ECE-GY 6263 Game Theory Fall 2019

Lecture 5 — 10/4, 2019

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1 Overview

In the last lecture we talked about **Braess's paradox**, **correlated equilibrium** and the learning method (**No-regret learning**) leading to correlated equilibrium, and **extensive-form games**.

In this lecture we will continue to talk about **extensive-form games**, over which the definition of mixed strategies and behavioral strategies will be discussed, **stakelberg game**, along with **Kuhn's theorem**.

2 Extensive-form games

2.1 Games with imperfect information

Before we move on to give some formal definition about extensive-form game and information, let's start with considering the matching penny game to illustrate the game with imperfect information, where the normal form can be expressed as table 1.

		II: H	II: T
I	H	$(1, -1)$	$(-1, 1)$
I	T	$(-1, 1)$	$(1, -1)$

Table 1: matching penny

There are many ways of alternative extensive-form expression, two of them are as 1 and 2 show.

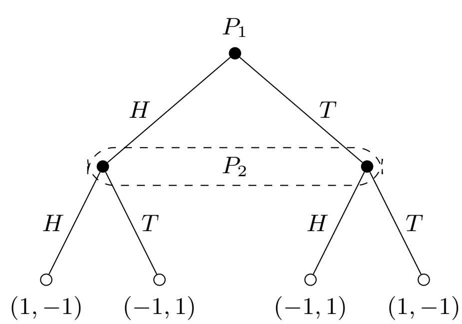


Figure 1: form 1

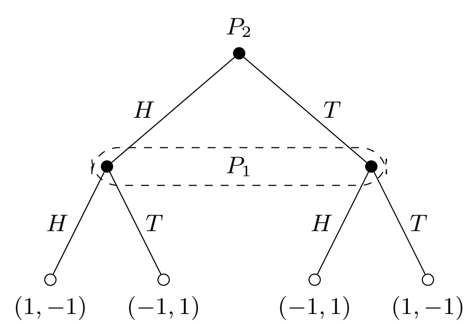


Figure 2: form 2

One can verify that the forms above are equivalent game, although the moves do not happen simultaneously, take form 1 for example, P_2 are not able to distinguish whether P_1 played H or

T, thus the game is equivalent to the bi-matrix game 1, while 1 does not carry all the information needed for description. To explicitly display the dynamic character of the problem, one has to define what signal device the players have access to, namely the information set, here we formly give the definition of extensive-form game with perfect information then information set.

Definition 1 (extensive-form games with perfect information and chance moves in [3])**.** *A game in extensive form (with perfect information and chance moves) is a vector*

$$G = \left(N, V, E, x^0, (V_i)_{i\in N\cup(0)}, (p_x)_{x\in V_0}, O, u\right)$$

where,

- *N is the set of players;*
- *(V, E, x^0) is the game tree.*
- *$(V_i)_{i\in N\bigcup\{0\}}$ is a partition of the set of vertices that are not leaves.*
- *For every vertex $x \in V_0, p_x$ is a probability distribution over the edges emanating from x.*
- *O is the set of possible outcomes.*
- *u is a function mapping each leaf of the game tree to an outcome in O.*

Definition 2 (information set/sets in [3])**.** *Let $G = \left(N, V, E, x^0, (V_i)_{i\in N\cup\{0\}}, (p_x)_{x\in V_0}, O, u\right)$ be an extensive-form game, an information set of player i is a pair $(\eta_i, A(\eta_i))$ such that:*

- *$\eta_i = \left\{x_i^1, x_i^2, \ldots, x_i^j\right\}$ is a subset of V_i that satisfies the property that at each vertex in η_i player i has the same number of actions $l_i = l_i(\eta_i)$, i.e.,*

$$\left|A\left(x_i^j\right)\right| = l_i, \quad \forall j = 1, 2, \ldots, m$$

- *$A(\eta_i)$ is ta partition of the ml_i edges $\bigcup_{j=1}^m A\left(x_i^j\right)$ to l_i disjoint sets, each of which contains one element from the sets $\left(A\left(x_i^j\right)\right)_{j=1}^m$. We denote the elements of the partition by $a_i^1, a_i^2, \ldots, a_i^{l_i}$. The partition $A(\eta_i)$ is called the action set of player i in the information set η_i.*

Roughly speaking, information set is a set of decision nodes such that[4]:

1. Every node in the set belongs to one player.
2. When the game reaches the information set, the player with the move cannot differentiate between nodes within the information set, i.e. if the information set contains more than one node, the player to whom that set belongs does not know which node in the set has been reached.

The notion of information set was introduced by John von Neumann, motivated by studying the game of Poker. With a little bit abuse of notations, here in 1 the **information set** for P_2 simply is $\eta_2 = \{H, T\}$, where H, T means the decision nodes coupled together.

