Market Participants Neither Commit Predictable Errors nor Conform to REH: Evidence from Survey Data of Inflation Forecasts

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Abstract

We develop a novel characterization of participants' forecasts with a mixture of normal variables involving a Markov chain, and formulate four new behavioral specifications, including three implied by the diagnostic expectations approach. We also consider DE's time-invariant specification, originated by Gennaioli and Shleifer, as well as two full-information REH specifications. We derive several new predictions for Coibion and Gorodnichenko's regression of forecast errors on forecast revisions. Predictions of all seven specifications are inconsistent with the observed instability of CG's individual-level regressions based on inflation forecasts. Our findings suggest how to build on key insights of the REH and behavioral approaches.

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1 Introduction

Coibion and Gorodnichenko (CG, 2015) proposed regressing market participants' forecast errors on their forecast revisions, as measured by survey data, to test predictions of alternative theoretical specifications implied by the rational expectations hypothesis (REH). CG's (pp. 2651, 2653) ingenious idea was that predictions of full- and limited-information REH models "map" onto the constant term and the slope of their regression.[1] This mapping is direct, based on the assumption that the process driving outcomes and participants' forecasting strategies – how they form forecasts on the basis of available information – is time-invariant, as exemplified by a standard AR process.

Here, we extend the applicability of the CG regression to testing predictions of theoretical specifications of participants' forecasting strategies that allow for change in the process driving outcomes. Following a prevailing practice, we represent change in these specifications with a stationary Markov chain.[2]

Our approach rests on a novel characterization of participants' forecasting strategies involving a Markov chain with a mixture of normal probability density functions (pdfs). This characterization plays a key role in our formulation of alternative specifications of forecasting strategies, as well as in deriving these specifications' predictions for the constant term and the coefficient of forecast revision (the slope) in the CG regression.

We formulate three new specifications implied by the diagnostic expectations (DE) approach. By allowing for change, our specifications extend the original formulation of DE by Gennaioli and Shleifer (GS, 2018, p.155) and their co-authors, which constrain the process driving outcomes and how participants forecast them to be time-invariant.

We contrast GS's formulation of DE with a version of Barberis, *et al.*'s (1998) pre-DE behavioral model, as well as with of our alternative specification

[1] CG focused on two classes of limited-information REH models: the noisy information models, originated by Lucas (1973) , and the sticky-information model proposed by Mankiw and Reis (2002). CG (p. 2655) also considered predictions for their regression of "a number of extensions of [REH] models with information rigidities."

[2] Hamilton (1988) originated modeling of change in REH models with Markov chains. See Hamilton (2008) for an extensive review of subsequent developments.

of DE based on that model. Thus, including DE's time-invariant formulation, we test the predictions of five behavioral specifications of participants' forecasts for the coefficients of the CG regression.

We also specify and test the predictions of a full-information rational expectations (FIRE) model involving a Markov chain, as well as those of its time-invariant counterpart, for the CG regression.

Our dataset consists of time-series of inflation forecasts from 1969 to 2014 by 24 individuals included in the US Survey of Professional Forecasters (SPF), with over 50 observations by each individual for the one-quarter-ahead forecast revision. We estimate a CG regression for each of the forecasters.

Our estimation of the individual CG regressions yields two main findings. First, each of the five behavioral specifications that we consider appear to be inconsistent with survey data for each of the forecasters. Our estimates of the 24 individual CG regressions are also inconsistent with the time-invariant FIRE specification, as well as with FIRE allowing the process driving outcomes to evolve according to a Markov chain.

Because of the central importance of inflation expectations in macroeconomic models, the inflation forecast, as measured by the survey data, has been the most prominently studied variable.[3] However, earlier studies have primarily examined the aggregate of the survey data of the three-quarters-ahead forecasts of inflation.[4] CG (2015) show that hypothesizing that the time-invariant model represents the process driving inflation, and that the aggregate of participants' forecasts conforms to the sticky or noisy information REH-alternatives alternatives to FIRE, predicts a constant equal to zero and a positive slope in their regression. However, as they (p. 2652) emphasize, this prediction "obtains only when averaging across agents." Because we confront predictions of specifications of participants' forecasts with the individual sur-

[3] For examples, see Coibion and Gorodnichenko (2012), Angeletos, Huo and Sastry (2021), and Bianchi, Ludvigson, and Ma (2021).

[4] In Frydman and Stillwagon (2021), we report the estimates of the individual CG regressions for the three-quarters-ahead forecasts of inflation. They are essentially the same as those reported here for the one-quarter-ahead forecasts, thereby providing further evidence of the empirical inadequacy of the theoretical specifications of participants' forecasts that we consider.

vey data, we leave a reexamination of the REH-implied limited information specifications on the basis of aggregate data for a companion paper.

However, our theoretical framework reveals a difficulty inherent in relying on the CG regression to test the REH limited-information alternatives to FIRE on the basis of aggregate data. It also reveals the CG regression's inability to distinguish predictions of behavioral specifications, including DE, from those of FIRE involving a Markov chain, on the basis of both individual and aggregate data.

