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Deriving Maxwell's Equations from First Principles of Relativistic Charge Invariance and Space-Time Relations

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Outline of Presentation

- Introduction: Maxwell’s equations (Ampere’s and Faraday’s laws) can be derived based on charge invariance and simple relativistic space-time relations
- Special nature of light, and space-time relations.
- Relativistic transform relations between current and charge density.
- Relativistic transformation of Gauss' law of the electric field: Ampere's law
- Relativistic transformation of Gauss' law for the magnetic field: Faraday's law
- Conclusion

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Introduction

- Electromagnetic theory is traditionally established starting from Maxwell's equations, under different simplified conditions or in their complete form.
- Historical development of Maxwell's equations were based on a series of experimental discoveries, and their gradual mathematical comprehension, without understanding of their fundamental origin.
- Understanding of the basic nature of an electric (or equivalent magnetic) charge, and its invariance and continuity across all reference frames, together with understanding of special nature of light, would allow a deeper insight into as well as direct "derivation" of Maxwell's equations .
- The new approach would provide a modern, alternative perspective of electromagnetic principles, with their direct connection to special relativity, making Maxwell's equations less mysterious to comprehend.

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Special Nature of Light and Observer Independence of Light Speed

- Light (or any electromagnetic) wave can propagate in an empty space, without any material medium.
- This is unlike any mechanical wave, such as a sound wave or a water ripple, which always propagates in a specific material body (air or water), with respect to which the wave's speed is preferentially fixed to a particular value.
- In contrast, the empty-space medium in which the light propagates, by the very nature of the empty space, is not "attached" to any material body of reference, with respect to which the propagation speed of light could be preferentially fixed.
- In other words, whatever is the light speed observed with respect to a given body or frame of reference, there is no rational basis why the same speed could not as well be observed with respect to any other reference frame.
- Therefore, the light speed in an empty-space medium, measured at a given location and time, must be fundamentally fixed to the same value, for all reference frames of observation, independent of the relative speeds between the frames.
- Further, assuming that the nature of the empty space is uniform in time and space, the observer-independent light speed in the empty space, would also be independent of the time and location of observation.

Observer Independence of Light Speed, and Relativistic Space-Time Relations

- The fundamental observer (and location, time) independence of light speed would require adjustment of space-time relationships, as observed from two different reference frames.
- By enforcing the observer-independence of the light speed, c, and appropriate symmetry and conditions between two reference frames, the following space-time relations can be established, where the “primed’’ reference frame is moving with a velocity V along the x-direction as seen from the “unprimed’’ frame:

$$x=\frac{x'+Vt'}{\alpha},\;\; t=\frac{t'+(V/c^2)x'}{\alpha};\;\; x'=\frac{x-Vt}{\alpha},\;\; t'=\frac{t-(V/c^2)x}{\alpha}$$

$$y=y',\; z=z',\; \alpha=\sqrt{1-V^2/c^2}$$

- The required relativistic space-time relations must be valid for all physical phenomena, not just to model light propagation, for universal consistency.

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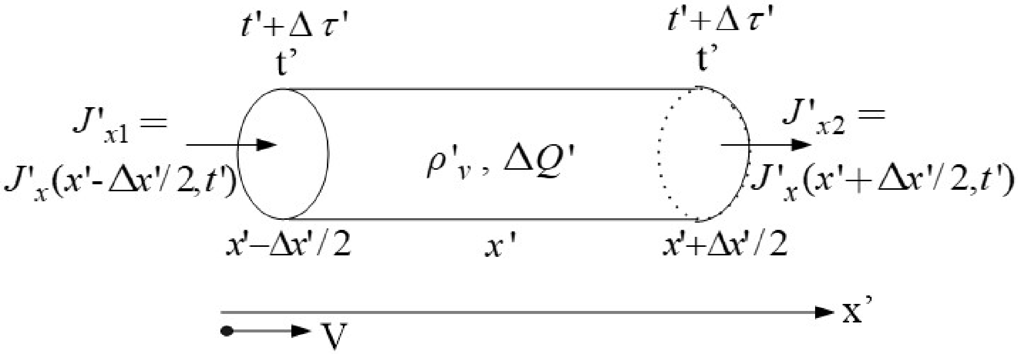
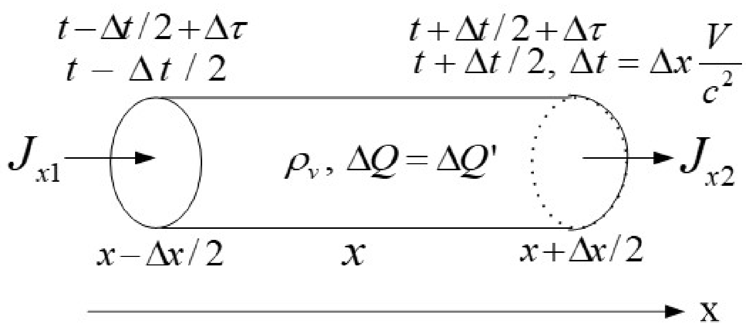
Charge Continuity and Invariance, and Relativistic Transform Relations Between Current and Charge Density



