Introducing Maxwell's Equations as Derived from Simple Relativity Transformation Principles

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***Abstract*—The paper presents a simple, rigorous approach to introduce electromagnetic theory, based on a new fundamental form of the Gauss' Laws using only the basic space-time transformation relations of the special relativity. More advanced concepts of relativistic mechanics, such as transformation of mass, momentum and force, are not necessary in this development, resulting in a fundamentally simple theory. The material would fill an educational need to introduce the modern theory of relativity in teaching engineering electromagnetics. The approach would be suitable for a senior (even junior) undergraduate or an introductory graduate engineering class, depending on the level of depth and rigor.**

I. Introduction

Traditionally, the basic level texts on engineering electromagnetics [1], [2], [3] have used the Maxwell's equations [4] under different special, simplified conditions, or in the complete form, as the starting point to study various electromagnetic fields and radiation problems. However, connecting Maxwell's equations to the special theory of relativity [5], [6] is not usually explored in the basic engineering texts. Simple relativistic treatment of the electromagnetic fields are available in some physics text books [7], [8] which are meant only to initiate an interesting, alternate mode of field analysis, but not really to provide a comprehensive derivation of Maxwell's equations from the relativistic theory. More involved relativistic treatments of Maxwell's equations at higher levels are available in [8], [9], [10], which can be mathematically and/or conceptually tasking, making them inaccessible to introductory, even some advanced level students. All the available approaches assume a decent level of prior study of the theory of special relativity, including mass, momentum and force transformation relations [11], [12]. That may be fine for advanced physics students, but usually not for introductory - even advanced - level engineering students. For the varied reasons, all the available approaches are inaccessible and unsatisfactory, and therefore have been rarely used in introductory engineering classrooms. Contributions in the general context of introducing special relativity to engineering electromagnetics include [13]-[15].

It would be valuable for students of introductory engineering electromagnetics to study the theory of special relativity at some essential level, and learn its direct connection to Maxwell's equations to provide a modern, alternative perspective of the electromagnetic principles. The new perspective would make Maxwell's equations less mysterious to understand, enriching the introductory learning process. The potential benefits may justify introduction of the new teaching, only if the material can be presented economically in a comprehensive manner, with minimal distraction into the involved mechanical principles of the special relativity. This is particularly considering the limited time and resources available for the engineering electromagnetics in a typical undergraduate electrical engineering curriculum.

To this end, we present a new theory to introduce the concepts of the special relativity at an early, basic level of teaching engineering electromagnetics. This is intended to provide an alternate interpretation as well as a complete "derivation" of Maxwell's equations. This is accomplished through basic relativistic transformations of the current continuity relations (section IV) and the Gauss' Laws (sections V, VI). A new form of the Gauss' Laws are introduced in the derivations, required in order to definitively enforce invariance of any electric or magnetic charge, across reference frames, as discussed in section VII.

It maybe noted, this fundamental approach to derive Maxwell's equations from basic physical principles is unlike the actual historical development of Maxwell's equations [4], [16], which were established through a series of experimental discoveries and their gradual mathematical comprehension, without physical understanding of their underlying fundamental origin. Interestingly, Maxwell's equations, which led to a fundamental understanding of the nature of light, historically served as the foundation for subsequent deduction of the space-time relations of special relativity [5], [6]. However, the space-time relations were then realized to be more fundamental, which could be simply established based on an independent understanding of the nature of light propagation, which in turn are used now to actually derive Maxwell's equations by assuming an invariant nature of charge. This turn of understandings, leading to the direct derivation of Maxwell's equations, is a significant development.

Before any rigorous derivation of Maxwell's equations, it may be useful to first justify and establish proper space-time relations between two reference frames, in section II, in order to be consistent with the special nature of light propagation in the empty space. This would be followed by a simple physical example in section III, introduced to inspire initial curiosity, where it is shown that the force applied on a given charge due to the electric field alone would lead to inconsistency, when viewed across two reference frames. Based on the space-time relations established in the section II, this example would motivate introduction of a suitable new field, referred to as the magnetic field, in order to resolve such inconsistency. Prompted by the initial curiosity in the above specific example, rigorous derivations for general conditions are presented in the subsequent sections, leading to the rigorous derivation of Maxwell's equations.

II. Special Nature of Light, and Relativistic Space-Time Transformation

It is a special nature of light (any electromagnetic wave), that it can propagate in the empty space without any material medium. This is unlike any mechanical wave such as a sound wave or a water ripple, which always propagates in a specific material body such as air or water (ideally with no internal relative motion), respectively, with respect to which the wave's speed is preferentially fixed to a particular value. In contrast, the empty space in which light propagates, due to the very nature of the empty space, is not "attached" to any particular body of reference. Therefore, the speed of the light wave, propagating in the empty space, could not be preferentially fixed with respect to any particular reference body or frame. In other words, whatever is the speed of light in the empty space, when it is observed at a given location and time with respect to any given frame, there is no rational basis why the same speed would not as well be observed with respect to any other frame. This would require that the speed of light in the empty space be fundamentally fixed to the same value, c, as observed from all reference frames, that are moving at any possible speed with respect to one another. Further, assuming that the empty space medium is ideally the same in its nature, at all locations and at all times, the above observer-independent fixed speed, c, of light propagation in the empty space would also be a fixed constant independent of its location and time of observation.

It may be noted, that the above rational understanding of the nature of light propagation in the empty space was historically far from evident. A long-held belief that the empty space could possibly consist of a mysterious "ether" medium, in which light would propagate with a preferential speed, had to be abandoned only after a definitive experiment by Michelson and Morley [17].