A finding of a positive coefficient of forecast revisions in the CG regression has traditionally been interpreted as an "underreaction," and a negative slope coefficient as an "overreaction." Once change in how outcomes unfold over time is represented with a Markov chain, FIRE predicts that the constant is equal to zero, as in time-invariant counterpart. However, unlike its time-invariant counterpart, FIRE predicts the slope as being either positive or negative. Thus, introducing a Markov chain into the process driving outcomes renders FIRE's predictions compatible with either underreaction, implied by REH's limited-information specifications on the aggregate level, or overreaction, implied GS's specification of DE on the individual level. Remarkably, FIRE's prediction of a negative coefficient of forecast revisions in the CG regression is the same as that of GS's specification of DE, which has been proposed to formalize market participants' representativeness-driven deviations away from FIRE.

Our approach avoids this difficulty by relying on the central implications of our theoretical framework: once *change* in the process driving outcomes and in how participants forecast them is represented with a stationary Markov chain, the predicted constant and slope coefficients in the CG regression *do not change* over time. Thus, subjecting the constant and the coefficient of forecast revisions in the CG regression to tests of structural breaks provides a hitherto unexplored way to test alternative models of expectations, including DE and other behavioral specifications, as well as FIRE. We find that these coefficients undergo structural breaks, thereby revealing the inconsistency of both REH and behavioral specifications with survey data on participants' forecasts.

The structural instability of the CG regression therefore points to a primary explanation of our findings. Despite their apparent differences, REH and behavioral specifications of participants' forecasting strategies typically rest on a shared premise: these strategies can be represented with a stationary stochastic process over an infinite past and indefinite future.

Our analysis also yields several novel theoretical implications regarding an increasingly accepted description advanced by GS (p. 11) that DE "builds on the famous representativeness heuristic of human judgment uncertainty... proposed by Daniel Kahneman and Amos Tversky." This has recently led them to assert that DE is "a *psychologically founded* non-Bayesian model of belief formation," which implies that participants systematically and predictably "overreact to news" about a payoff-relevant variable, relatively to REH-implied forecasts (Bordalo, *et al.*, 2020, p. 2749, emphasis added).

However, as we show here, the overreaction supposedly implied by DE is not a regularity. Rather, it is an artifact of GS's particular specification of DE, which rests on their assumption that how the representativeness heuristic impels participants to deviate from REH can be formalized with the REH-implied forecast revisions.

GS's "REH-like" specification of DE represents the representativeness-driven "distortion" of participants' forecasts, relative to REH, as being based on an "objective" process driving outcomes, as formalized by an economist's model. Because this "objective" process, according to Muth's (1961) hypothesis, underpins REH, DE's supposed overreaction, relative to REH, is driven *solely* by news about the payoff-relevant outcomes.

GS's specification of DE as being based on the pdf underpinning REH appears to be at odds with behavioral economists' empirical evidence. For example, Barberis, *et al.* (1998, pp. 310-317) provide an extensive and thorough review of evidence of how the representativeness heuristic, as well as other psychological mechanisms, influence market participants' assessment of uncertainty about stock returns. In contrast to GS's assumption that the "distortion" caused by the heuristic arises solely from news, Barberis, *et al.* conclude that,

> the psychological evidence *does not* tell us quantitatively what kind

> of information is strong and salient (and hence is overreacted to) and what kind of information is low in weight (and hence is underreacted to) (p. 317, emphasis added).

Adhering to the pre-DE behavioral models' core premise, Barberis, *et al.* argue (p. 318) that, "If our model is to generate the [pattern] of returns documented in the empirical studies, the investor must be using the wrong model to form expectations." They formulate such a "wrong" model by assuming that, while an economist's model specifies earnings to evolve according to a random walk, the investor "thinks that the world moves between two 'states' or 'regimes' and that there is a different model governing earnings in each regime." As is typical in the literature, they formalize this assumption with the two-state Markov chain.

Our alternative specification of DE builds on Barberis, *et al.*'s model. Once we acknowledge the relevance of behavioral economists' findings, DE no longer implies the regularity of overreaction. Depending on the values of the Markov chain's transition probabilities, other model parameters, and the realizations of payoff-relevant variables, DE overreacts in some periods and underreacts in others periods, relative to the REH-implied forecast.

To be sure, our finding that behavioral specifications are inconsistent with the survey data is compatible with behavioral economists' compelling evidence that psychological and other non-fundamental factors have a substantial influence on participants' forecasts.[5] What our findings of structural instability reject is behavioral models' formalization of this evidence with a stationary stochastic process.