$$\Delta Q' = \rho'_v \Delta x' \Delta A' = \Delta Q = \alpha \rho'_v \Delta x \Delta A;$$
$$\Delta t' = 0,\ \Delta t = \Delta x V / c^2,\ \Delta x' = \alpha \Delta x.$$

$$(J_{x2} - J_{x1})\Delta A = \left(\frac{\partial J_x}{\partial x}\Delta x + \frac{\partial J_x}{\partial t}\Delta t\right)\Delta A$$
$$= \left(-\frac{\partial \rho_v}{\partial t}\Delta x + \frac{\partial J_x}{\partial t}\Delta x V / c^2\right)\Delta A$$
$$= -\frac{\partial \Delta Q}{\partial t} = -\alpha \frac{\partial \rho'_v}{\partial t}\Delta x \Delta A,$$



$$J_{x1}, J_{x2} = J_x(x \mp \Delta x/2, t \mp \Delta t/2)$$
$$= J_x(x,t) \mp \frac{\partial J_x}{\partial x}\Delta x/2 \mp \frac{\partial J_x}{\partial t}\Delta t/2$$

$$\frac{\partial \rho_v}{\partial t} - \frac{\partial J_x}{\partial t} V / c^2 = \alpha \frac{\partial \rho'_v}{\partial t}$$
$$\rho_v - J_x V / c^2 = \alpha \rho'_v;\ \rho'_v + J'_x V / c^2 = \alpha \rho_v$$
$$\rho_l - I_x V / c^2 = \alpha \rho'_l;\ \rho'_l + I'_x V / c^2 = \alpha \rho_l$$

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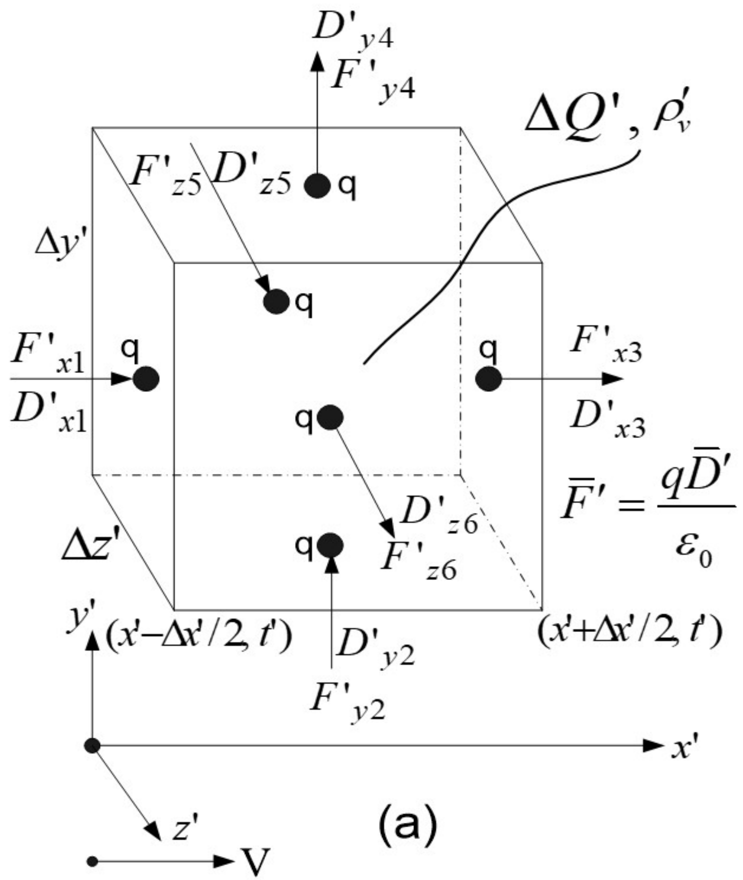
Relativistic Transformation of Gauss' Law for the Electric Field: A Gauss' Law Experiment Observed from the "Primed" Frame