The space-time relations between reference frames must be properly adjusted, in order to accommodate the above fundamental requirement of independence of the light speed. The same space-time relations must be applicable as well in observation of any other physical phenomena, not just for light, in order to maintain universal consistency. The required space-time relations, between two frames that are moving with a given speed V with respect to each other along a particular direction x, maybe established mathematically by assuming suitable linear relationships between the space-time variables, and then enforcing certain basic constraints. The frames maybe referred to as primed and unprimed frames, respectively represented by primed and unprimed space-time variables, where the primed frame is assumed to move with a velocity V along the positive x direction, with respect to the unprimed frame. The relative motion along the x direction would bias any relative measurement of only the x coordinate, keeping the other coordinates along the y and z directions unchanged.

$$x = ax' + bt',\ t = gx' + ht',\ y = y',\ z = z'. \tag{1}$$

The above choice of the linear relations assumes that the reference coordinates $(x, y, z, t) = (0, 0, 0, 0)$ in the unprimed frame are initialized with the coordinates $(x', y', z', t') = (0, 0, 0, 0)$ in the primes frame. The linearity constants a, b, g and h may be solved by enforcing the following necessary requirements: (1) The origin $x = 0$ in the unprimed coordinate must move with velocity $-V$ as measured in the primed frame. Accordingly, for $x = 0$ we must have $x' = -Vt'$, which would relate $b = Va$. (2) Conversely, for $x' = 0$, we must have $x = Vt$, which would relate $b = Vh$, leading to $a = h$ using the first relation. (3) For a point (x', t') moving with the light speed c in the primed frame, that is $x' = ct'$, we must have the corresponding point (x, t) in the unprimed frame to also move with the same light speed c, that is $x = ct$. This would relate $ca + b = c(cg + h)$, which would then lead to $g = (V/c^2)h$ by combining with the above two relations. (4) For $x' = 0$, we have $t = ht'$, and by symmetry we also must have $t' = ht$ for $x = 0$. Combining with the above three relations, this would complete solutions for all linearity constants with $a = b/V = g(c^2/V) = h = 1/\sqrt{(1 - (v/c)^2)}$.

$$\begin{gathered} x = 0, x' = -Vt', b = Va; \\ x' = 0, x = Vt, b = Vh; a = h = b/V; \\ x' = ct', x = ct, cat' + bt' = c(cgt' + ht'); \\ ch + Vh = c(cg + h), g = (V/c^2)h; \\ x' = 0, t = ht'; x = 0, t' = ht, t' = h(-g(b/a)t' + ht'), \\ 1 = h^2(1 - (V/c)^2),\ h = \frac{1}{\sqrt{1-(V/c)^2}}. \end{gathered} \tag{2}$$

$$\begin{gathered} x = \frac{x'+Vt'}{\alpha},\ t = \frac{t'+x'V/c^2}{\alpha},\ y = y',\ z = z', \\ \alpha = \sqrt{1 - \frac{V^2}{c^2}}. \end{gathered} \tag{3}$$

Similarly, the inverse-relationship for the prime variables in terms of the unprimed variables may also be expressed, by simply substituting V by $-V$ and interchanging the primed and unprimed variables in the above space-time relations.

$$x' = \frac{x-Vt}{\alpha},\ t' = \frac{t-xV/c^2}{\alpha},\ y' = y,\ z' = z. \tag{4}$$

Our initial assumption of the liner relationships (1) between the space-time variables would be mathematically valid, in principle, to model any possible general space-time relations, linearized for sufficiently small ranges of distance and time duration. However, the consequent final results (3,4) now confirm the initial assumption of linearity, if the relative velocity V remains fixed, in which case the initial linearity assumption (1) and the consequent results (3,4) may be declared to be rigorously valid for any general range of distance and time duration. The rigorous conclusion is based on the light speed c to be a fixed constant independent of space and time, which we have already established earlier as a fundamental condition.

III. A Line Current as Viewed from Two Reference Frames: The Need for a Magnetic Field

Fig.1 shows a charge q at a distance r from a line current, which is a superposition of a positive and a negative line-charge distributions, as viewed from two reference frames with a relative velocity V between the frames. In the primed frame, the charge q is stationary, and the line charge densities $+\rho'_{l1}$ and $-\rho'_{l2}$ are selected to be negative of each other, leading to the total line charge density $\rho'_l = \rho'_{l1} - \rho'_{l2} = 0$. This results in zero total electric field $\overline{E}' = 0$, and therefore zero total force $\overline{F}' = q\overline{E}' = 0$ on the stationary charge q. Accordingly, when viewed in the unprimed frame, the same charge q, which is now moving with a velocity V, should also be expected to experience zero total force $\overline{F} = \overline{F}' = 0$.

The positive line-charge is selected to be stationary in the primed frame, whereas the negative line-charge is selected to be stationary in the unprimed frame. Accordingly, the positive line-charge would be seen to be moving with a velocity V along the positive x-axis in the unprimed frame, whereas the negative line-charge would be seen to be moving with a velocity V along the negative x-axis in the primed frame. Using the space-time relations (3,4), the different line-charge densities as seen in the two frames maybe related to each other by suitable length scaling. The charge Δq in a given length Δx of the positive line-charge with density ρ_{l1}, as seen in the unprimed frame, would be equal to $\rho_{l1}\Delta x$, when measured at a fixed time t ($\Delta t = 0$). The fixed timing of the measurement in the unprimed frame, where the line charge is moving, is important. Otherwise, some charge would escape out of the segment Δx during any non-zero differential timing of measurement at the two ends of the line segment, making the relation $\Delta q = \rho_{l1}\Delta x$ invalid. The same charge Δq would be seen with a different length segment $\Delta x' = \Delta x/\alpha$ in the primed frame, as per the space-time relation (4). The charge Δq would be measured equal to $\rho'_{l1}\Delta x'$ in the primed frame, irrespective of any timing of measurement, because the line charge is stationary in this frame. This would result in the measured line charge density $\rho'_{l1} = \Delta q/\Delta x' = \rho_{l1}\Delta x/\Delta x' = \rho_{l1}\alpha$. Similar transformation of the negative line-charge density $-\rho'_{l2}$ in the primed frame in which it is moving, can be made into the unprimed frame in which it is stationary, resulting in the relation $\rho_{l2} = \rho'_{l2}\alpha$.

Now, due to the above transformations, the line charge densities $+\rho_{l1}$ and $-\rho_{l2}$ in the unprimed frame are no longer negative of each other, even though the respective densities $+\rho'_{l1}$ and $-\rho'_{l2}$ were in the primed frame. This leads to a non-zero total line charge density $\rho_l = \rho_{l1} - \rho_{l2} \neq 0$. This would result in a non-zero electric field $\overline{E} = \rho_l/(2\pi r\epsilon_0)$, which can be derived using the Gauss' Law [1], and therefore a non-zero force on the charge q in the unprimed frame, if only the electric field is used for the force calculation. However, the total force on the charge q is expected to be zero in the unprimed frame, as discussed earlier, because the total force in the primed frame is known to be zero. This would lead to a fundamental contradiction, which could be resolved by introducing a suitable new field, called the magnetic field $\overline{B}$. The new magnetic field needs to be defined to apply a force on the charge q only when it is moving (in the unprimed frame), directed perpendicular to the charge velocity, and is to be produced by any moving charge or current distribution (non zero line current $I = \rho_{l1}V$ in the unprimed frame). This new force is in addition to the conventional force on the charge q due to the electric field, which is applied irrespective of the motion of the charge.