Moreover, although our findings are inconsistent with REH, they should not be interpreted as a rejection of the relevance of Muth's (1961) hypothesis in specifying participants' forecasts. As we discuss in Section 14, to build models that rest on Muth's hypothesis and yet recognize the relevance of behavioral findings requires acknowledging that the process driving market outcomes un-

[5] For early reviews of this evidence, see Barberis, *et al.* (1998) and Shleifer (2000). Throughout their book, GS (2018) extensively discuss subsequent studies documenting the influence of non-fundamental factors on participants' forecasts.

dergoes change that cannot be represented with a stationary stochastic process, such as a Markov chain. Our findings suggest that, as Knight (1921) emphasized, market participants recognize the uncertainty that such unforeseeable change engenders and revise their forecasting strategies accordingly.

The plan of the paper is as follows. Sections 2 and 3 provide a formal overview of the DE approach in the context of the Linda experiment and highlight the key steps in applying the approach in macroeconomic and finance models. Building on GS's formulation, Section 4 presents a general definition of overreaction in terms of the means of the "objective" and reference pdfs, which underpin an economist's specification of DE.

Using this definition, Sections 5 and 6 show that the regularity of overreaction is an artifact of GS's specification of the reference pdf as being based on the “objective” process represented by an economists' model. Section 7 formulates a Markov counterpart of GS's time-invariant specification of DE and characterizes the “objective” and reference distributions with mixtures of normal pdfs. This characterization implies that, as with GS's time-invariant specification, overreaction is assumed to be of the same sign and a fixed proportion of the REH-implied forecast revision. Section 8 formulates the specification of DE based on Barberis, *et al.*'s (1998) model and shows that DE no longer implies the regularity of overreaction.

Relying on the characterization of the REH, reference, and DE specifications with mixtures of normal pdfs established in the foregoing sections, Sections 9, 10 and 11 derive predictions of the theoretical specifications of participants' forecasts for the coefficients of the CG regression. Section 12 summarizes these predictions, and Section 13 presents our findings that the predictions of all of the behavioral and FIRE specifications considered in the paper are inconsistent with the survey data on forecasts of inflation by each of the 24 forecasters. Section 14 addresses the implications of our findings for building macroeconomic and finance models. The proofs are presented in Online Appendix A. The sketch of the econometric methodology and detailed estimates of the CG regression for each of the 24 forecasters are in Online Appendix B.

2 Diagnostic Expectations in the Linda Experiment

Here, we follow Gennaioli and Shleifer (2018) and provide an overview of the main concepts underpinning their DE approach in the context of the Linda experiment. The simplicity of the experiment enables us to highlight a difficulty overlooked by GS, but which is inherent in any application of the representativeness heuristic in economic models: events that in some contexts appear representative of other events may, in other contexts, appear unrepresentative of those events. As Kahneman and Tversky (1972, p. 431) acknowledged, "Representativeness, like perceptual similarity, is easier to assess than to characterize. In both cases, no general definition is available."

2.1 An Overview of the Linda Experiment

The Linda experiment features a fictitious 31-year-old woman who currently works as a bank teller. As a college student, Linda engaged in "progressive" activities, including opposing discrimination, advocating for social justice, and participating in anti-nuclear demonstrations. We treat the set of 31-year-old women who graduated from college as a population, which we denote with W. We denote the subset of those who engaged in progressive activities while in college with $H^p \subset W$.

Tversky and Kahneman (TK, 1983 p. 297) presented the following statements to their experiment's subjects:

- Linda is a bank teller, places her among individuals in the set $T \subset W$.
- Linda is a bank teller who is also active in the feminist movement (the set F), which places her among the individuals comprising the intersection $T \cap F \subset W$.

Kahneman and Tversky asked the subjects whether it was more or less probable that Linda is among the bank tellers who are also active in the feminist movement (in $T \cap F$) than that she is among generic bank tellers (in T). An overwhelming majority of subjects responded that it is more probable that Linda is in $T \cap F$ than that she is in T. This finding was then replicated in many Linda-like experiments in a variety of contexts.

2.2 Representativeness in an Experimental Setting

TK (pp. 296-297, 299) hypothesized that their findings could be explained by subjects' reliance on a psychological mechanism, which they called the representativeness heuristic and operationalized in terms of the ratio of the relevant frequencies.[6]

Definition 1 *"An attribute is representative of a class if it is very diagnostic, that is, if the relative frequency of this attribute is* much higher *in that class than in a relevant reference class"*

For example, in the context of the Linda experiment, TK consider the event $T \cap F$ as an "attribute," H^p as a "class," and individuals who do not have a history of progressive activities, H^{np}, as a "reference class." The idea underpinning TK's operationalization of Definition 1 was that one would expect feminist bank tellers to be more prevalent among the individuals who, like Linda, have a progressive history, $f(T \cap F|H^p)$, than among the individuals who do not have that history, $f(T \cap F|H^{np})$.[7] It is this apparently much greater prevalence that TK referred to in describing $T \cap F$ as being "very diagnostic" of H^p, which they formalized with $\frac{f(T \cap F|H^p)}{f(T \cap F|H^{np})} >> 1$

We assume that the uncertainty about the events in the Linda experiment can be represented with a probability measure on the space $\Omega = H^p \cup H^{np}$. Thus, we operationalize Definition 1 in terms of the ratio of the conditional probabilities:

$$R(A|C, C^{ref}) = \frac{P(A|C)}{P(A|C^{ref})}, \tag{1}$$

where, in the context of our foregoing example, $A = T \cap F \subset \Omega$, $C = H^p$, and $C^{ref} = H^{np}$. According to Definition 1, A "is representative" of C if it is "very diagnostic," that is, if

$$R(A|C, C^{ref}) > d >> 1. \tag{2}$$

[6] All citations only to page numbers refer to TK (1983).