$$t' = \frac{t - x\frac{V}{c^2}}{\alpha}, \quad \Delta t' = 0, \; \Delta t = \Delta x \frac{V}{c^2}$$



$$\frac{\epsilon_0}{q}\left(\frac{\partial F'_x}{\partial x'} + \frac{\partial F'_y}{\partial y'} + \frac{\partial F'_z}{\partial z'}\right)\Delta x' \Delta y' \Delta z' = \Delta Q' = \rho'_v \Delta x' \Delta y' \Delta z'$$

$$\frac{\epsilon_0}{q}\left(\frac{\partial F'_x}{\partial x'} + \frac{\partial F'_y}{\partial y'} + \frac{\partial F'_z}{\partial z'}\right) = \rho'_v$$

$$\frac{\partial D'_x}{\partial x'} + \frac{\partial D'_y}{\partial y'} + \frac{\partial D'_z}{\partial z'} = \overline{\nabla} \cdot \overline{D}' = \rho'_v$$

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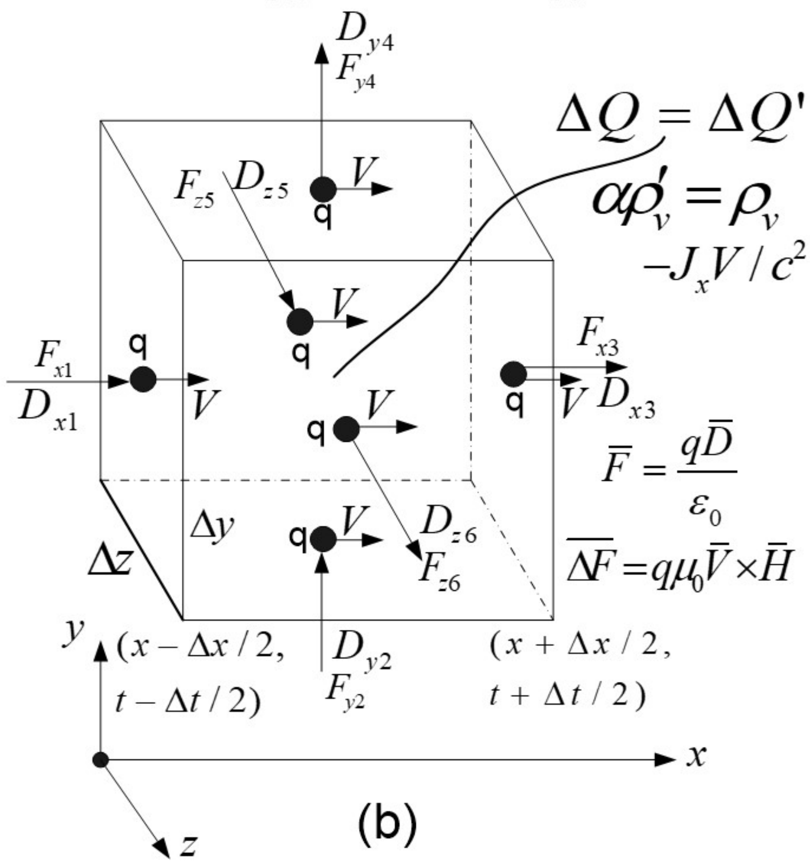
Relativistic Transformation of Gauss' Law for the Electric Field (Continued): A Gauss' Law Experiment as Observed from the "Unprimed" Frame

$$D_{x1}, D_{x3} = D_x(x \mp \Delta x/2, t \mp \Delta t/2) =$$

$$D_x(x,t) \mp \frac{\partial D_x}{\partial x}\Delta x/2 \mp \frac{\partial D_x}{\partial t}\Delta t/2$$



