Deriving Maxwell's Equations from First Principles of Relativistic Charge Invariance and Space-Time Relations

Nirod K. Das
Department of Electrical and Computer Engineering, Tandon School of Engineering
New York University, Brooklyn, New York, USA

***Abstract*—The paper introduces a simple, rigorous approach to derive Maxwell's equations from first principles, based on an invariant form of Gauss' law using only basic space-time transformation relations of special relativity. More advanced concepts of relativistic mechanics, such as transformation of mass, momentum and force, are not necessary in this development, resulting in a simple theory. The new derivation is of fundamental theoretical significance in electromagnetic theory, making Maxwell's equations less mysterious to comprehend, and would fill an educational need to introduce the modern theory of special relativity to derive electromagnetic theory from the first principles. The approach would be suitable for a senior (even junior) undergraduate or an introductory graduate engineering student, depending on the level of depth and rigor.**

I. Introduction

Maxwell's equations [1], [2] were historically established following a series of experimental discoveries and their gradual mathematical comprehension, without much physical insight into their underlying fundamental origin. Modern textbooks [3]- [9] follow similar historical steps to introduce electromagnetic principles, essentially as a mathematical deductive theory, by simply accepting Maxwell's equations, or the constituent Ampere's and Faraday's laws, as the starting point of deduction. Although the theory has proved remarkably successful in predicting experimental observations, the underlying physical principles remain mysterious to comprehend, and therefore academically unsatisfactory.

The present paper introduces a new approach to establish electromagnetic theory from first principles. This is based on basic space-time relations of special relativity, which is supported by a fundamental understanding of the nature of light propagation in empty space to start with, and on the basic concept of an invariant charge that remains independent of motion or frame of observation. The new approach provides a simple physical interpretation as well as a complete "derivation" of Maxwell's equations, accomplished through basic relativistic transformations of charge continuity (section III) and Gauss' laws (sections IV, V). A new basic form of Gauss' laws is introduced in the derivations, required in order to unambiguously enforce invariance of any electric or magnetic charge across reference frames, allowing a simple and direct derivation of Maxwell's equations. The key principles of the approach, consisting of four foundational steps, are briefly explained in the following. Further details would be available in related references [10], [11].

II. Special Nature of Light, and Relativistic Space-Time Transformation

It is a special nature of light, which is unlike any mechanical wave such as the sound or a water ripple, that it can propagate in the empty space without any material medium to which the propagation speed could be preferentially fixed to a given value. A closer deliberation on this special nature would lead to the conclusion that whatever is the speed of light c, measured at a given time or location in the empty space, it must be independent of any given body or frame of reference of observation. And, the measured speed c would also be independent of the time and location of the measurement, for an ideal empty space whose nature is assumed to be uniform in space and time. Algebraically enforcing these conditions from two reference frames would lead to the following simple relationships between space-time coordinates observed from the two frames, where the primed frame is moving along the x-axis with a velocity V with respect to the unprimed frame. The space-time relationship must apply for all physical phenomena, not just for the light measurement, for universal consistency.

$$x = \frac{x'+Vt'}{\alpha},\ t = \frac{t'+x'V/c^2}{\alpha},\ y = y',\ z = z',\\ \alpha = \sqrt{1-\frac{V^2}{c^2}}. \tag{1}$$

III. Relativistic Transformation of Current and Charge Density

Measurements are conducted in the primed frame around a closed space, ensuring charge continuity in a current flow. The measurements need to be timed simultaneously ($\Delta t' = 0$) at all locations of the closed space, but would be seen from the unprimed frame with differential timing at different x coordinates, as per (1). Enforcing invariance of the charge measurements in both frames of observation, it maybe shown that the volume-charge density ρ_v and volume-current density J_x in the x-direction, as measured in the unprimed frame, be related to the volume-charge density ρ'_v measured in the primed frame. By symmetry between the coordinates, a similar relationship would also be established by exchanging the primed and unprimed variables, and substituting V with $-V$.