[7] $f(T \cap F|H^p) = \frac{n(T \cap F \cap H^p)}{n(H^p)}$, $f(T \cap F|H^{np}) = \frac{n(T \cap F \cap H^{np})}{n(H^{np})}$, and $n(\cdot)$ stands for a number of individuals in a respective set.

2.3 Diagnostic Probabilities

GS (pp. 144-152) introduce DE in the context of the Linda experiment. They represent subjects' assessment of uncertainty with a so-called distorted probability measure and specify how representativeness distorts subjective probabilities (p. 148) as follows:

$$P^{DE}(A|C) = P(A|C)\left[R(A|C, C^{ref}))\right]^{\theta} Z, \tag{3}$$

where $P^{DE}(\cdot|\cdot)$ specifies a distorted (subjective) probability on the space Ω, which we refer to as a diagnostic probability, $P(\cdot|\cdot)$ is the "objective" probability, and $\theta > 0$ formalizes the degree of distortion. Z ensures that (3) specifies a well-defined probability.

3 From the Laboratory to Real-World Markets

To operationalize how the representativeness heuristic "distorts" market participants' assessment of uncertainty, an economist would specify the probability distribution of outcomes (an analog of the attribute $T \cap F$) that he aims to explain in terms of a set of causal variables (an analog of the class H^P), usually called information available to participants. Because any formal economic model rests on the premise that it specifies the "objective" process driving outcomes, an economist, relying on Muth's (1961) hypothesis, can then represent a participant's "rational" assessment of uncertainty, and her REH forecasts, with the "objective" distribution, as specified by the economist's model.

However, there does not appear to be a theoretical argument that would enable an investigator – an experimental psychologist or an economist – to specify the reference class. By providing information to the subjects that Linda has a progressive history, H^P, TK (p. 300) aimed to influence them to compare her to those who do not have that history, thereby considering H^P as the relevant reference class.

In real-world market settings, by contrast, an economist has no way to influence participants' interpretation of the context within which they assess representativeness of uncertain events. However, empirical evidence on how market participants actually assess uncertainty provides a basis for specifying the reference class that participants might have considered relevant. Behavioral economists have provided compelling evidence that participants' forecasts do not conform to REH, and formalizing this evidence could provide the basis for specifying the reference class.

GS proposed DE to provide a unified approach to explain such behavioral findings. However, GS chose to formalize their argument – that the representativeness heuristic impels participants to overreact to information, relative to the REH forecasts – with a specification of the reference class of outcomes that is based on the "objective" probability distribution, which underpins REH. However, as we show in Section 8, once we specify the probability distribution of the reference class of outcomes on the basis of behavioral economists' findings, the supposedly "distorting" influence of the representativeness heuristic, as formalized with the analog of (3), does not result in the regularity of overreaction. DE overreacts to information in some periods and underreacts in others.

4 Representativeness in Macroeconomics and Finance Models

In contrast to the Linda experiment, the concept of representativeness in macroeconomic and finance models involves continuous random variables. To fix ideas, we consider a payoff-relevant variable $x_{t+1} = \ln \widetilde{x}_{t+1}$, and formalize an "attribute" (an analog of $A = T \cap F$ in (1)) with the measurable event, $x_{t+1} \in A \subset \mathbb{R}^{+}$, and a "class" (an analog of $C = H^{p}$) with an event $x_t \in C \subset \mathbb{R}^{+}$. We also operationalize the "reference class" (an analog of $C^{ref} = H^{np}$) with an event $x_t^{ref} \in C^{ref} \subset \mathbb{R}^{+}$.

GS (p. 154) define $x_{t+1} \in A$'s representativeness of x_t, relative to x_t^{ref}, in terms of the ratio of conditional probability density functions (pdfs), as follows:

$$R^{gs}(x_{t+1}|x_t, x_t^{ref}) = \frac{f(x_{t+1}|x_t)}{f^{ref}(x_{t+1}|x_t^{ref})} > 1, \; x_{t+1} \in A, x_t \in C, x_t^{ref} \in C^{ref} \qquad (4)$$

where $f(x_{t+1}|x_t)$ is the "objective" (conditional) pdf of x_{t+1}, as hypothesized by an economist's model.[8] We refer to $f^{ref}(x_{t+1}|x_t^{ref})$ as a (conditional) reference pdf, which is assumed by an economist to characterize the reference class that participants consider relevant. We note that GS's (p. 154) specification of the reference class of outcomes specifies $x_t^{ref} = x_{t-1}$.