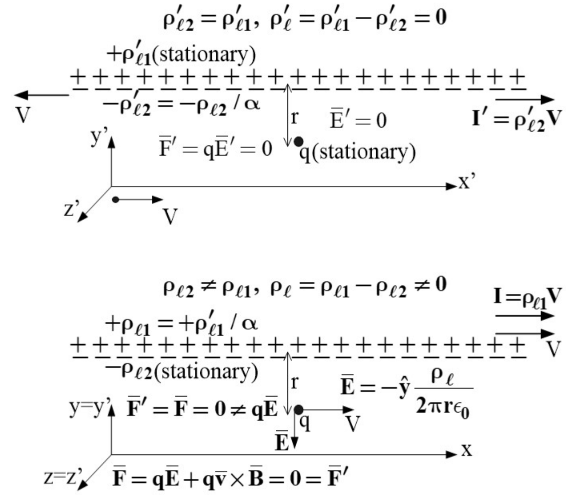

Fig. 1. The force on an electric charge near a line current, as viewed from two frames.

$$
\begin{aligned}
\rho_l &= \rho_{l1} - \rho_{l2} = \rho_{l1} - \rho'_{l2}\alpha = \rho_{l1} - \rho'_{l1}\alpha \\
&= \rho_{l1} - \rho_{l1}\alpha^2 = \rho_{l1}V^2/c^2 = IV/c^2,\ I = \rho_{l1}V, \\
\overline{E} &= -\hat{y}\frac{\rho_l}{2\pi r\epsilon_0} = -\hat{y}\frac{IV}{2\pi r\epsilon_0 c^2}.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\overline{F} &= q\overline{E} + q\overline{V}\times\overline{B} = 0 = \overline{F}', \\
\overline{V}\times\overline{B} &= V\hat{x}\times\overline{B} = -\overline{E} = \hat{y}\frac{IV}{2\pi r\epsilon_0 c^2}, \\
\overline{B} &= -\hat{z}\frac{I}{2\pi r\epsilon_0 c^2} = -\hat{z}\frac{\mu_0 I}{2\pi r},\ \mu_0 = \frac{1}{c^2\epsilon_0}.
\end{aligned} \tag{6}
$$

This above specific relationship between the the magnetic field $\overline{B}$ and the line current I, required in order to establish consistency of force across the reference frames, can be verified using the Ampere's Law, to be established in the section V under general conditions of current distribution.

The above specific example clearly establishes the necessity of a suitable magnetic field for relativistic consistency. Motivated by the specific initial study, a complete set of relations must be established for relativistic consistency of charge and force measurements across reference frames, valid under most general conditions. General relationships between any charge and current distributions across reference frames would be derived first in section IV. The general results from the section IV may be used to verify the relationships between specific line charge and current densities of section III. This would be followed by derivation of general relationships between the electric field and the new magnetic field, in sections V and VI, in order to establish relativistic consistency of force measurements across reference frames. These required field relationships constitute Maxwell's equations.

IV. Relativistic Transformation of Current and Charge Density

Let us consider a current distribution in the x direction, which is along the relative velocity V between two inertial frames. An experiment is conducted in the (x', y', z', t') coordinates (referred to as the primed reference frame), as shown in Fig.2, in order to verify charge conservation or continuity in the current distribution J'_x. It is implemented by counting the number of charges entering and exiting a small volume element over a time interval $\Delta\tau'$, and equating the difference with the charge accumulation due to time-variation of the charge density ${\rho'}_v$ inside the volume. The experiment may be mathematically expressed as:

$$\begin{gathered}\Delta\tau'[J'_x(x'+\Delta x'/2,t') - J'_x(x'-\triangle x'/2,t')]\Delta A = \\ -\frac{\partial\triangle Q'}{\partial t'}\Delta\tau' = -\frac{\partial\rho'_v}{\partial t'}(x',t')\Delta\tau'\Delta A\Delta x', \\ \frac{J'_x(x'+\Delta x'/2,t') - J'_x(x'-\triangle x'/2,t')}{\Delta x'} = -\frac{\partial\rho'_v}{\partial t'}(x',t'), \\ \frac{\partial J'_x}{\partial x'} = -\frac{\partial\rho'_v}{\partial t'}. \qquad (7)\end{gathered}$$

If the experiment was independently conducted in the (x, y, z, t) coordinates (referred to as the unprimed frame), then it would lead to a similar relationship for parameters in the unprimed frame.

$$\frac{\partial J_x}{\partial x} = -\frac{\partial\rho_v}{\partial t}. \qquad (8)$$

Now, let the experiment originally conducted in the primed frame be "observed" from the unprimed frame, as shown in the Fig.2. Measurements were done at the two ends, $x' - \Delta x'/2$ and $x' + \Delta x'/2$, of the volume element at the same time t' ($\Delta t' = 0$). In the unprimed reference frame, the associated distance and time increments Δx and Δt may be related using relativistic transform relations (3,4). Note that Δt is not zero, meaning the above measurements at the two ends, $(x - \Delta x/2)$ and $(x + \Delta x/2)$, are not seen at the same time in the unprimed frame.

$$\begin{gathered}t' = \frac{t - xV/c^2}{\alpha},\ x' = \frac{x - Vt}{\alpha},\ y = y',\ z = z', \\ \alpha = \sqrt{1 - \frac{V^2}{c^2}},\ \triangle t' = 0,\ \triangle t = \triangle x\frac{V}{c^2},\ \triangle x' = \alpha\triangle x. \qquad (9)\end{gathered}$$

The charges ΔQ and $\Delta Q'$, as measured or observed in the two frames, are equal, assuming a basic invariant nature of the charge. The transverse cross-section area ΔA is the same in both frames.

$$\triangle Q = \triangle Q' = \rho'_v\triangle A\triangle x' = \alpha\rho'_v\triangle A\triangle x. \qquad (10)$$

It maybe noted, the invariance of the amount of charge under any relative motion of observation is a fundamental condition, which is also required to ensure a neutral material nature, that we take for granted. Otherwise, when a particular pair of positive and a negative charges (proton and electron), originally assumed with equal magnitudes constituting a neutral atomic material, undergoes different relative motions of natural charge re-configurations (different atomic or molecular state transitions, or different electronic current flows in bulk matter), it would have effectively resulted in having different relative magnitudes for the negative and positive charges. This would have led to having different effective non-zero values of the total charge, under the different charge re-configurations, violating the charge neutrality condition originally assumed.