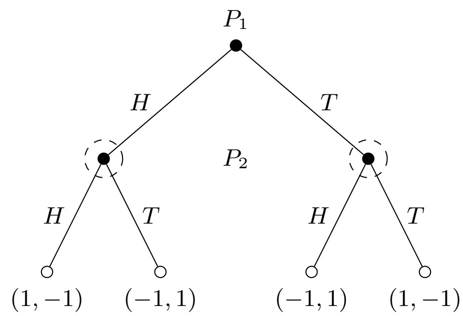


Figure 3: form 1

2.2 Extension of strategy space

The sequential play depicted by 1 can be modified as 3 shows.

Given the information set above, P_2 now is able to distinguish between different act of P_1, according to the "device" sitting somewhere indicating the action of P_1, denoted as u_1, P_2 now have different choice of strategy, which are:

$$\gamma_2^1 = \begin{cases} H & \text{if } u_1 = H \\ H & \text{if } u_1 = T \end{cases}$$

$$\gamma_2^2 = \begin{cases} T & \text{if } u_1 = H \\ T & \text{if } u_1 = T \end{cases}$$

$$\gamma_2^3 = \begin{cases} H & \text{if } u_1 = H \\ T & \text{if } u_1 = T \end{cases}$$

$$\gamma_2^4 = \begin{cases} T & \text{if } u_1 = H \\ H & \text{if } u_1 = T \end{cases}$$

Thus the normal form will be extended to table 2, where HH, HT, TH, TT denote respectively

		II			
		HH	HT	TH	TT
I	H	$(1, -1)$	$(-1, 1)$	$(1, -1)$	$(-1, 1)$
	T	$(-1, 1)$	$(1, -1)$	$(1, -1)$	$(-1, 1)$

Table 2: matching penny

γ_2^1, γ_2^2, γ_2^3, γ_2^4, one can see that the modification of the information set extends P_2's strategy space to a set of 4 pure strategies, and thus there's a dominate strategy for P_2, testifing the claim that **knowledge is power**. *when people have more choice, they have more freedom, which is usually the case but not generally true.*

The mixed strategy in extensive-form game is then mixing over the pure strategies, roughly speaking, assigning a distribution over the strategy space. We now give the formal definition.

Definition 3 (mixed strategy). *Let $G = \left(N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N}\right)$ be a strategic-form game(replace action space with strategy space in previous games) in which the set of strategies of each player is finite. A mixed strategy of player i is a probability distribution over his set of strategies Γ_i . Denote by*

$$\Delta_i^j := \left\{ p_j : \Gamma_i \to [0,1] \ s.t. \sum_{\gamma_i^j \in \Gamma_i} p_j(\gamma_i^j) = 1 \right\}$$

Note that this mixed strategy is for open-loop strategic-form game, which means we have to transform the extensive-form game into strategic-form game in order to get the strategies that it mixed over, moreover we can make extension to the strategic-form game in order to involve the payoff function, which we omit here.

Another important strategy form is behavioral strategy, which arises naturally from the information sets.

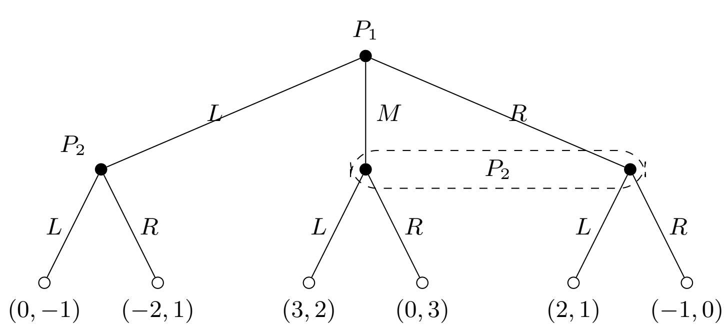
Definition 4 (behavioral strategy). *A behavioral strategy for player i is a function $\gamma_i(\cdot) : I_i \to \bigcup_{\eta_i \in I_i} \Delta(A(\eta_i))$ such that $\gamma_i(\eta_i) \in \Delta(A(I_i))$ for all $\eta_i \in I_i$, where I_i is the information sets, $\eta_i \in I_i$ is one of the information set, $A(\eta_i)$ is the action space at η_i, $\Delta(A)$ represents the set of all feasible distribution over action space A, this function maps each of his information sets to a probability distribution over the set of possible actions at that information set.*

To summarize, The player can randomly choose a pure strategy for the whole play at the outset of the game; this type of randomization yields in fact the concept of mixed strategy in an extensive-form game. Alternatively, at every one of his information sets, the player can randomly choose one of his available actions; this type of randomization yields the concept of behavior strategy.[3] Roughly a mixed strategy randomly chooses a **deterministic path** through the game tree, while a behavior strategy can be seen as a **stochastic path**.