TK define representativeness in terms of probabilities (or, equivalently frequencies of discrete events), which for continuous variables can be written as

$$R(x_{t+1}|x_t, x_t^{ref}) = \frac{\int_A f(x_{t+1}|x_t) dx_{t+1}}{\int_A f^{ref}(x_{t+1}|x_t^{ref}) dx_{t+1}} > 1, \; x_t \in A, x_t^{ref} \in C^{ref}. \qquad (5)$$

However, if the ratio of "objective" and reference pdfs satisfies (4), there exists an event $x_{t+1} \in A$, which is representative of x_t, relative to x_t^{ref}, in the sense that (5)

4.1 Tractable Specification

To render the operationalization in (4) tractable in deriving the testable predictions of macroeconomic and finance models, GS (p. 155) specify the "objective" pdf of x_{t+1}, conditional on x_t as

$$f(x_{t+1}|x_t) = \frac{1}{\sigma_{t+1|t}\sqrt{2\pi}} \exp\left[-\frac{(x_{t+1} - m_{t+1|t})^2}{2\left(\sigma_{t+1|t}\right)^2} \right], \; x_{t+1} \in A, x_t \in C, \qquad (6)$$

[8] In addition to x_t, an economist's model typically specifies the conditioning set to include other relevant information (such as realizations of the model's variables) up to time t. Allowing for such a larger information set would not alter any of our conclusions here.

where $m_{t+1|t}$ and $\left(\sigma_{t+1|t}\right)^2$ denote the conditional mean and the variance. GS (p. 155) also assume that the reference class that underpins participants' assessment of x_{t+1}'s representativeness can be characterized with the normal pdf:

$$f^{ref}(x_{t+1}|x_t^{ref}) = \frac{1}{\sigma_{t+1|t}^{ref}\sqrt{2\pi}} \exp\left[-\frac{(x_{t+1}-m_{t+1|t}^{ref})^2}{2\left(\sigma_{t+1|t}^{ref}\right)^2}\right], \; x_{t+1}\in A, \; x_t^{ref}\in C^{ref}, \tag{7}$$

where $m_{t+1|t}^{ref}$ and $\left(\sigma_{t+1|t}^{ref}\right)^2$ denote the conditional mean and variance.

4.2 Diagnostic Expectations

Using (4), GS (p. 154) specify the "distorted" pdf of x_{t+1} in the class x_t:

$$f^{de}(x_{t+1}|x_t) = f(x_{t+1}|x_t)\left[R^{gs}(x_{t+1}|x_t, x_t^{ref})\right]^{\theta} Z(\theta, x_t, x_{t-1}), \tag{8}$$

where, we refer to $f^{de}(x_{t+1}|x_t)$ as the diagnostic pdf, $\theta > 0$, and $Z(\theta, x_t, x_{t-1})$ is specified to ensure that $f^{de}(x_{t+1}|x_t)$ integrates to 1. We denote the conditional mean of a diagnostic density with $m_{t+1|t}^{de}$. GS call $m_{t+1|t}^{de}$ a diagnostic expectation (DE) of x_{t+1}, conditional on x_t.
GS's Proposition 5.1. (p. 155), which we restate here, provides the basis for their argument that DE implies the regularity of overreaction.

Proposition 2 *Suppose that, as specified in (6) and (7), the "objective," and reference (conditional) pdfs underpinning representativeness, in (4), are normal. Then, provided that* $(1+\theta)\left(\sigma_{t+1|t}^{ref}\right)^2 > \theta\left(\sigma_{t+1|t}\right)^2$, *there exists* $Z(\theta, x_t, x_{t-1})$ *that renders the diagnostic pdf,* $f^{de}(x_{t+1}|x_t)$ *in (8), a well-defined normal pdf with the following conditional mean and variance,*

$$m_{t+1|t}^{de} = m_{t+1|t} + \gamma\left(m_{t+1|t} - m_{t+1|t}^{ref}\right), \tag{9}$$

$$\left(\sigma_{t+1|t}^{de}\right)^2 = \frac{\gamma\left(\sigma_{t+1|t}^{ref}\right)^2}{\theta}, \tag{10}$$

where

$$\gamma = \theta \frac{\left(\sigma_{t+1|t}\right)^2}{\left(\sigma^{ref}_{t+1|t}\right)^2 + \theta\left[\left(\sigma^{ref}_{t+1|t}\right)^2 - \left(\sigma_{t+1|t}\right)^2)\right]} > 0. \tag{11}$$

Proof: GS (pp. 217-19).