The same relationships would work as well, if the volume-charge and volume-current densities are substituted by the respective line-charge densities (ρ_l, ρ'_l) and line-currents. (I_x, I'_x).

$$\rho_v - \frac{V}{c^2}J_x = \alpha\rho'_v,\ \rho'_v + \frac{V}{c^2}J'_x = \alpha\rho_v,$$
$$\rho_l - \frac{V}{c^2}I_x = \alpha\rho'_l,\ \rho'_l + \frac{V}{c^2}I'_x = \alpha\rho_l. \quad (2)$$

IV. Relativistic Transformation of Gauss' Law for the Electric Field: Ampere's Law

Enforcing invariance of charge would require Gauss' law that fundamentally defines the charge be properly defined across reference frames. In its basic form, Gauss' law is implemented using measurements of force on stationary test-charges placed around a closed surface. In order to ensure that the same physical charge is unambiguously captured inside the closed surface, Gauss' law measurements must be performed in a primary frame (primed) at a fixed time ($\Delta t' = 0$) for all locations of the closed surface, which are then observed from any other frame (unprimed) of reference.

The measurements are observed from the unprimed frame, with differential timings as per (1), and with the test charges in motion. In order that the unprimed observer must conclude the same value for the total enclosed charge as the primed observer, the moving test-charges must experience a new kind of force that depends on the charge's velocity. Accordingly, a new force-field, recognized as the magnetic field, that properly defines the above required new force, must also satisfy certain required relations, recognized as Ampere's law. In other words, the magnetic field governed by Ampere's law is a synthesized new field, that ensures fundamental invariance of the electric charge.

$$\epsilon_0(\tfrac{\partial F_x}{\partial x} + \tfrac{\partial F_x}{\partial t}\tfrac{V}{c^2} + \tfrac{\partial F_y}{\partial y} + \tfrac{\partial F_z}{\partial z}) = \alpha\rho'_v = \rho_v - J_x\tfrac{V}{c^2},$$
$$\overline{F} = \overline{E} + \mu_0 V(\hat{x}\times\overline{H}),\ q = 1,$$
$$F_x = E_x,\ F_y = E_y - \mu_0 V H_z,\ F_z = E_z + \mu_0 V H_y,$$
$$\Delta t' = 0,\ \Delta t = \Delta x\tfrac{V}{c^2},\ \Delta x' = \alpha\Delta x,\ c^2 = \tfrac{1}{\mu_0\epsilon_0},$$
$$\epsilon_0(\tfrac{\partial E_x}{\partial x} + \tfrac{\partial E_y}{\partial y} + \tfrac{\partial E_z}{\partial z}) = \epsilon_0\overline{\nabla}\cdot\overline{E} = \rho_v,$$
$$\tfrac{\partial H_z}{\partial y} - \tfrac{\partial H_y}{\partial z} = \epsilon_0\tfrac{\partial E_x}{\partial t} + J_x,$$
$$\overline{\nabla}\times\overline{H} = \epsilon_0\tfrac{\partial\overline{E}}{\partial t} + \overline{J}. \quad (3)$$

V. Relativistic Transformation of Gauss' Law for the Magnetic Field: Faraday's Law

The new magnetic field, like the original electric field, must also satisfy its own version of Gauss's law. The new charge that is theoretically associated with the new magnetic field is referred to as the magnetic charge, which may or may not physically exist. Now, Gauss' law for the magnetic field and any associated magnetic charge, are required to be invariant to any motion or frame of observation, like the counter parts for the electric field. Similar transformation relations of (2) would be satisfied for the magnetic volume-charge ρ_{mv}, ρ'_{mv}, and volume-current M_x distributions. Gauss' law for the magnetic field needs to be implemented in a primary (primed) frame of reference, similar to that for the electric field discussed above, in the form of measurement of force on suitable test-elements of electric current (moving charges, but with zero total charge), that would fundamentally define the magnetic field. The test-current elements maybe chosen along the x direction for measuring y- and z-directed magnetic fields, but along y for measuring the x-directed magnetic field.