2.3 Open-loop and Closed-loop equilibria

2.3.1 A single-act game with dynamic information

Consider following example of extensive-form game where both players are minimizers:



Here the information sets $I_2 = \{\{L\}, \{M, R\}\}$ and by backward induction we can easily find out the optimal behavioral strategy for P_2 and P_1, which are

$$\gamma_2^*(\eta_2) = \begin{cases} L & \text{if } \eta_2 = \{L\} \\ R & \text{otherwise} \end{cases} \qquad \gamma_1^*(\eta_1) = R$$

As we can write the normal form (or strategic-form) of subgame induced by $\eta_2 = \{M, R\}$:

I \ II	L	R
M	$(3, 2)$	$(0, 3)$
R	$(2, 1)$	$(-1, 0)^\star$

For which $\{R, R\}$ is the Nash equilibrium, the optimal payoff pair is $J_1^* = -1,\ J_2^* = 0$. The strategy space $\Gamma_1 = \{L, M, R\}$ and $\Gamma_2 = \{LL, LR, RL, RR\}$, denoting the following strategies

$$\gamma_2^1 = \begin{cases} L & \text{if } u_1 = L \\ L & \text{if } u_1 = M, R \end{cases}$$

$$\gamma_2^2 = \begin{cases} L & \text{if } u_1 = L \\ R & \text{if } u_1 = M, R \end{cases}$$

$$\gamma_2^3 = \begin{cases} R & \text{if } u_1 = L \\ R & \text{if } u_1 = M, R \end{cases}$$

$$\gamma_2^4 = \begin{cases} R & \text{if } u_1 = L \\ L & \text{if } u_1 = M, R \end{cases}$$

Thus the open-loop is then table 3

I \ II	LL	LR	RR	RL
L	$(0, -1)^\star$	$(0, -1)$	$(-2, 1)^\star$	$(-2, 1)$
M	$(3, 2)$	$(0, 3)$	$(0, 3)$	$(3, 2)$
R	$(2, 1)$	$(-1, 0)^\star$	$(-1, 0)$	$(2, 1)$

Table 3: open-loop

We observe that there are **3 NE**[1] in open-loop form but only **1 NE** according to the backward analysis of the sequential extensive-form, it can be remarked that the open-loop game is generally not equivalent to the feedback game, while 2 of the NE are for "open-loop" solutions, and 1 equilibrium is from **"backward induction"** so that we are selecting equilibrium according to some "nice feature", which are:

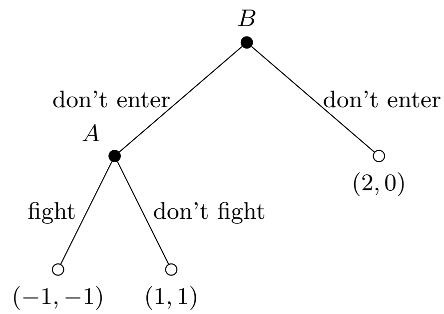
1 The NE is credible, which means that it represents a Nash equilibrium of every subgame of the original dynamic game.

2 It satisfies trembling hands property, which means the equilibrium takes in the negligible possiblities of players choosing unintended actions.

[1]Nash Equilibria

2.3.2 Market entry game

Consider the situation when firm B is deciding whether to enter the market which A already occupied, both players are maximizers.



The strategy spaces $\gamma_1 \in \{E, D\}$, $\gamma_2 \in \{F, A\}$, and the open-loop form is table 4

		II	
		E	D
I	F	$(-1, -1)$	$(2, 0)^\star$
	A	$(1, 1)^\star$	$(2, 0)$

Table 4: enter/not enter

While (A, E) and (F, D) are both NE, (F, D) is not credible because knowing the outcome ahead of time, B will always choose to enter, which is not actually a unilaterely deviation since A will always choose not to fight after B enters.

3 Stakelberg game[1]

As stated in [1], the Nash equilibrium solution concept provides a reasonable noncooperative equilibrium solution for nonzero-sum games when the roles of the players are **symmetric**, i.e., when no single player **dominates** the decision process. However, there are yet other types of noncooperative decision problems wherein one of the players has the ability to enforce his strategy on the other player(s), and for such decision problems one has to introduce a **hierarchical equilibrium solution concept**.

3.1 Stakelberg game with ambiguity

The market entry game discussed before can be viewed as a case of such game where B is dominating the decision process, and A's optimal reaction to B is unique. But what if the followers have multiple optimal strategy? Let's do a thought experiment.