(b)

$$\Delta t' = 0,\ \Delta x' = \alpha\Delta x,\ \Delta t = \Delta x V/c^2$$

$$\frac{\epsilon_0}{q}\left(\frac{\partial F_x}{\partial x}\Delta x + \frac{\partial F_x}{\partial t}\frac{V}{c^2}\Delta x\right)\Delta y\Delta z + \frac{\epsilon_0}{q}\frac{\partial F_y}{\partial y}\Delta x\Delta y\Delta z$$

$$+\frac{\epsilon_0}{q}\frac{\partial F_z}{\partial z}\Delta x\Delta y\Delta z = \Delta Q = \Delta Q' = \rho'_v\,\Delta x'\Delta y'\Delta z'$$

$$= \rho'_v\,\alpha\Delta x\Delta y\Delta z = \left(\rho_v - J_x\frac{V}{c^2}\right)\Delta x\Delta y\Delta z$$

$$\frac{\epsilon_0}{q}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_x}{\partial t}\frac{V}{c^2} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = \alpha\rho'_v = \rho_v - J_x\frac{V}{c^2}$$

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Relativistic Transformation of Gauss' Law for the Electric Field (Continued): A Gauss' Law Experiment as Observed from the "Unprimed" Frame

- Timings of force measurements in the unprimed frame are different for different x coordinates

$$\Delta t' = 0 = \frac{\Delta t - \frac{V}{c^2}\Delta x}{\alpha}, \ \Delta t = \Delta x V / c^2$$

- Lengths in x coordinates in the unprimed frame are different from those in the primed frame

$$\Delta t' = 0, \ \Delta x = \frac{\Delta x' + V\Delta t'}{\alpha}, \ \Delta x' = \alpha \Delta x$$

- Charge measured in the unprimed frame is equal to that in the primed frame

$$\Delta Q = \Delta Q' = \rho'_v \Delta x' \Delta y' \Delta z' = \rho'_v \alpha \Delta x \Delta y \Delta z$$

- Charge density in the primed frame is transformed in terms of charge and current densities in the unprimed frame.

$$\alpha \rho'_v = \rho_v - J_x \frac{V}{c^2}$$

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Relativistic Transformation of Gauss' Law for the Electric Field (Continued): Ampere's Law

Gauss' Law Force Measurement, Transformed from "Primed" to "Unprimed" Frame:

$$\frac{\epsilon_0}{q}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_x}{\partial t}\frac{V}{c^2}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)=\alpha\rho'_v=\rho_v-J_x\frac{V}{c^2}$$

Modified Forces in the "Unprimed" Frame:

$$\overline{F}=\frac{q}{\epsilon_0}\overline{D}+\overline{\Delta F}=\frac{q}{\epsilon_0}\overline{D}+\frac{q}{\epsilon_0 c^2}V(\hat{x}\times\overline{H})=\frac{q}{\epsilon_0}\overline{D}+q\mu_0 V(\hat{x}\times\overline{H}),\ \mu_0=\frac{1}{\epsilon_0 c^2}$$

$$F_x=\frac{q}{\epsilon_0}D_x,\ F_y=\frac{q}{\epsilon_0}\left(D_y-H_z\frac{V}{c^2}\right),\ F_z=\frac{q}{\epsilon_0}\left(D_z+H_y\frac{V}{c^2}\right)$$

Transformed Gauss' Law with Modified Forces: Derivation of Ampere's Law:

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_x}{\partial t}\frac{V}{c^2}\right)+\left(\frac{\partial D_y}{\partial y}-\frac{\partial H_z}{\partial y}\frac{V}{c^2}\right)+\left(\frac{\partial D_z}{\partial z}+\frac{\partial H_y}{\partial z}\frac{V}{c^2}\right)=\left(\rho_v-J_x\frac{V}{c^2}\right)$$

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=\overline{\nabla}\cdot\overline{D}=\rho_v\quad \text{(Gauss' Law)}$$

$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x+\frac{\partial D_x}{\partial t},\ \overline{\nabla}\times\overline{H}=\overline{J}+\frac{\partial\overline{D}}{\partial t}\quad \text{(Amper's Law)}$$