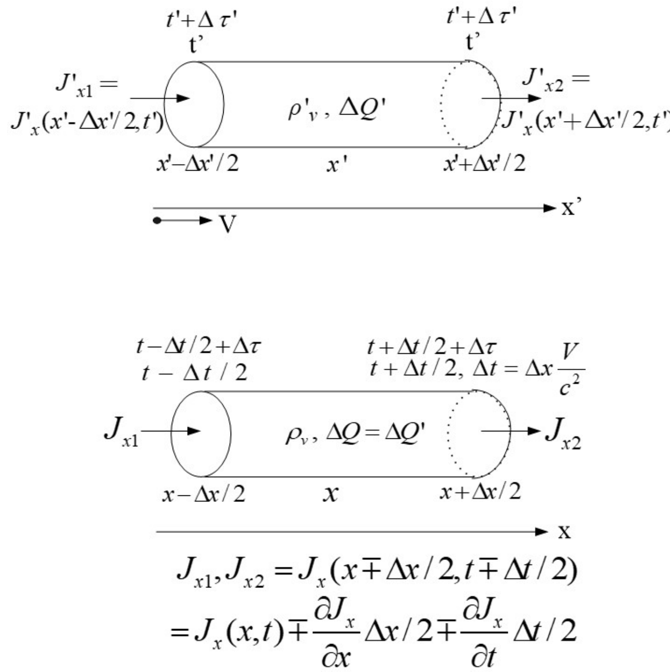

Fig. 2. Current continuity experiment conducted in the primed frame, and observed in the unprimed frame

The outcome or conclusion from the basic, physical experiment conducted in the primed reference frame at a given $t'(\Delta t' = 0)$, as implemented in (7), should be replicated when the same experiment is observed from the unprimed reference frame, as implemented in the following:

$$\begin{gathered}\Delta\tau[J_x(x+\Delta x/2, t+\Delta t/2) - J_x(x-\triangle x/2, t- \\ \triangle t/2)]\Delta A = -\frac{\partial\Delta Q}{\partial t}\Delta\tau = [-\alpha\frac{\partial\rho'_v}{\partial t}(x,t)\Delta x\Delta A]\Delta\tau, \\ (J_x + \frac{\partial J_x}{\partial x}\Delta x/2 + \frac{\partial J_x}{\partial t}\Delta t/2) - (J_x - \frac{\partial J_x}{\partial x}\Delta x/2 - \\ \frac{\partial J_x}{\partial t}\Delta t/2) = -\alpha\frac{\partial\rho'_v}{\partial t}(x,t)\triangle x, \\ \frac{\partial J_x}{\partial x}\triangle x + \frac{\partial J_x}{\partial t}\triangle t = -\alpha\frac{\partial\rho'_v}{\partial t}\triangle x, \\ -\frac{\partial\rho_v}{\partial t} + \frac{\partial J_x}{\partial t}\frac{V}{c^2} = -\alpha\frac{\partial\rho'_v}{\partial t},\ \rho_v - \frac{V}{c^2}J_x = \alpha\rho'_v. \qquad (11)\end{gathered}$$

Equations (8,9,10) are used in the above derivation. A similar result, $\rho'_v + VJ'_x/c^2 = \alpha\rho_v$, would be obtained by simply interchanging primed and unprimed coordinates, and substituting V with $-V$ for the relative velocity. For line charge and current distributions along the x-axis, similar relationships may also be obtained by substituting ρ_v, ρ'_v, and J_x, J'_x in the above derivations with the associated line charge densities ρ_l, ρ'_l, and line currents I_x, I'_x, respectively.

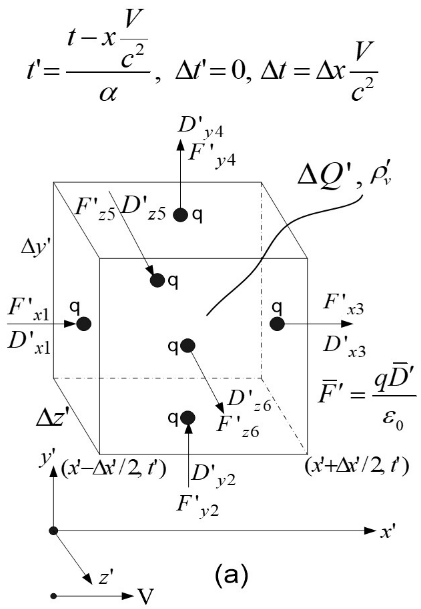

Fig. 3.

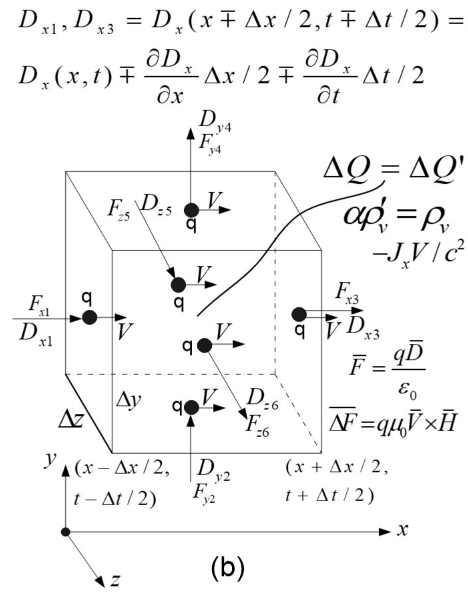

Fig. 3. (a) Electric field divergence (Gauss' Law) experiment in the primed reference frame, and (b) the same experiment as seen by an observer from the unprimed reference frame.

$$\rho_l - \frac{V}{c^2} I_x = \alpha \rho_l', \;\; \rho_l' + \frac{V}{c^2} I_x' = \alpha \rho_l. \tag{12}$$

If the current is charge free or neutral as seen in the primed frame, then the transformation relations (11,12) would yield

$$\rho_v' = 0 = \rho_l', \;\; \rho_v = \frac{V}{c^2} J_x, \;\; \rho_l = \frac{V}{c^2} I_x. \tag{13}$$

Accordingly, a charge-free (neutral) current in the primed frame would look charged in the unprimed frame. This is an important understanding.