Thought Experiment. *Given the stakelberg game* 4 *where both players are minimizers, is there any existence of Stakelberg equilibria?*

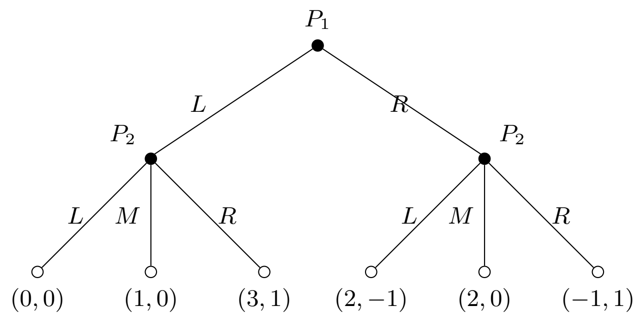


Figure 4: Stakelberg game

		II		
		L	M	R
I	L	$(0,0)^\star$	$(1,0)^\star$	$(3,1)$
	R	$(2,-1)$	$(2,0)$	$(-1,1)$

Table 5: Stakelberg game

The answer is yes if we stipulate that the leader's attitude is towards securing his possible losses against the choices of the follower within the class of his optimal responses, rather than towards taking risks. Then, under such a mode of play, P_1's secured cost level corresponding to his strategy L would be 1,and the one corresponding to R would be 2. **In contracdiction to NE, for which the saddle point solutions in mixed strategy always exist, the solutions for SE [2] might not exist in mixed strategy, but always exist in pure strategy.**

Now we provide the precise definition of Stakelberg solution within the context of 2-person finite games which do not incorporate chance moves.

Definition 5. *(Rational response) Optimal Response set of P_2 to strategy $\gamma_1 \in \Gamma_1$ of P_1 is:*

$$R_2(\gamma_1) = \{\xi \in \Gamma_2 : J_2(\gamma_1, \xi) \leqslant J_2(\gamma_1, \gamma_2)\} \tag{1}$$

R_2 is nonempty if the game is finite.

Definition 6. *In a two-person finite game with P_1 as the leader, a strategy $\gamma_1 \in \Gamma_1$ is called a Stakelberg equilibrium for the leader if*

$$\max_{\gamma_2 \in R_2(\gamma_1)} J_1(\gamma_1^*, \gamma_2) = \min_{\gamma_1 \in \Gamma_1} \max_{\gamma_2 \in R_2(\gamma_1)} J_1(\gamma_1, \gamma_2) \equiv J_1^* \tag{2}$$

where J_1^* is the equilibrium cost of the leader.

Remark. *The Stackelberg strategy for the leader does not necessarily have to be unique. But nonuniqueness of the equilibrium strategy does not create any problem here (as it did in the case of Nash equilibria), since the Stackelberg cost for the leader is unique.*

Remark. *If $R_2(\gamma_1)$ is a singleton set for each $\gamma_1 \in \Gamma_1$, then there exists a mapping $T_2 : \Gamma_1 \to \Gamma_2$ such that $\gamma_2 \in R_2(\gamma_1)$ implies that $\gamma_2 = T_2(\gamma_1)$, This corresponds to the case when the optimal*

[2] Stakelberg Equilibrium

response of the follower (which is T_2) is unique for every strategy of the leader, and it leads to the following simplified version:

$$J_1(\gamma_1^*, T_2(\gamma_1^*)) = \min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, T_2(\gamma_1)) \equiv J_1^* \tag{3}$$

Thus J_1^* becomes a cost value that the leader can truely attain.

Definition 7 (Stakelberg outcome)**.** *Let $\gamma_1^* \in \Gamma_1$ be a Stakelberg strategy for leader P_1, then, any element $\gamma_2^* \in R_2(\gamma_1^*)$ is an optimal strategy for the followers, $\{\gamma_1^*, \gamma_2^*\}$ is a Stakelberg solution for the game with P_1 as the leader and $J_1(\gamma_1^*, \gamma_2^*)$ and $J_2(\gamma_1^*, \gamma_2^*)$ are the corresponding Stakelberg outcome.*

Remark. *In the preceding definition, the cost level $J_1(\gamma_1^*, \gamma_2^*)$ could in fact be lower than the Stackelberg cost J_1^*, a feature which has already been observed within the context of the example. However, if $R_2(\gamma_1^*)$ is a singleton, then these two cost levels have to coincide.*

3.2 Mixed strategies for Stakelberg games

In the case of the Stackelberg solution of two-person finite games an equilibrium always exists, there seems to be no need for making extension to mixed strategies or behavioral strategies. Such an argument, however, is not always valid as in [1], and there are cases when the leader can actually do better (in the average sense) with a proper mixed strategy, than the best he can do within the class of pure strategies.