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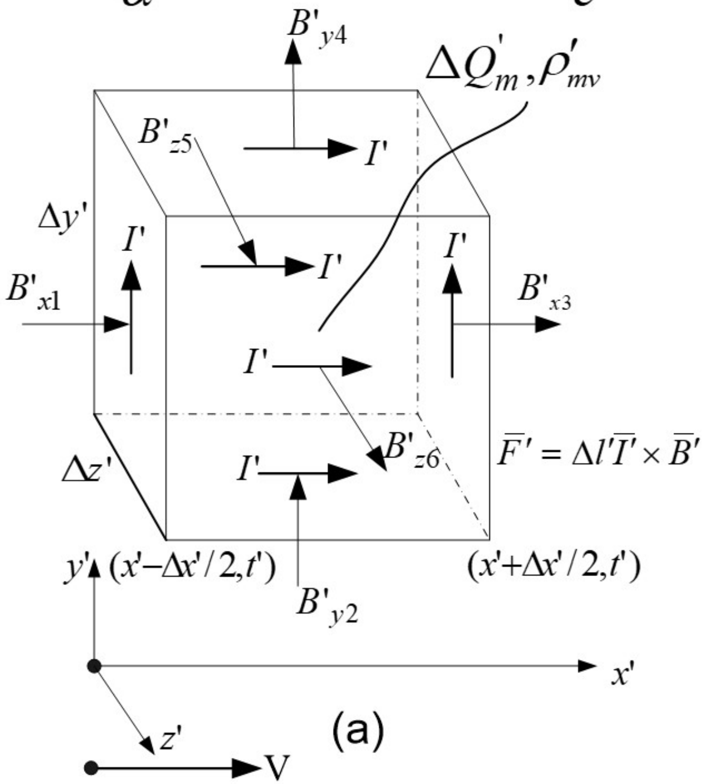
Relativistic Transformation of Gauss' Law for the Magnetic Field: A Gauss’ Law Experiment Observed from the ‘’Primed’’ Frame

$$t'=\frac{t-x\frac{V}{c^2}}{\alpha},\ \Delta t'=0,\ \Delta t=\Delta x\frac{V}{c^2}$$



(a)

$$[\frac{1}{I'_y\Delta l'_y}\frac{\partial(-F'_z)}{\partial x'}+\frac{1}{I'_x\Delta l'_x}\frac{\partial F'_y}{\partial y'}+\frac{1}{I'_x\Delta l'_x}\frac{\partial F'_z}{\partial z'}]\Delta x'\Delta y'\Delta z'$$
$$=\Delta Q'_m=\rho'_{vm}\Delta x'\Delta y'\Delta z'$$

$$\frac{1}{I'_y\Delta l'_y}\frac{\partial(-F'_z)}{\partial x'}+\frac{1}{I'_x\Delta l'_x}\frac{\partial F'_z}{\partial y'}+\frac{1}{I'_x\Delta l'_x}\frac{\partial(-F'_y)}{\partial z'}=\rho'_{vm}$$

$$\frac{\partial B'_x}{\partial x'}+\frac{\partial B'_y}{\partial y'}+\frac{\partial B'_z}{\partial z'}=\overline{\nabla}\cdot\overline{B}'=\rho'_{vm}$$