In the above derivation, for simplicity we have considered only a x-directed current (along the relative velocity). The basic derivation may be generalized for currents along an arbitrary direction, with additional y and z components (with a rectangular instead of a cylindrical box in Fig.2). It maybe shown that the above derivation would lead to the same final relationships (11-13) between the charge densities ρ_v, ${\rho'}_v$, and the current components J_x, J_x', even after inclusion of the additional y and z current components.

V. Relativistic Transformation of the Gauss' Law for the Electric Field: The Ampere's Law

A. Gauss's Law for the Electric Field, Applied to a Rectangular Box

Consider the Gauss's Law for the electric field, applied to a rectangular box in the primed coordinate system (x', y', z', t'). The box is enclosed by the surface consisting of six faces, as shown in Fig.3a.

$$\begin{aligned} &(D_{x3}' - D_{x1}')\Delta y' \Delta z' + (D_{y4}' - D_{y2}')\Delta x' \Delta z' + \\ &(D_{z6}' - D_{z5}')\Delta y' \Delta x' = \Delta Q' = \rho_v' \Delta x' \Delta y' \Delta z', \\ &\frac{\partial D_x'}{\partial x'} + \frac{\partial D_y'}{\partial y'} + \frac{\partial D_z'}{\partial z'} = \rho_v', \end{aligned} \tag{14}$$

where $\Delta Q'$ is the total charge inside the box, and ρ_v' is the associated charge density. If the law is applied to a box in the unprimed frame, a relation similar to (14) would be obtained for respective parameters in the unprimed frame.

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v. \tag{15}$$

B. A Basic Gauss's Law Experiment in the Primed Reference Frames

Let an observer in the primed coordinate system conduct an experiment to verify or validate the Gauss' Law relation in (14), for a given rectangular box placed in a free-space medium. In the experiment, the flux density $\overline{D}'$ is measured in terms of the force $\overline{F}' = q\overline{D}'/\epsilon_0$ experienced by a given stationary charge q, as shown in Fig.3a.

$$\begin{aligned} &\frac{\epsilon_0}{q}(F_{x3}' - F_{x1}')\Delta y' \Delta z' + \frac{\epsilon_0}{q}(F_{y4}' - F_{y2}')\Delta x' \Delta z' + \\ &\frac{\epsilon_0}{q}(F_{z6}' - F_{z5}')\Delta y' \Delta x' = \rho_v' \Delta x' \Delta y' \Delta z'. \end{aligned} \tag{16}$$

C. The Gauss's Law Experiment as Observed in the Unprimed Reference Frames

In the above experiment, the force measurements are conducted in the primed frame at a given time t' ($\Delta t' = 0$). Let the same experiment be observed from the unprimed coordinate system, as shown in Fig.3b. Now, the force measurements on the six faces would not be observed simultaneously in the unprimed frame ($\Delta t \neq 0$, see (9)).

Following a new general principle of relativity for the Gauss' Law, the basic Gauss' Law experiment as measured in the primed frame should find the same charge $\Delta Q' = \Delta Q$ when the measurements are "observed" from the unprimed frame. This assumes that the given charge, as fundamentally defined by the Gauss' Law, is relativistically invariant as measured or observed in the two different frames.

$$\frac{\epsilon_0}{q}(F_{x3} - F_{x1})\Delta y \Delta z + \frac{\epsilon_0}{q}(F_{y4} - F_{y2})\Delta x \Delta z + \frac{\epsilon_0}{q}(F_{z6} - F_{z5})\Delta y \Delta x = \Delta Q = \Delta Q' = \alpha \rho_v' \Delta x \Delta y \Delta z = (\rho_v - J_x V/c^2)\Delta x \Delta y \Delta z. \quad (17)$$

Equations (10,11) are used in the above derivation.

D. Need for a New Force Field

If we express the forces in the unprimed frame using only the electric fields ($\overline{F} = q\overline{D}/\epsilon_0$), the above required relationship (17) will be inconsistent with (15), due to the $J_x V/c^2$ term in (17). An additional force field, which is proportional to the charge velocity V, would be required to restore consistency. The required force field must result in force components directed normal to the faces 2, 4, 5 and 6 of Fig.3b, which is normal to the velocity of the charges q placed on these faces. Accordingly, we may define a new field $\overline{H}$, such that any additional force $\overline{\Delta F}$ is expressed in terms of a cross-product of the field $\overline{H}$ and the charge velocity vector $\bar{V}$ as follows:

$$\frac{q}{\epsilon_0 c^2}\overline{V} \times \overline{H} = \overline{\Delta F} = q\mu_0 \overline{V} \times \overline{H}, \quad \mu_0 = \frac{1}{\epsilon_0 c^2}. \quad (18)$$

It maybe mentioned, introducing an alternate voltage-dependent force $\overline{\Delta F}$ directed along the charge velocity, which would modify the forces normal to the faces 1 and 3 of Fig.3b, instead of using (18) that modifies normal forces on the faces 2, 4, 5 and 6, would fail to enforce the required relationship (17). This may be shown by going through the similar following steps. Anyway, assuming that the final field solution is unique, a successful solution with the anticipated new field (18), without any force component along the charge velocity, would be complete.

Now, using the anticipated new force field $\overline{H}$ (18), in addition to the conventional electric force $\overline{F} = q\overline{D}/\epsilon_0$, the different force components in (17) can be expressed (see Fig.4).

$$\begin{aligned} F_{x1} &= \frac{q}{\epsilon_0}D_x(x - \triangle x/2, t - \triangle t/2) = \frac{q}{\epsilon_0}(D_x - \frac{\partial D_x}{\partial x}\Delta x/2 \\ &\quad - \frac{\partial D_x}{\partial t}\Delta t/2) = \frac{q}{\epsilon_0}(D_x - \frac{\partial D_x}{\partial x}\Delta x/2 - \frac{\partial D_x}{\partial t}\frac{V}{c^2}\Delta x/2), \\ F_{x3} &= \frac{q}{\epsilon_0}D_x(x + \triangle x/2, t + \triangle t/2) = \frac{q}{\epsilon_0}(D_x + \frac{\partial D_x}{\partial x}\Delta x/2 \\ &\quad + \frac{\partial D_x}{\partial t}\Delta t/2) = \frac{q}{\epsilon_0}(D_x + \frac{\partial D_x}{\partial x}\Delta x/2 + \frac{\partial D_x}{\partial t}\frac{V}{c^2}\Delta x/2), \\ F_{y2} &= \frac{q}{\epsilon_0}(D_y - \frac{\partial D_y}{\partial y}\Delta y/2) - q\mu_0 V H_{z2}, \\ F_{y4} &= \frac{q}{\epsilon_0}(D_y + \frac{\partial D_y}{\partial y}\Delta y/2) - q\mu_0 V H_{z4}, \\ F_{z5} &= \frac{q}{\epsilon_0}(D_z - \frac{\partial D_z}{\partial z}\Delta z/2) + q\mu_0 V H_{y5}, \\ F_{z6} &= \frac{q}{\epsilon_0}(D_z + \frac{\partial D_z}{\partial z}\Delta z/2) + q\mu_0 V H_{y6}. \end{aligned} \quad (19)$$