4.2.1 REH-Implied Specification of Participants' Forecasts

Muth (1961, p. 316) advanced the pathbreaking hypothesis that an economist could formally relate a participant's forecasts to "the way the economy works" by specifying them as being consistent with an economic model's specification of the process driving outcomes. Muth implemented his hypothesis in a model that assumed that how outcomes have unfolded over an infinite past and will unfold over an indefinite future can be represented with a stationary stochastic process. It was this implementation that came to be known as the rational expectations hypothesis (REH).

Adopting Muth's hypothesis, the conditional mean and variance of the pdf characterizing the REH forecast are the same as their "objective" counterparts, that is, $m^{reh}_{t+1|t} = m_{t+1|t}$ and $\sigma^{reh}_{t+1|t} = \sigma_{t+1|t}$. This implies that $m^{de}_{t+1|t}$ in (9) can be written as

$$m^{de}_{t+1|t} = m^{reh}_{t+1|t} + \gamma\left(m^{reh}_{t+1|t} - m^{ref}_{t+1|t}\right), \tag{12}$$

where γ in (11) is defined accordingly. GS (p.155) refer to $m^{de}_{t+1|t} > m^{reh}_{t+1|t}$ ($m^{de}_{t+1|t} < m^{reh}_{t+1|t}$) as the "overreaction" ("underreaction") of DE, relative to the REH forecast. Proposition 2 shows that if $f(x_{t+1}|x_t)$ and $f^{ref}(x_{t+1}|x_t)$ are normal, then DE overreacts if and only if $m^{reh}_{t+1|t} > m^{ref}_{t+1|t}$.

5 Representing Deviations from REH as Driven by Revision of the REH Forecast

GS proposed DE as a new approach to specifying forecasts in behavioral-finance models that aimed to explain empirical findings that participants' forecasts do not conform to REH. However, their specification of the reference pdf shares a key feature with its REH counterpart: both are based on the "objective" process driving outcomes, as formalized by an economist's model.

However, in contrast to the REH forecast, which is conditional on x_t, GS (p. 154) specified the mean of the reference pdf, $m^{ref}_{t+1|t}$ as conditional on x_{t-1}. We refer to this specification as REH-like and denote it with $m^{reh}_{t+1|t-1}$. We state this key assumption of GS's specification of DE as follows:

Assumption 3 *The "distorting" influence of the representativeness heuristic on participants' forecasts, $m^{reh}_{t+1|t} - m^{ref}_{t+1|t}$, is driven solely by the revision of its REH counterpart, which we formally state as follows*

$$m^{de}_{t+1|t} - m^{reh}_{t+1|t} = \gamma \left(m^{reh}_{t+1|t} - m^{ref}_{t+1|t} \right) = \gamma \left(m^{reh}_{t+1|t} - m^{reh}_{t+1|t-1} \right). \tag{13}$$

This assumption implies that the supposed regularity of overreaction is in fact generated by a well-known property of REH forecasts: *by design*, the revision of such a forecast is driven solely by the time-t realization of news about x_t.

6 Overreaction as an Artifact of the REH-like Specification of the Reference PDF

GS (p.174) illustrate their argument that DE implies an overreaction in the context of the following standard AR(1) model,

$$X_{t+1} = \rho X_t + \mu + \varepsilon_{t+1}, \tag{14}$$

where $0 < \rho < 1$ and μ are constants, and $\varepsilon_t \sim iidN(0, \sigma^2)$. In the context of this section, this model specifies the "objective" process driving a payoff-relevant variable x_t. Thus, according to Muth's hypothesis,

$$m^{reh(gs)}_{t+1|t} = E(X_{t+1}|x_t) = \rho x_t + \mu \tag{15}$$

$$= \rho^2 x_{t-1} + (1+\rho)\mu + \rho e_t, \tag{16}$$

$$\left(\sigma^{reh(gs)}_{t+1|t}\right)^2 = \sigma^2, \tag{17}$$

where e_t, in (16), denotes the realization of ε_t.

Furthermore, according to Assumption 3, the mean and the variance of the reference pdf, in (7), are given by

$$m_{t+1|t}^{ref(gs)} = E(X_{t+1}|x_{t-1}) = \\ = \rho^2 x_{t-1} + (\rho+1)\mu, \tag{18}$$

$$\left(\sigma_{t+1|t}^{ref(gs)}\right)^2 = (1+\rho^2)\sigma^2. \tag{19}$$

Because the "objective" and reference pdfs are normal and $\left(\sigma_{t+1|t}^{ref(gs)}\right)^2 > \left(\sigma_{t+1|t}^{reh(gs)}\right)^2$, Proposition 2 holds, which together with Assumption 3, implies that

$$m_{t+1|t}^{de(gs)} - m_{t+1|t}^{reh(gs)} = \gamma^{(gs)}\left(m_{t+1|t}^{reh} - m_{t+1|t-1}^{reh}\right) = \left(m_{t+1|t}^{reh} - m_{t+1|t-1}^{reh}\right) = \gamma^{(gs)}\rho e_t, \tag{20}$$

where $\gamma^{(gs)} = \frac{\theta}{(1+\rho^2)(1+\theta)}$, and e_t is the realization of ε_t.