3.2.1 A zero-sum single-act game

Consider game 5, which has a subgame in normal form 6

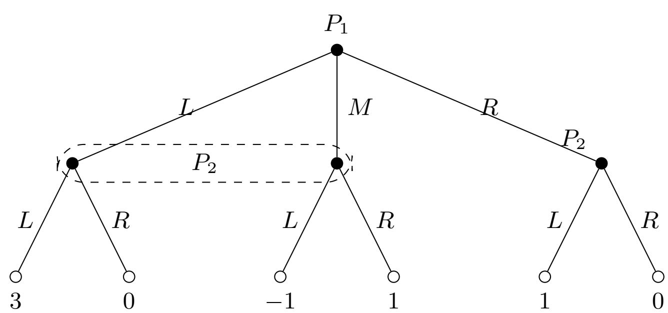


Figure 5: Zero-sum stakelberg game

		II: L	II: R
I	L	3	0
I	M	−1	1

Table 6: Zero-sum stakelberg game

The strategy space for P_2 is $\Gamma_2 = \{LL, LR, RL, RR\}$ and the elements are representing $\gamma_2^1,\ \gamma_2^2,\ \gamma_2^3,\ \gamma_2^4$ respectively, where

$$\gamma_2^1 = \begin{cases} L & \text{if } u_1 = L, M \\ L & \text{if } u_1 = R \end{cases}$$

$$\gamma_2^2 = \begin{cases} L & \text{if } u_1 = L, M \\ R & \text{if } u_1 = R \end{cases}$$

$$\gamma_2^3 = \begin{cases} R & \text{if } u_1 = L, M \\ R & \text{if } u_1 = R \end{cases}$$

$$\gamma_2^4 = \begin{cases} R & \text{if } u_1 = L, M \\ L & \text{if } u_1 = R \end{cases}$$

The subgame admits the mixed saddle-point solution $(y_L^* = \frac{2}{5}, y_M^* = \frac{3}{5})$ for P_1 and $(z_L^* = \frac{1}{5}, z_R^* = \frac{4}{5})$ for P_2, So the optimal strategy for P_1 is

$$\hat{\gamma}_1^* = \begin{cases} L & \text{w.p. } \frac{2}{5} \\ M & \text{w.p. } \frac{3}{5} \\ R & \text{w.p. } 0 \end{cases}$$

and the rational response for P_2 is

$$\hat{\gamma}_2^* = R_2(\hat{\gamma}_1) \begin{cases} L \quad \text{w.p. } 1 \quad \text{if } u^1 = R \\ \left.\begin{array}{ll} L & \text{w.p. } \frac{1}{5} \\ R & \text{w.p. } \frac{4}{5} \end{array}\right\} \text{otherwise} \end{cases}$$

And they satisfy

$$\hat{J}(\hat{\gamma_1^*}, \hat{\gamma_2}) \leqslant \hat{J}(\hat{\gamma_1^*}, \hat{\gamma_2^*}) \leqslant \hat{J}(\hat{\gamma_1}, \hat{\gamma_2^*}) \qquad \forall \hat{\gamma_1^*} \in \hat{\Gamma}_1,\ \hat{\gamma_2^*} \in \hat{\Gamma}_2 \tag{4}$$

3.3 Behavioral strategies (Kuhn's Theorem)

Again consider the example in 3.2.1, but seeking a solution in behavioral strategy.

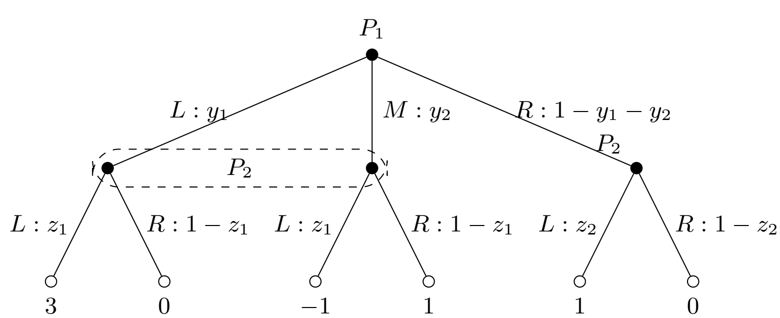


Figure 6: Zero-sum stakelberg game

Thus the payoff is a probabilistic weighted sum of all the path outcome,

$$\hat{J}(y_1, y_2, z_1, z_2) = 3 \cdot y_1 \cdot z_1 + 0 \cdot y_1 \cdot (1 - z_1) + (-1) \cdot z_1 \cdot y_2 + 1 \cdot y_2 \cdot (1 - z_1) + 1 \cdot (1 - y_1 - y_2) \cdot z_2$$

The problem

$$\min_{y_1, y_2} \max_{z_1, z_2} \hat{J}(z_1, z_2, y_1, y_2)$$

can be easily solved by taking partial deravative and induces the optimal solution $(y_1 = \frac{2}{5},\ y_2 = \frac{3}{5}),\ 1 - y_1 - y_2 = 0)$ and $(z_1 = \frac{1}{5},\ 1 - z_1 = \frac{4}{5},\ z_2 = 1,\ 1 - z_2 = 0)$ which is the same as mixed strategy. Thus we observed that these 2 strategies are equivalent and give rise to **Kuhn's theorem**.