Using the force expressions of (19) in the equation (17), we get,

$$\Delta y \Delta z(\frac{\partial D_x}{\partial x} + \frac{\partial D_x}{\partial t}\frac{V}{c^2} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}) - \frac{V}{c^2}(H_{z4}\Delta z - H_{z2}\Delta z - H_{y6}\Delta y + H_{y5}\Delta y) = (\rho_v - J_x\frac{V}{c^2})\Delta y \Delta z. \quad (20)$$

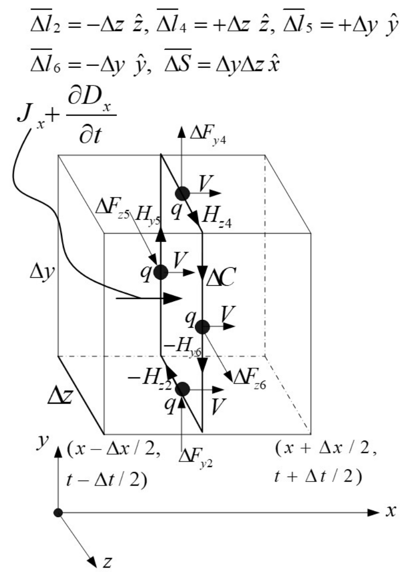

Fig. 4. Deduction of the Ampere's Law from the Electric field divergence (Gauss' Law) experiment of Fig.3, as seen by an observer from the unprimed reference frame.

Further, combining (20) with (15) we get,

$$-H_{z2}\Delta z + H_{z4}\Delta z + H_{y5}\Delta y - H_{y6}\Delta y = \sum_{\triangle C}\overline{H}_i \cdot \overline{\Delta l}_i = (\frac{\partial D_x}{\partial t} + J_x)\Delta y \Delta z. \quad (21)$$

$$\sum_{\triangle C}\overline{H}_i \cdot \overline{\Delta l}_i = (\overline{J} + \frac{\partial \overline{D}}{\partial t}) \cdot \overline{\Delta S}. \quad (22)$$

The loop ΔC, and the associated surface element $\overline{\Delta S}$, are shown in Fig.4. The choice of the relative velocity $\overline{V}$ to be along $\hat{x}$ is arbitrary. Therefore, by performing the above analysis with the $\overline{V}$ in any other direction, it may be recognized that the equation (22) would apply for an elemental closed loop ΔC, and its enclosed surface $\overline{\Delta S}$, having any general orientation.

The equation (22) is the Ampere's Law expressed for an elemental loop. The force field described by the vector $\overline{H}$ in (18) is a new "synthesized" field, which may be recognized as the magnetic field. The relationship (22) for the Ampere's Law may also be expressed using the curl operator.

$$\sum_{\triangle C}\overline{H}_i \cdot \overline{\Delta l}_i = (\overline{\nabla} \times \overline{H}) \cdot \overline{\Delta S} = (\overline{J} + \frac{\partial \overline{D}}{\partial t}) \cdot \overline{\Delta S},$$
$$\overline{\nabla} \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}. \quad (23)$$

VI. Relativistic Transformation of the Gauss' Law for the Magnetic Field: The Faraday's Law

A. Gauss's Law for the Magnetic Field, Applied to a Rectangular Box

Now, analogous to the Gauss' Law for the electric field, we may establish a Gauss' Law for the new magnetic field $\overline{H}$ in terms of an associated magnetic flux density $\overline{B}$ defined as $\overline{B} = \mu_0 \overline{H}$. The total magnetic flux over a closed surface,

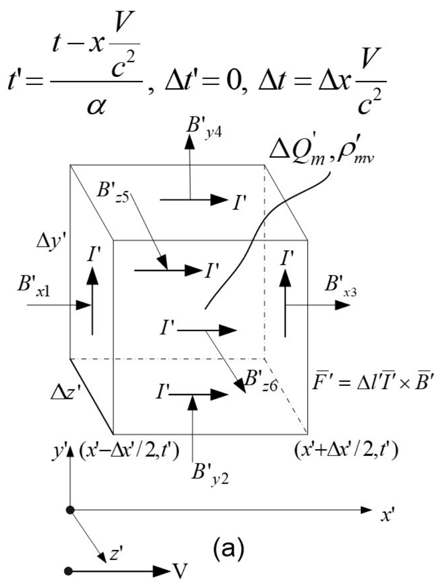

Fig. 5.

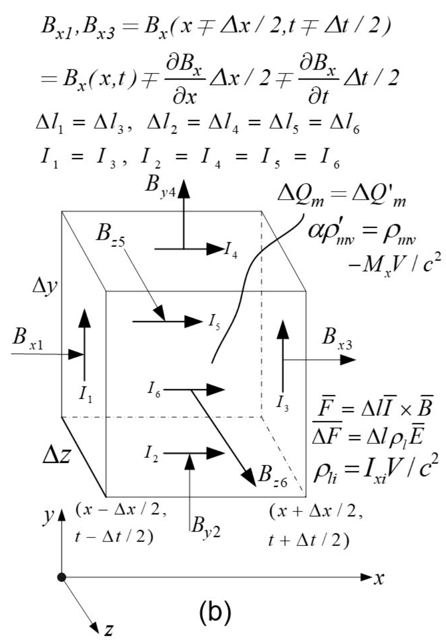

Fig. 5. (a) Magnetic field divergence (Gauss' Law) experiment in the primed reference frame, and (b) the same experiment as seen by an observer from the unprimed reference frame.

as calculated using the above magnetic flux density, may be referred to as the magnetic charge Q_m present inside the surface.