GS (p. 155) refer to $e_t > 0$ ($e_t < 0$) as good (bad) news about the payoff-relevant outcome x_t. Expression (20) shows that the supposed regularity of overreaction, relative to REH, is an artifact of GS's Assumption 3: good (bad) news leads participants to overreact in the same direction and in the proportionately (predictable) magnitude as the REH forecast revision.

7 Allowing for Change in the REH-like Specification of DE

The AR process, in (14), exemplifies the usual structure of macroeconomic and finance models, an overwhelming majority of which assume away change in the process driving outcomes. Models that recognize change in this process typically represent it with a Markov chain. Such representations imply that the news about x_t comprises the realizations of ε_t as well as the realized state of the Markov chain at t.

We show here that the REH-like specification of DE involving a Markov component in both the REH and reference pdfs implies overreaction to news.

As with the time-invariant specification, this overreaction is an artifact of GS's Assumption 3 that the "distorting" influence of the representativeness heuristic on participants' forecasts can be represented with a revision of the REH forecast.

7.1 A Markov Specification of the Change in the Process Driving An Outcome

We follow the prevailing practice in a particularly simple way by allowing the mean of the process, in (14), to change over time, which we formally state as follows:

$$X_{t+1} = \rho X_t + \mu_{t+1} + \varepsilon_{t+1}, \tag{21}$$

where μ_t evolves according to a Markov chain, which switches between two states, $\mu^{(1)}$ and $\mu^{(2)}$ with the transition probabilities p_{12} and p_{21} . Here $0 < \rho < 1$ is a constant, and $\varepsilon_t \sim iidN(0, \sigma^2)$. It follows from (21) that, while x_t and μ_{t-i} for $i = 0, 1...$ are dependent, x_t and μ_{t+i} for $i = 1, 2....$are independent.[9]

Macroeconomic and finance models typically constrain the parameters of a Markov chain, such as $(\mu^{(1)}, \mu^{(2)}, p_{12}, p_{21})$, to remain unchanging over an infinite past and indefinite future. Thus, in the context of these models, the unconditional distribution of μ_t eventually converges to a steady-state (stationary) probability distribution (Lawler, 2006, p. 15). In accordance with the usual practice, we make the following assumption:

Assumption 4 *The distribution of the Markov process μ_t is stationary: $P(\mu_t = \mu^{(1)}) = \pi$, $P(\mu_t = \mu^{(2)}) = (1 - \pi)$, for all t.*

This assumption implies that for all t

$$E(\mu_t) = \pi\mu^{(1)} + (1 - \pi)\mu^{(2)}, \tag{22}$$

$$V(\mu_t) = \pi(1 - \pi)\left(\mu^{(1)} - \mu^{(2)}\right)^2, \tag{23}$$

where the expression for $V(\mu_t)$ is derived in the proof of Lemma 6 in Online Appendix A.

[9] To save on notation, (lower case) μ_r denotes a random variable.

7.2 REH Forecasts as a Mixture of Normal PDFs

Allowing μ_t to evolve according to a Markov chain implies that, in contrast to (14), the conditional pdf of the REH-implied forecasts is no longer simply normal. However, the following lemma shows that the pdf of x_{t+1}, conditional on x_t, implied by (21) is a mixture of the two normal pdfs with the following means and variances:

$$m_{t+1|t}^{(mk,i)} = E(X_{t+1}|x_t, \mu_{t+1} = \mu^{(i)}) = \rho x_t + \mu^{(i)}, \tag{24}$$

$$\left(\sigma^{(mk,i)}\right)^2 = \sigma^2 + E(\mu_t - \mu^{(i)})^2, \tag{25}$$

where "mk" in the superscript "(mk,i)" denotes that the process driving x_{t+1} has a Markov component, and "i " denotes whether $\mu_{t+1} = \mu^{(1)}$ or $\mu_{t+1} = \mu^{(2)}$.

Lemma 5 *Suppose that (21) characterizes the process driving x_{t+1}. Then, conditional on x_t, the "objective" pdf of x_{t+1}, denoted with $g^{reh(mk)}(x_{t+1}|x_t)$, is the following mixture of the two conditional normal pdfs with the means and variances in (24) and (25):*

$$g^{reh(mk)}(x_{t+1}|x_t) = \pi f^{(1)}(x_{t+1}|x_t, \mu_{t+1} = \mu^{(1)}) + (1-\pi) f^{(2)}(x_{t+1}|x_t, \mu_{t+1} = \mu^{(2)}), \tag{26}$$

Proof in Online Appendix A.

