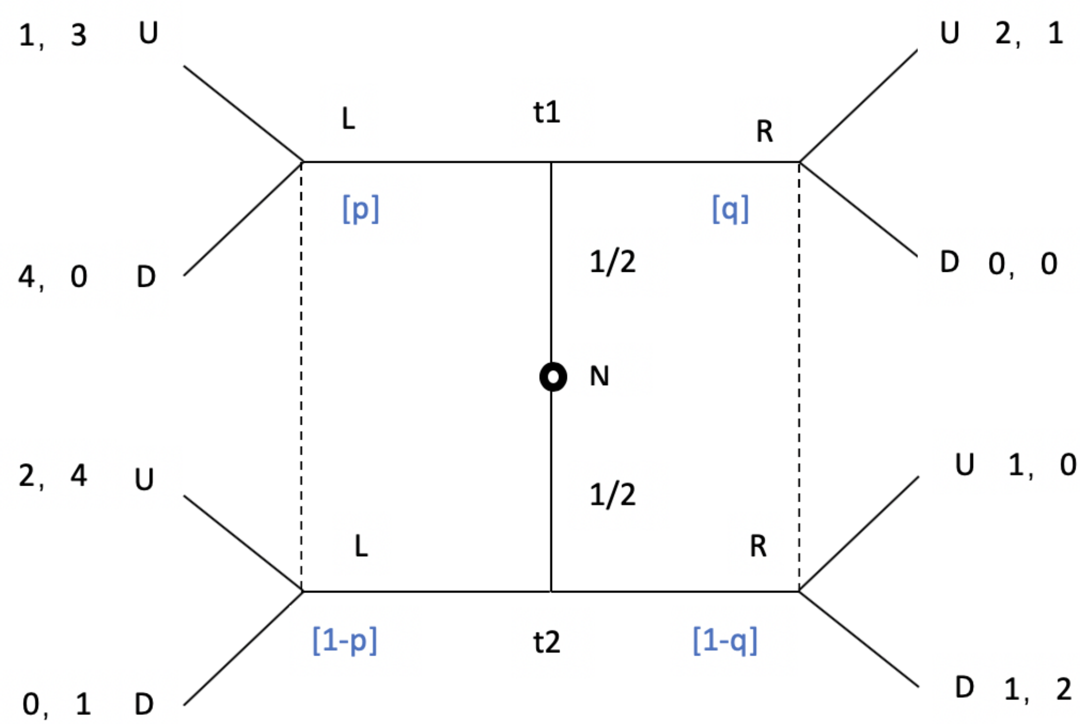

Figure 3: Example Signaling Game

Given the receiver's objective function is:

$$\max_{a_k} \sum_{t_i \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k) \tag{1}$$

1. When $m_j = L$, we have:

$$\max_{U,D} \mu(t_i|L)U_R + \mu(t_2|L)U_R = \max U, DpU_R + (1-p)U_R$$

$$\Rightarrow \max\{p \cdot 3 + (1-p) \cdot 4(U), p \cdot 0 + (1-p) \cdot 1(D)\}$$

$$\Rightarrow \max\{4-p, 1-p\} = 4-p$$

   Therefore, the receiver chooses U.

2. Similarly, when $m_j = R$, we have:

$$\max_{U,D} \mu(t_1|R)U_R + \mu(t_2|R)U_R$$

$$\Rightarrow \max\{q \cdot 1 + (1-q) \cdot 0(U), q \cdot 0 + (1-q) \cdot 2(D)\}$$

$$\Rightarrow \max\{q, 2-2q\}$$

   Therefore, the receiver chooses U if $q \geq 2 - 2q \Leftrightarrow q \geq \frac{2}{3}$, and D otherwise.

Also, given the sender's objective function:

$$\max_{L,R} U_S(t_i, m_j, a^*(m_j)) \tag{2}$$

1. When $t_i = t_1$:

$$\max_{L,R} U_S(t_1, m_j, a^*(m_j))$$

$$\Rightarrow \max\{U_S(t_1, L, a^*(L)) = 1, U_S(t_1, R, a^*(R)) = \begin{Bmatrix} 2 \ if \ q \geq \frac{2}{3} \\ 0 \ ifq < \frac{2}{3} \end{Bmatrix}$$

   Therefore, the sender chooses R if $q \geq \frac{2}{3}$, and chooses L otherwise.

2. When $t_i = t_2$:

$$\max\{U_S(t_2, L, a^*(L)), U_S(t_2, R, a^*(R))\}$$
$$\Rightarrow \max\{U_S(t_2, L, a^*(L)\} = 2$$

   Therefore, the sender choose L.

In summary of above results, we have:

$$m^*(t_2) = L, \ m^*(t_1) = \begin{cases} R \ if \ q \geq \frac{2}{3} \\ L \ if \ q < \frac{2}{3} \end{cases}$$

$$a*(L) = U, \ a*(R) = \begin{cases} U \ if \ q \geq \frac{2}{3} \\ D \ if \ q < \frac{2}{3} \end{cases}$$

Therefore:

1. $q \geq \frac{2}{3}$, the equilibrium is (R,L) for the sender, and (U,U) for the receiver.
2. $q < \frac{2}{3}$, the equilibrium is (L,L) for the sender, and (U,D)for the receiver.