Consider the Gauss's Law for the magnetic field, as applied to a rectangular box in the primed coordinate system (x', y', z', t'). The box is enclosed by six surfaces, as shown in Fig.5a.

$$(B'_{x3} - B'_{x1})\Delta y'\Delta z' + (B'_{y4} - B'_{y2})\Delta x'\Delta z' +$$
$$(B'_{z6} - B'_{z5})\Delta y'\Delta x' = \rho'_{mv}\triangle x'\triangle y'\triangle z',$$
$$\frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'} = \rho'_{mv}, \tag{24}$$

where $\Delta Q'_m$ is the total magnetic charge inside the box, and ρ'_{mv} is the associated magnetic charge density. If the law is applied to a box in the unprimed frame, a relation similar to (24) would be obtained for respective parameters in the unprimed frame.

$$\frac{\partial Bx}{\partial x} + \frac{\partial By}{\partial y} + \frac{\partial Bz}{\partial z} = \rho_{mv}. \tag{25}$$

It may be noted, no magnetic charge has been found to exist naturally, in which case the magnetic charge used in the Gauss' Law for the magnetic field, and any associated magnetic current $\overline{M}$, would be zero ($\Delta Q_m = 0, \overline{M} = 0$). However, in the following analyses the magnetic charge and current are still maintained to be non-zero for theoretical generality and symmetry with any non-zero electric charge and current. As a practical benefit, this may be useful for a general theoretical treatment involving both electric current and voltage sources, where an electric voltage source maybe equivalently modeled as a magnetic current source [18], as a dual condition to an electric current source.

B. A Basic Gauss's Law Experiment for the Magnetic Field in the Primed Reference Frame

Let an observer in the primed coordinate system conduct an experiment to verify or validate the Gauss' Law of (24) for a rectangular box, placed in a free-space medium, as shown in Fig.5a. The magnetic field $\overline{H}'_i$ or the flux density $\overline{B}'_i = \mu_0\overline{H}'_i$ are fundamentally defined in terms of measured force experienced by a given current element $\overline{I}'$ of a given length $\Delta l'$. This is as per the definition of the magnetic field in (18), as applied to the primed reference frame, where $q\overline{V}$ in (18) may be equivalently substituted by $\overline{I}'\Delta l'$.

$$\overline{\Delta F}' = \Delta l'\mu_0\overline{I}' \times \overline{H}' = \Delta l'\overline{I}' \times \overline{B}',\quad \overline{I}'\triangle l' = q\overline{V}. \tag{26}$$

$$\frac{1}{I'\Delta l'}[(-F'_{z3} + F'_{z1})\Delta y'\Delta z' + (F'_{z4} - F'_{z2})\Delta x'\Delta z' +$$
$$(-F'_{y6} + F'_{y5})\Delta x'\Delta y'] = \Delta Q'_m = \rho'_{mv}\triangle x'\triangle y'\triangle z'. \tag{27}$$

It may be noted, all the above force measurements are conducted simultaneously in the primed coordinates, at a fixed time t', $\Delta t' = 0$.

C. The Experiment as Observed from the Unprimed Frame

Now, like the Gauss' Law for the electric field, the Gauss' Law for the magnetic field is a fundamental charge-field relationship. The magnetic charge and the defining Gauss's Law may also be assumed relativistically invariant, as measured and "observed" in different inertial frames.

Accordingly, the governing relationship (27) for the experiment performed in the primed frame (Fig.5a) is expected to remain invariant, when observed from the unprimed frame (Fig.5b). The magnetic charge measured and observed from the different frames would remain equal ($\Delta Q'_m = \Delta Q_m$).

$$\Big[(-\frac{F_{z3}}{I_3\Delta l_3}+\frac{F_{z1}}{I_1\Delta l_1})\Delta y\Delta z+(\frac{F_{z4}}{I_4\Delta l_4}-\frac{F_{z2}}{I_2\Delta l_2})\Delta x\Delta z+ (-\frac{F_{y6}}{I_6\Delta l_6}+\frac{F_{y5}}{I_5\Delta l_5})\Delta x\Delta y\Big]=\Delta Q_m=\Delta Q'_m= \alpha\rho'_{mv}\triangle x\triangle y\triangle z=(\rho_{mv}-M_xV/c^2)\triangle x\triangle y\triangle z. \quad (28)$$

Continuity relationships equivalent to (10,11), as extended for magnetic charge density ρ_{mv} and magnetic current density M, are used in the above derivation.

Charged Reference Currents, and Force Transformation Using Electric and Magnetic Fields: The reference current elements $I'\Delta l'$ used in the original experiment are required to be free of electric charge (charge-neutral), so that the forces experienced by the current elements in the primed reference frame are contributed only due to the magnetic fields. However, when these charge-free (neutral) currents are observed from the unprimed frame, they would look charged if the current is directed along x, but remain charge free (neutral) if directed along y or z. This is governed by (11-13).

For the two faces 1 and 3 in (28), Fig.5, the test current elements are directed along y, and therefore do not appear charged, as mentioned above. Consequently, the forces experienced by these current elements can be expressed using magnetic field alone. This is as per (26), as applied to the unprimed frame. The above two faces differ in their x coordinates, and therefore the forces measured on these faces in the primed coordinates at a given time (t', $\Delta t'=0$) are observed in the unprimed coordinates at different times t, $\Delta t\neq 0$. This is in accordance with the differential space-time relationship (9).

$$\begin{aligned} -F_{z1}&=\Delta l_1I_1B_{x1}=\Delta l_1I_1B_x(x-\triangle x/2,t-\triangle t/2)\\ &=\Delta l_1I_1(B_x(x,t)-\frac{\triangle x}{2}\frac{\partial B_x(x,t)}{\partial x}-\frac{\triangle xV}{2c^2}\frac{\partial B_x(x,t)}{\partial t}),\\ -F_{z3}&=\Delta l_3I_3B_{x3}=\Delta l_3I_3B_x(x+\triangle x/2,t+\triangle t/2)\\ &=\Delta l_3I_3(B_x(x,t)+\frac{\triangle x}{2}\frac{\partial B_x(x,t)}{\partial x}+\frac{\triangle xV}{2c^2}\frac{\partial B_x(x,t)}{\partial t}). \quad (29)\end{aligned}$$

On the other hand, for the faces 2, 4, 5 and 6, the reference currents are directed along x, and therefore they would appear charged as per (13), as discussed earlier. Consequently, the forces experienced by these currents are to be expressed using both magnetic and electric fields. The centers of measurement for the above four faces have the same x coordinates. Therefore, the experiment in the primed coordinates at a fixed time (t', $\Delta t'=0$) are observed in the unprimed coordinates also at a fixed time t, $\Delta t=0$, as per the differential space-time relationship of (9).