Theorem 8 (Kuhn 1957)**.** *In every game in extensive form, if player i has* **perfect recall***, then for every mixed strategy of player i, there exists an equivalent behavioral strategy.*

Definition 9 (Perfect recall[3])**.** *Player i has perfect recall if the following conditions are satisfied:*

- *a Every information set of player i intersects every path from the root to a leaf at most once.*
- *b Every two paths from the root that end in the same information set of player i pass through the same information sets of player i, and in the same order, and in every such information set the two paths choose the same action. In other words, for every information set* η_i *of player i and every pair of vertices* $x, \hat{x}$ *in* η_i*, if the decision vertices of player i on the path from the root to* x *are* $x_i^1, x_i^2, ..., x_i^L = x$ *and his decision vertices on the path from the root to* $\hat{x}$ *are* $\hat{x}_i^1, \hat{x}_i^2, ..., \hat{x}_i^{\hat{L}} = \hat{x}$*, then* $\hat{L} = L$*, and* $\eta_i(x_i^l) = \eta_i(\hat{x}_i^l)$*, and* $a_i(x_i^l \to x) = a_i(\hat{x}_i^l \to \hat{x})\ \ \forall l \in \{1, 2, ..., L\}$*.*

A game is called a game with perfect recall if all the players have perfect recall.

Intuitively, a basically states that player i "remember" what stage he has steped onto if an oracle sitting somewhere tells him which path he was in; While b states that the moves that make him finally fall into the same situation are the same for every trajectory, that is to say, no matter what path he was in, he knows exactly what he had done before. A simple example of imperfect recall is the absent driver game as illustrated by 7.

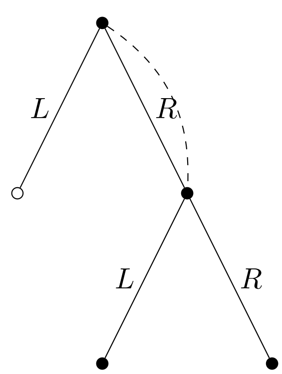


Figure 7: Absent driver game(imperfect recall)

Remark. *Showing Kuhn's theorem can be sketched as 3 steps:*

- *Suppose there's a mixed strategy p_i, let x be a vertex in η_i, $\forall a_i \in A(x)$, the collection $\Gamma_i^*(x^{a_i})$ contains all the pure strategy $\gamma_i(\cdot)$ s.t. $\gamma_i(\eta_i) = a_i$ define a behavioral strategy b_i: The behavioral strategy is defined as:*

$$b_i\left(a_i; \eta_i\right) := \frac{\sum_{\gamma_i \in \Gamma_i^*(x^{a_i})} p_i\left(\gamma_i\right)}{\sum_{\gamma_i \in \Gamma_i^*(x)} p_i\left(\gamma_i\right)}, \quad \forall a_i \in A(x)$$

  thus representing the conditional probabilty of playing a after reaching x;

- *Prove $b_i(\eta_i)$ is a behavioral strategy(i.e. a probabilistic distribution) $\forall \eta_i \in I_i$ over the action space $A(\eta_i)$, which boils down to showing that the summation over $A(\eta_i)$ is 1.*

- *Showing that b_i is equivalent to γ_i, equivalent to showing that their probabilities of visiting every vertex are the same.*

An elaborate proof can be found in **[3] chapter 6. Behavior strategies and Kuhn's Theorem***, along with exhaustive definitions regarding equivalence between behavioral and mixed strategy.*

Kuhn's theorem answer many questions induced by the confusion of mixed/behavioral strategies, while in practice using behavioral strategies might be **cheaper** (don't have to enlarge the strategy space, just mix over action space thus require fewer parameters) and **more "natural"** (players choose randomly between their actions at each information set at which they find themselves, rather than making one grand random choice of a "master plan" (i.e., a pure strategy) for the entire game, all at once[3].)

4 Lecture Scribe Revisit

This part will be a supplement of this lecture, showing some successive relevant results.

4.1 Equilibria in Behavior Strategy

As the definition of **perfect recall** and **Kuhn's theorem** is given, people kept thinking if there's a Nash Equilibrium in behavioral strategy, the answer is yes when the game satisfies perfect recall property. In other words, when all players have perfect recall, as there exists a vector of mixed strategies under which no player has a profitable deviation to another mixed strategy, by equivalence of these two kinds of strategy, an equilibrium in behavior strategies is a vector of behavior strategies under which no player has a profitable deviation to another behavior strategy.