Recall the Bayesian Probability formula:

$$\mu(t_i|m_j) = \frac{p(t_i)\sigma(m_j|t_i)}{\sum_{t_i \in T} p(t'_i)\sigma(m_j|t'_i)} \tag{3}$$

we can therefore calculate the receiver's belief, value of p and q, as below:

- p: $\mu(t_1|L) = \frac{\mu(L|t_1)p(t_1)}{\mu(L|t_1)p(t_1) + \mu(L|t_2)p(t_2)} = \begin{cases} \frac{0 \cdot 1/2}{0 \cdot 1/2 + 1 \cdot 1/2} = 0 \ if \ (R,L) \\ 1/2 \ if \ (L,L) \end{cases}$

- q: $\mu(t_1|R) = \frac{\mu(R|t_1)p(t_1)}{\mu(R|t_1)p(t_1) + \mu(R|t_2)p(t_2)} = \begin{cases} \frac{1 \cdot 1/2}{1 \cdot 1/2 + 0 \cdot 1/2} = 0 \ if \ (R,L) \\ \frac{0}{0+0} \Rightarrow Any \ Belief \ if \ (L,L) \end{cases}$

4 Related studies of PBNE and Signaling Game

4.1 Estimation Control

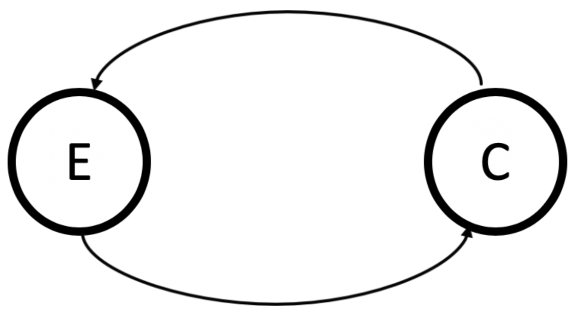

Figure 4: Causal loop of Estimation Control

4.2 Grandfather Paradox[2]

The grandfather paradox is a paradox of time travel in which inconsistencies emerge through changing the past. The name comes from the paradox's common description: a person travels to the past and kills their own grandfather before the conception of their father or mother, which prevents the time traveller's existence. Despite its title, the grandfather paradox does not exclusively regard the contradiction of killing one's own grandfather to prevent one's birth. Rather, the paradox regards any action that alters the past, since there is a contradiction whenever the past becomes different from the way it was.

4.3 Self-fulfilling Prophecy[3]

Self-fulfilling prophecy refers to the socio-psychological phenomenon of someone ”predicting” or expecting something, and this "prediction" or expectation comes true simply because one believes it will, and their resulting behaviors align to fulfil those beliefs. This suggests peoples' beliefs influence their actions. The principle behind this phenomenon is people create consequences regarding people or events, based on their previous knowledge toward that specific subject. Additionally, self-fulfilling prophecy is applicable to negative and positive outcomes.

5 Multistage N Person Game

In this section, we generalize the signaling game to a multistage N-person game:

1. N players
2. Each player has a type, $\theta_1, \theta_2, \ldots, \theta_N$, and $p(\theta) = \prod_{i=1}^{N} p_i(\theta_i)$.
3. t = 0, 1,...,T.

4. $a_i^t \in A_i(h^t)$, where h^t represents the history of time t. $h^t = (a^0, a^1, \ldots, a^{t-1})$, $a^t = (a_1^t, a_2^t, \ldots, a_N^t)$
5. No additional observation:
$$\mu_i(\theta_{-i}|h^t, \theta_i) = \prod_{j \neq i} \mu_i(\theta_j|h^t)$$
6. Behavioral Strategy: $\sigma_i(a_i|h^t, \theta_i)$
7. Payoffs: $U_i(h^{t-1}, \theta) = \sum_{t=0}^{k} U_i(t, \{\sigma_i(\cdot|h^t, \theta_i)\}_{i=1}^N, \{\mu_i(\cdot|h^t, \theta_i)\}_{i=1}^N)$
8. Dynamic Programming: Given $[\mu]$, analyze $\sigma_i[\mu]$. Often use backward induction decision, and refer to **Sequential Rationality**.
9. **Consistence of the belief**, Causal Forward Estimation.
$$\mu_i(\theta_j|h^t, a^t) = \frac{\mu_i(\theta_j|h^t)\sigma_j(a_j^t|h^t, \theta_j)}{\sum_{\bar{\theta_j} \in \theta_j} \mu_i(\bar{\theta_j}|h^t)\sigma_j(a_j^t|h^t, \bar{\theta_j})}$$
10. Structure:
    - $\mu_i(\theta_k|h^t, a^t) = \mu_j(\theta_k|h^t, a^t), \forall h^t, \theta_k,\ i \neq j \neq k$. Both estimating third-party information, and if information is the same, the estimation should be the same.
    - $\mu_i(\theta_j|h^t, a^t) = \mu_i(\theta_j|h^t, \hat{a^t})$ if $a_j^t = \hat{a_j^t}, \forall h^t, i, j$. $a^t = [a_1^t, \ldots, a_j^t, \ldots, a_N^t]$, and $\hat{a^t} = [a_1^t, \ldots, \hat{a_j^t}, \ldots, \hat{a_N^t}]$, so only a_j^t is the same for these two actions sets, other are different.

    Then, by assumption, they share the same estimation of player j.

Definition 5. *PBNE[1]*
(σ, μ) *is a Perfect Bayesian Nash Equilibrium(PBNE) if 8. and 9. satisfy.*

References

[1] Drew Fudenberg, Jean Tirole. *Game Theory*, ch6-7, pp207-314, 1991.

[2] `https://en.wikipedia.org/wiki/Grandfather_paradox`

[3] `https://en.wikipedia.org/wiki/Self-fulfilling_prophecy`