$$\begin{aligned} \overline{F}&=(\triangle l\overline{I}\times\overline{B})+\triangle l\rho_l\overline{E},\ \ \rho_l=IV/c^2,\\ F_{z2}&=\Delta l_2I_2(B_y-\frac{\triangle y}{2}\frac{\partial B_y}{\partial y})+\Delta l_2I_2\frac{V}{c^2}E_{z2},\\ F_{z4}&=\Delta l_4I_4(B_y+\frac{\triangle y}{2}\frac{\partial B_y}{\partial y})+\Delta l_4I_4\frac{V}{c^2}E_{z4},\\ -F_{y5}&=\Delta l_5I_5(B_z-\frac{\triangle z}{2}\frac{\partial B_z}{\partial z})-\Delta l_5I_5\frac{V}{c^2}E_{y5},\\ -F_{y6}&=\Delta l_6I_6(B_z+\frac{\triangle z}{2}\frac{\partial B_z}{\partial z})-\Delta l_6I_6\frac{V}{c^2}E_{y6}. \quad (30)\end{aligned}$$

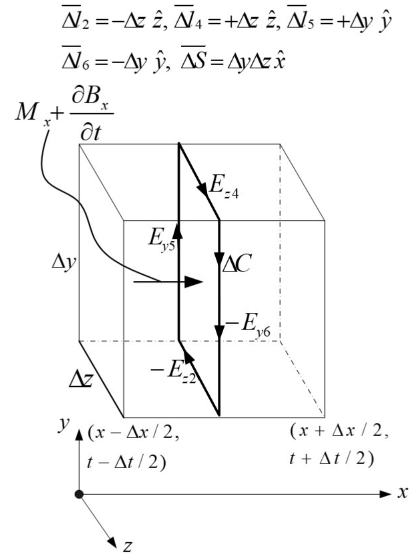

Fig. 6. Deduction of the Faraday's Law from the magnetic field divergence (Gauss' Law) experiment of Fig.5, as seen by an observer from the unprimed reference frame.

D. The Faraday's Law, Deduced from the Force Transformation

Now, using the equations (29,30) in the force equation (28), we get a relationship between the electric and magnetic fields in the unprimed reference frame.

$$\begin{aligned} &\Big[(\frac{\partial B_x}{\partial x}+\frac{V}{c^2}\frac{\partial B_x}{\partial t})\triangle x\Delta y\Delta z+(\frac{\partial B_y}{\partial y}\triangle y+\frac{V}{c^2}E_{z4}- \frac{V}{c^2}E_{z2})\Delta x\Delta z+(\frac{\partial B_z}{\partial z}\triangle z-\frac{V}{c^2}E_{y6}+ \frac{V}{c^2}E_{y5})\Delta x\Delta y\Big]=(\rho_{mv}-M_x\frac{V}{c^2})\triangle x\triangle y\triangle z,\\ &(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}+\frac{V}{c^2}\frac{\partial B_x}{\partial t})\triangle x\Delta y\Delta z+ \frac{V}{c^2}\Big[(-E_{y6}+E_{y5})\Delta x\Delta y+(E_{z4}-E_{z2})\Delta x\Delta z\Big]= (\rho_{mv}-M_x\frac{V}{c^2})\triangle x\triangle y\triangle z. \quad (31)\end{aligned}$$

Then, using (25) in (31) we would get

$$\Big[-E_{y6}\Delta y+E_{y5}\Delta y+E_{z4}\Delta z-E_{z2}\Delta z\Big]= -(\frac{\partial B_x}{\partial t}+M_x)\triangle y\triangle z. \quad (32)$$

The above relation (32) may be recognized in terms of line and surface integrals over an elemental rectangular loop ΔC and its enclosed surface $\overline{\Delta S}$, respectively, as shown in Fig.6.

$$\sum_{\triangle C}\overline{E}_i\cdot\overline{\triangle l}_i=-(\frac{\partial\overline{B}}{\partial t}+\overline{M})\cdot\overline{\triangle S}. \quad (33)$$

The relative velocity between the two frames was selected to be along the $\hat{x}$ direction, which is an arbitrary choice. Therefore, the expression (33) would apply to an elemental loop with any general orientation.

Equation (33) may be recognized as the Faraday's Law as applied to an elemental loop ΔC. The above equation (33) for the Faraday's Law may also be expressed using the curl operator.

$$\sum_{\Delta C} \overline{E_i} \cdot \overline{\Delta l_i} = (\overline{\nabla} \times \overline{E}) \cdot \overline{\Delta S} = -(\overline{M} + \frac{\partial \overline{B}}{\partial t}) \cdot \overline{\Delta S},$$
$$\overline{\nabla} \times \overline{E} = -\overline{M} - \frac{\partial \overline{B}}{\partial t}. \quad (34)$$

VII. Discussions

A. Application in a Material Medium

It may be noted, that we have explicitly assumed a free-space medium for the derivations of the Ampere's Law in (22,23), as well as the Faraday's Law in (33,34). The derivations, in principle, are theoretically complete. The results may be extended to any material medium by using suitable permittivity and permeability parameters, to represent different induced effects in the internal charge structure of the material.

B. A New form of the Gauss' Laws and Charge Invariance

The conventional Gauss' Laws are valid independently in the primed (equations (14,24)) and unprimed (equations (15,25)) frames. The experiments to implement these conventional Gauss' Laws are conducted simultaneously, at a given time, in the respective frames. In addition, a new form of the Gauss' Laws are introduced, implemented in the experiments (17,28). Here, a Gauss' Law experiment originally conducted in the primed frame at a given time t', $\Delta t' = 0$, is invariant as observed from the unprimed frame with different timing ($\Delta t \neq 0$) for the individual measurements. These new Gauss' Law experiments, definitively timed in a particular frame (primed frame), unambiguously measure the same amount of charge (electric or equivalent magnetic), assuming the charge is invariant to any relative motion. On the other hand, the conventional form of the Gauss' Law experiments are not guaranteed to measure the same amount of charge in two frames, because a part of the charge may be moving and might escape from the measurement box during the different, independent timings in the two frames. In other words, the new form of the Gauss' Laws is the only unambiguous way to ensure invariance of charge (electric or equivalent magnetic) across reference frames, and therefore is more fundamental.

Enforcing this new fundamental form of the Gauss' Laws naturally allows a direct "derivation" of the Ampere's and the Faraday's Laws (22,33), as additional required conditions for the enforcements. This is a significant development.

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