Theorem 10 (existence)**.** *If all the players in an extensive-form game have perfect recall then the game has a Nash equilibrium in behavior strategies.*

The proof is immediately following Nash's theorem in finite extensive-form game with finite player and finite action space, when a game has perfect recall, at each equilibrium in mixed strategies no player has a profitable deviation to a behavior strategy, and at each equilibrium in behavior strategies no player has a profitable deviation to a mixed strategy.

4.2 Kuhn's Theorem for infinite games

Since there are sequential move games with infinite game tree, such as in backgammon and Monopoly where there exists a infinitely long path that may never end, we wonder if Kuhn's theorem can also be generalized to such games, there are two technique challenges in doing such:

- Since the set of pure strategies has a cardinality of a continunn, we need to define a -algebra over the collection of all pure strategies in order to be able to define probability distributions over this set.
- Instead of probability distributions induced over a finite set of leaves, in the case of an infinite game we need to deal with probability distributions induced over the set of paths in the game tree, which as we showed above has the cardinality of the continuum. This requires defining a measurable space over the set of plays of the game, that is, over the set of paths (finite and infinite) starting at the root of the tree.

4.2.1 Machineries

A bunch of definitions and theorems would be helpful.

Definition 11 (σ-algebra)**.** *Let X be a set. A collection Y of subsets of X is a σ-algebra over X if*

a. $\emptyset \in Y$*,*

b. $X \backslash Y \in Y \ \forall Y \in \mathcal{Y}$*(closed under complement),and*

c. $\cup_{i \in \mathbb{N}} Y_i \in \mathcal{Y}$*(closed under countable unions)*

A σ-algebra is closed under complement and countable unions, there are at least three key motivators for σ-algebras: defining measures, manipulating limits of sets, and managing partial information characterized by sets. De Morgan's Laws imply that a σ-algebra is also closed under countable intersections.

Definition 12. *Let $(X_n)_{n \in \mathbb{N}}$ be sequence of finite sets, and let $X^\infty := \times_{n \in \mathbb{N}} X_n$. A set $B \in X^\infty$ is called a cylinder set if there exist $N \in \mathbb{N}$ and $(A_n)_{n=1}^{N}, A_n \subseteq X_n$ for all $n \in \{1, 2, \ldots, N\}$, such that $B = \left(\times_{n=1}^{N} A_n\right) \times \left(\times_{n=N+1}^{\infty} X_n\right)$. The σ -algebra of cylinder sets is the σ-algebra $\mathcal{Y}$ generated by the cylinder sets in X^∞ .*

Cylinder set is useful in such tree topology to define a probability distribution over a product space of finite and infinite sets, as the σ-algebras will be generated from the cylinder sets over the infinite product space.

Definition 13 (measurable space and its probability function)**.** *A measurable space is a pair $(X, \mathcal{Y})$ such that X is a set and $\mathcal{Y}$ is a σ-algebra over X. A probability distribution over a measurable space $(X, \mathcal{Y})$ is a function $p : \mathcal{Y} \to [0, 1]$ satisfying:*

a. $p(\emptyset) = 0$,

b. $p(X \backslash Y) = 1 - p(Y)$ *for every* $Y \in \mathcal{Y}$*,and*

c. $p\left(\cup_{n\in\mathbb{N}} Y_n\right) = \sum_{n\in\mathbb{N}} p\left(Y_n\right)$ *for any sequence* $\left(Y_n\right)_{n\in\mathbb{N}}$ *of pairwise disjoint sets in* $\mathcal{Y}$

Theorem 14. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of finite sets. Suppose that for each* $N \in \mathbb{N}$ *there exists a probability distribution* p^N *over* $X_N := \times_{n=1}^{N} X_n$ *that satisfies*

$$p^N(A) = p^{N+1}\left(A \times X_{N+1}\right), \quad \forall N \in \mathbb{N}, \forall A \subseteq X^N$$

Let $X^\infty := X_{n\in\mathbb{N}} X_n$ *and let* $\mathcal{Y}$ *be the* σ *-algebra of cylinder sets over* X^∞*. Then there exists a unique probability distribution* p *over* $(X^\infty, \mathcal{Y})$ *extending* $\left(p^N\right)_{N\in\mathbb{N}}$*, i.e.*

$$p^N(A) = p\left(A \times X_{N+1} \times X_{N+2} \times \cdots\right), \quad \forall N \in \mathbb{N}, \forall A \subseteq X^N$$

Thus the example in tree structured infinite space can be given as following theorem:

Theorem 15. *Let* $\left(V, E, x^0\right)$ *be a (finite or infinite) game tree such that* $|C(x)| < \infty$*(set of children) for each vertex* x*. Denote by* H *the set of maximal paths. Let* $p : V \to [0, 1]$ *be a function satisfying* $p(x) = \sum_{x'\in C(x)} p\left(x'\right)$ *for each vertex* $x \in V$ *that is not a leaf. Then there exists a unique probability distribution* $\widehat{p}$ *over* $(H, \mathcal{H})$ *satisfying* $\widehat{p}(H(x)) = p(x)$ *for all* $x \in V$*.*

The proof is directly following theorem 14, as one only need to find out the corresponding relation in order to show there's an unique probability distribution that extends the distribution over verices, and We have also to make use of the theorem which states that if there is an infinite tree such that each vertex x in the tree has an associated probability distribution $p(x)$.

4.2.2 Definition of pure strategy, mixed strategy, and behavioral strategy

- A pure strategy of player i is a function that associates each information set of player i with a possible action at that information set, the set of strategies: $S_i = \times_{I_i\in\mathcal{I}_i} A\left(I_i\right)$
- Mixed strategies need a σ-algebra to be defined, let $\mathcal{S}_\rangle$ denote the cylinder sets of S_i, Thepair $(S_i, \mathcal{S}_\rangle)$ is a measurable space and the set of probability distributions over it is the set of mixed strategies Σ_i of player i.
- A behavioral strategy is easy to define as it's just a function associating each one of player's information sets with a probability distribution over the set of possible actions at that information set: $\mathcal{B}_i = \times_{I_i\in\mathcal{I}_i} \Delta\left(A\left(I_i\right)\right)$

4.2.3 Equivalence between mixed strategies and behavior strategies

Step by step, In a finite game of depth T, a mixed strategy σ_i^T is equivalent to a behavior strategy b_i^T if: $\rho\left(x; \sigma_i^T, \sigma_{-i}^T\right) = \rho\left(x; b_i^T, \sigma_{-i}^T\right)$ for every mixed/behavior strategy vector σ_{-i}^T of the other players and every vertex x in the game tree. In this section we extend the definition of equivalence between mixed and behavior strategies to infinite games.

The term $\rho_i(x;\sigma_i)$ and $\rho_i(x;b_i)$ are the probability player i uses either strategy leading to the vertex x at each vertex along the path from the root to x that is in his information sets.

Behavioral strategy:

$$\rho_i(x;b_i) := \Pi_{l=1}^{L_i^x} b_i\left(a_i; I_i^l\right)$$

where L_i^x is the number of vertices along the path from the root to x that are in player i's information sets.

Mixed strategy:

For any $T \in \mathbb{N}$ let G^T be the game that includes the first T stages of the game G.

- The set of vertices V^T of G^T contains all vertices of G with depth at most T.
- The information sets of each player i in G^T are all nonempty subsets of V^T that are obtained as the intersection of an information set in $\mathcal{I}_i^T$ with V^{T-1}

Thus define for each vertex x:

$$\rho_i(x;\sigma_i) := \rho_i\left(x;\sigma_i^T\right)$$

where T is greater than or equal to the depth of x. Finally, define, for each mixed/behavior strategy vector σ,

$$\rho(x;\sigma) := \prod_{i \in N} \rho_i(x;\sigma_i)$$

This is the probability that the play of the game reaches vertex x when the players implement the strategy vector σ.

Following theorem states that every vector of strategies uniquely defines a probability distribution over the set of infinite plays.

Theorem 16. *Let σ be a mixed/behavioral strategy vector in a (finite or infinite) extensive-form game.Then there exists a unique probability distribution μ_σ over $(H, \mathcal{H})$ satisfying $\mu_\sigma(H(x)) = \rho(x;\sigma)$ for every vertex x.*

Definition 17. *A mixed strategy i of player i is equivalent to a behavior strategy b_i of player i if, for every mixed/behavior strategy vector σi of the other players, $\mu_{(\sigma_i,\sigma_{-i})} = \mu_{(b_i,\sigma_{-i})}$*

Theorem 18. *A mixed strategy σ_i of player i is equivalent to his behavior strategy b_i if for every mixed/behavior strategy vector σi of the other players and every vertex x we have: $\rho(x;\sigma_i,\sigma_{-i}) = \rho(x;b_i,\sigma_{-i})$*

4.2.4 Kuhn's Theorem for infinite games

If player i has perfect recall in a game G, then he also has perfect recall in the game G^T for all $T \in \mathbb{N}$

Theorem 19. *Let G be an extensive-form game with an infinite game tree such that each vertex in the game tree has a finite number of children. If player i has perfect recall, then for each mixed strategy of player i there is an equivalent behavior strategy and for each behavior strategy of player i there is an equivalent mixed strategy.*

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