The x-directed test currents with no total electric charge (neutral), as seen in the primed frame, would be seen in the unprimed frame with a non-zero electric charge $q = V/c^2$ for a unit current element $I_x\Delta l = 1$, as per (2). These charged currents would experience additional force due to the electric field. Therefore, the force measurements for Gauss' law for the magnetic field, that are contributed only due to the magnetic field in the primed frame, will now involve additional forces due to the electric field as seen in the unprimed frame. Accordingly, the required invariance of Gauss' law for the magnetic field, enforced across the reference frames, would require additional relationships for the electric field, recognized as Faraday's law.

$$\tfrac{\partial B_x}{\partial x} + \tfrac{\partial B_x}{\partial t}\tfrac{V}{c^2} + \tfrac{\partial(B_y + E_z V/c^2)}{\partial y} + \tfrac{\partial(B_z - E_y V/c^2)}{\partial z}$$
$$= \alpha\rho'_{mv} = \rho_{mv} - M_x\tfrac{V}{c^2},$$
$$\overline{F} = q\overline{E} + (\overline{I}\times\overline{B})\Delta l,\ \overline{B} = \mu_0\overline{H},$$
$$\overline{I}\Delta l = 1\hat{x},\ q = V/c^2,\text{ for } B_y, B_z \text{ measurements},$$
$$F_z = B_y + E_z V/c^2,\ -F_y = B_z - E_y V/c^2,$$
$$\overline{I}\Delta l = 1\hat{y},\ q = 0,\text{ for } B_x \text{ measurement},\ -F_z = B_x,$$
$$\tfrac{\partial B_x}{\partial x} + \tfrac{\partial B_y}{\partial y} + \tfrac{\partial B_z}{\partial z} = \overline{\nabla}\cdot\overline{B} = \rho_{mv},$$
$$\tfrac{\partial E_z}{\partial y} - \tfrac{\partial E_y}{\partial z} = -\tfrac{\partial B_x}{\partial t} - M_x,\ \overline{\nabla}\times\overline{E} = -\tfrac{\partial\overline{B}}{\partial t} - \overline{M}. \quad (4)$$

References

[1] James Clark Maxwell. *A Treatise on Electricity and Magnetism, Vol. I and II (Reprint from 1873)*. Dover Publications, 2007.

[2] James Clark Maxwell and W. D. Niven. *The Scientific Papers of James Clerk Maxwell, Vol. I and II*. Dover Publications, 2003.

[3] D. K. Cheng. *Fundamentals of Engineering Electromagnetics*. Addison-Wesley Publishing, 1993.

[4] Jr. D. H. Hayt. *Engineering Electromagnetics*. McGraw-Hill, New York, 1995.

[5] N. N. Rao. *Elements of Engineering Electromagnetics*. Prentice Hall, New Jersey, 2000.

[6] E. M. Purcell. *Electricity and Magnetism (Berkeley Physics Course, Vol.2, 2 Ed.)*. Prentice Hall, New Jersey, 1984.

[7] Richarrd P. Feynman, Robert B. Leighton, and Mathew Sands. *Lectures on Physics, Vol.II, Ch.25,26*. Addision Wesley, 1964.

[8] J. D. Jackson. *Classical Electrodynamics, 2 Ed.* John Wiley and Sons, New York, 1975.

[9] R. S. Elliott. *Electromagnetics: History, Theory and Applications*. Wiley-IEEE Press, 1999.

[10] N. Das. Introducing Maxwell's Equations as Derived from Simple Relativity Transformation Principles. *IEEE Antennas and Propagation Magazine (to appear)*, 2021.

[11] N. Das. *Introduction to Electromagnetic Fields, Radiation and Antenna Theory*. UEGM Publishing, ISBN:978-1734063004, 2021.