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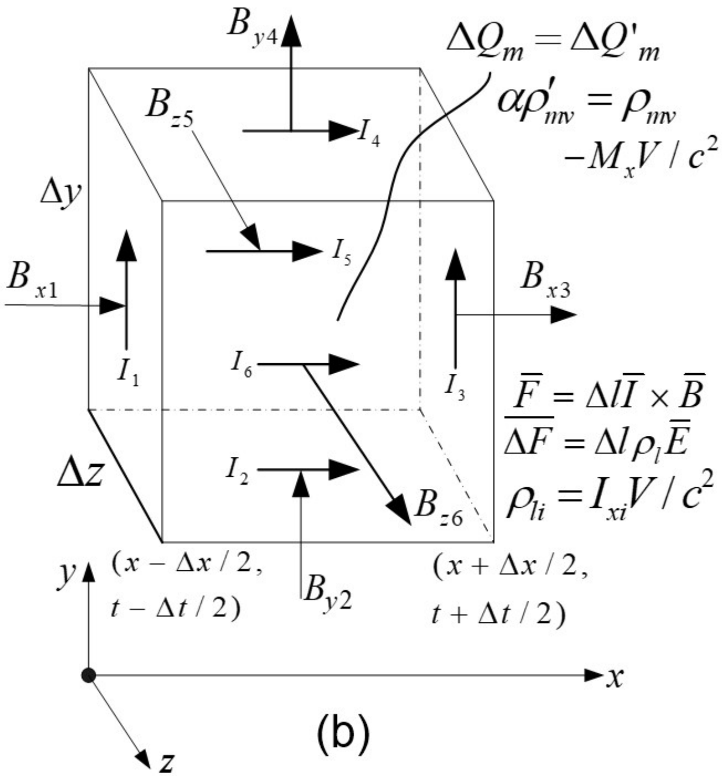
Relativistic Transformation of Gauss' Law for the Magnetic Field (Continued): A Gauss’ Law Experiment as Observed from the ‘’Unprimed’’ Frame

$$B_{x1}, B_{x3} = B_x(x \mp \Delta x/2, t \mp \Delta t/2)$$

$$= B_x(x,t) \mp \frac{\partial B_x}{\partial x}\Delta x/2 \mp \frac{\partial B_x}{\partial t}\Delta t/2$$

$$\Delta l_1 = \Delta l_3,\ \Delta l_2 = \Delta l_4 = \Delta l_5 = \Delta l_6$$

$$I_1 = I_3,\ I_2 = I_4 = I_5 = I_6$$



(b)

$$\Delta t' = 0,\ \Delta x' = \alpha\Delta x,\ \Delta t = \Delta x V/c^2$$

$$\frac{1}{I_y\Delta l_y}\left(\frac{\partial(-F_z)}{\partial x}\Delta x + \frac{\partial(-F_z)}{\partial t}\frac{V}{c^2}\Delta x\right)\Delta y\Delta z + \frac{1}{I_x\Delta l_x}\frac{\partial F_z}{\partial y}\Delta x\Delta y\Delta z$$
$$+\frac{1}{I_x\Delta l_x}\frac{\partial(-F_y)}{\partial z}\Delta x\Delta y\Delta z = \Delta Q_m = \Delta Q'_m = \rho'_{vm}\Delta x'\Delta y'\Delta z'$$
$$= \rho'_{vm}\alpha\Delta x\Delta y\Delta z = \left(\rho_{vm} - M_x\frac{V}{c^2}\right)\Delta x\Delta y\Delta z$$

$$\frac{1}{I_y\Delta l_y}\left(\frac{\partial(-F_z)}{\partial x} + \frac{\partial(-F_z)}{\partial t}\frac{V}{c^2}\right) + \frac{1}{I_x\Delta l_x}\frac{\partial F_z}{\partial y} + \frac{1}{I_x\Delta l_x}\frac{\partial(-F_y)}{\partial z}$$
$$= \alpha\rho'_{vm} = \rho_{vm} - M_x\frac{V}{c^2}$$

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Relativistic Transformation of Gauss' Law for the Magnetic Field (Continued): A Gauss' Law Experiment as Observed from the "Unprimed" Frame

- As in the Gauss' law experiment for the electric field, similar transform relations across frames also work for Gauss' law experiments for the magnetic field

$$\Delta t',\ \Delta t = \Delta x V / c^2,\ \Delta x' = \alpha \Delta x$$

$$\Delta Q_m = \Delta Q'_m = \rho'_{vm} \Delta x' \Delta y' \Delta z' = \rho'_{vm} \alpha \Delta x \Delta y \Delta z$$

$$\alpha \rho'_{vm} = \rho_{vm} - M_x \frac{V}{c^2}$$

- The test currents are to be perpendicular to the magnetic field being measured. We choose x-directed current elements for measuring y- and z-directed magnetic fields, but y-directed current elements for measuring x-directed magnetic field

$$\text{Test currents } I_x \Delta l_x \text{ (or, } I'_x \Delta l'_x\text{): for } \mathrm{B}_y, \mathrm{B}_z \text{ (or, } \mathrm{B}'_y, \mathrm{B}'_z\text{) measurements}$$

$$I_y \Delta l_y \text{ (or, } I'_y \Delta l'_y\text{): for } \mathrm{B}_x \text{(or, } \mathrm{B}'_x\text{) measurement}$$

- All the test electric current elements in the primed frame are chosen to be charge-free (neutral), so that force on the test element will depend only on the magnetic field. These test currents may look charged in the unprimed frame

$$\rho_l - I_x \frac{V}{c^2} = \alpha \rho'_l = 0,\ \rho_l = I_x \frac{V}{c^2},\ q = \Delta l \rho_l = \Delta l_x I_x \frac{V}{c^2},\ \text{for } \mathrm{B}_y, \mathrm{B}_z \text{ Measurements}$$

$$\rho_l = \rho'_l = 0,\ q = \Delta l_y \rho_l = 0,\ \text{for } \mathrm{B}_x \text{ Measurement}$$

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Relativistic Transformation of Gauss' Law for the Magnetic Field (Continued): Faraday's Law

Gauss' Law Force Measurement, Transformed from "Primed" to "Unprimed" Frame:

$$\frac{1}{I_y \Delta l_y}\left(\frac{\partial(-F_z)}{\partial x}+\frac{\partial(-F_z)}{\partial t}\frac{V}{c^2}\right)+\frac{1}{I_x \Delta l_x}\frac{\partial F_z}{\partial y}+\frac{1}{I_x \Delta l_x}\frac{\partial(-F_y)}{\partial z}=\alpha\rho'_{vm}=\rho_{vm}-M_x\frac{V}{c^2}$$

Forces in the "Unprimed" Frame:

$$\overline{F}=\Delta l\overline{I}\times\overline{B}+q\overline{E}$$

$$-F_z=I_y\Delta l_y B_x,\ \text{q=0},\ \text{for } B_x \text{ Measurement}$$

$$F_z=I_x\Delta l_x B_y+I_x\Delta l_x E_z\frac{V}{c^2},\ \ -F_y=I_x\Delta l_x B_z-I_x\Delta l_x E_y\frac{V}{c^2},\ \text{for } B_y, B_z \text{ Measurements}$$

$$\rho_l-I_x\frac{V}{c^2}=\alpha\rho'_l=0,\ \rho_l=I_x\frac{V}{c^2},\ q=\Delta l\rho_l=\Delta l_x I_x\frac{V}{c^2}$$

Transformed Gauss' Law with Modified Forces: Derivation of Faraday's Law:

$$\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_x}{\partial t}\frac{V}{c^2}\right)+\left(\frac{\partial B_y}{\partial y}+\frac{\partial E_z}{\partial y}\frac{V}{c^2}\right)+\left(\frac{\partial B_z}{\partial z}-\frac{\partial E_y}{\partial z}\frac{V}{c^2}\right)=\left(\rho_{vm}-M_x\frac{V}{c^2}\right)$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=\overline{\nabla}\cdot\overline{B}=\rho_{vm}\ \ \text{(Gauss' Law)}$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-M_x-\frac{\partial B_x}{\partial t},\ \overline{\nabla}\times\overline{E}=-\overline{M}-\frac{\partial\overline{B}}{\partial t}\ \ \text{(Faraday's Law)}$$

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Conclusion

- Charge invariance across reference frames, based on simple relativistic space-time relations, is the fundamental basis of all electromagnetic principles.
- A new form of Gauss' law, based on force measurements over a closed surface performed in a given frame, and "observed'' from any other frame, is required in order to ensure charge invariance across reference frames. The new Gauss' law would be more fundamental than conventional Gauss' law, which is only established in each individual frame.
- The new relativistically invariant form of Gauss' law allows simple, direct derivation of Maxwell's equations from first principles.
- Relativistic transformation of Gauss's law for the electric field leads to the derivation of Ampere's law. The magnetic field in the Ampere's law is simply a "synthesized field", introduced in order to enforce charge invariance across reference frames.
- Similarly, relativistic transformation of Gauss' law for the magnetic field would lead to the derivation of Faraday's law. Like the electric charge, any non-zero magnetic charge and associated magnetic current would also be fundamentally required to satisfy continuity and invariance across reference frames. However, no magnetic charge or current has been observed in nature, in which case they may be equated to zero in the derivation of the Faraday's law.