This lemma implies that the "objective" pdf is normal. Furthermore, from (26), (24), (25), (22), and (23), the (conditional) mean and variance of $g^{reh(mk)}(x_{t+1}|x_t)$ are given by

$$m_{t+1|t}^{reh(mk)} = \rho x_t + \mu^{(2)} + \pi\left(\mu^{(1)} - \mu^{(2)}\right) = \rho x_t + E(\mu_t), \tag{27}$$

$$. \left(\sigma_{t+1|t}^{reh(mk)}\right)^2 = \sigma^2 + V(\mu_t) \tag{28}$$

7.3 A Mixture Specification of the Reference PDF

GS assumed that, like REH, the reference pdf is based on the "objective" process, in (21). However, they specified the reference class of outcomes as

x_{t-1}. Using the same approach as in Section 7.2, the following lemma shows that allowing for the Markov component in the "objective" process implies that the reference pdf is a mixture of the four normal pdfs with the following means:[10]

$$m_{t+1|t}^{(mk,i,j)} = E(X_{t+1}|x_{t-1}, \mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}) = \rho^2 x_{t-1} + \rho\mu^{(j)} + \mu^{(i)},\ i,j = 1,2. \tag{29}$$

Lemma 6 *Suppose that (21) characterizes the process driving* x_{t+1}*. Then, conditional on* x_{t-1}*, the REH-like reference pdf of* x_{t+1}*, denoted with* $g^{ref(mk)}(x_{t+1}|x_{t-1})$*, is the mixture of the four normal pdfs with means specified in (29),*

$$g^{ref(mk)}(x_{t+1}|x_{t-1}) = \sum_{i,j=1}^{2} p_{ji}\pi_j f^{(i,j)}(x_{t+1}|x_{t-1}, \mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}), \tag{30}$$

where p_{ji}*,* $j,i = 1,2$ *are transition probabilities and* $\pi_j = P(\mu_{t+1} = \mu^{(j)})$ *for all* t*. Furthermore, the mean and variance of* $g^{ref(mk)}(x_{t+1}|x_{t-1})$ *are given by*

$$m_{t+1|t}^{ref(mk)} = \rho^2 x_{t-1} + (1+\rho)E(\mu_t), \tag{31}$$

$$\left(\sigma_{t+1|t}^{ref(mk)}\right)^2 = (1+\rho^2)\left[\sigma^2 + V(\mu_t)\right] + 2\rho\left\{E(\mu_{t+1}\mu_t) - \left[E(\mu_t)\right]^2\right\}. \tag{32}$$

Moreover,

$$(1+\theta)\left(\sigma_{t+1|t}^{ref}\right)^2 > \theta\left(\sigma_{t+1|t}^{reh}\right)^2 \tag{33}$$

holds, for any values of the model parameters $\left(\theta, \rho, \mu^{(1)}, \mu^{(2)}, p_{12,} p_{21}\right)$.

Proof in Online Appendix A.

[10] Because the explicit expression for $\left(\sigma^{(mk,i,j)}\right)^2$ plays no role in our argument, we omit it to save space. The derivation of this expression is analogous to that for $\left(\sigma_{t+1|t}^{ref(mk)}\right)^2$ in (32) below.

7.4 Overreaction

The normality of the REH and reference mixtures of pdfs and (33) ensures that Proposition 2 holds for the mixture specification of DE. Moreover, analogously to GS's time-invariant specification, DE's REH-like specification of the reference pdf that includes a Markov component is tantamount to assuming that the participants' overreaction, *relative to the REH forecast*, can be represented with *the revision of the REH forecast*. We state this conclusion and that of the previous section with a proposition:

Proposition 7 *Suppose that an economist assumes that while the reference pdf is based on the ""objective" normal pdf, which underpins REH, participants assessing an event's* $x_{t+1} \in A$*'s representativeness consider* x_{t-1} *the reference class of outcomes. Then, DE's REH-like specification overreacts to good (bad) news, regardless of whether it is time-invariant or involves a Markov component.*

Proof in Online Appendix A.

8 The Irregularity of DE's Overreaction in Pre-DE Behavioral Models

GS (pp. 137-152) argue that their specification of the reference pdf formalizes Kahneman and Tversky's findings in the Linda-like experiments in a variety of contexts. However, as we discussed in Section 3, Kahneman and Tversky (1972, p. 431) emphasized that there appears to be no theoretical basis for specifying the reference class that participants might consider relevant in assessing the representativeness of uncertain events.

Moreover, GS's specification of the reference pdf, and thus of DE, as being based on the "objective" pdf underpinning REH appears to be at odds with behavioral economists' empirical evidence. For example, the seminal pre-DE behavioral-finance model of Barberis, *et al.* (1998) is based on extensive evidence about how psychological influences drive market participants' assessment of uncertainty about stock returns. One of the main psychological mechanisms underpinning their model's specification of participants' forecasts is the representativeness heuristic. However, they argue that TK's and other empirical findings do not provide a basis for specifying how the news drives the overreaction or underreaction of participants' forecasts. As they put it,

> Unfortunately, the psychological evidence *does not* tell us quantitatively what kind of information is strong and salient (and hence is overreacted to) and what kind of information is low in weight (and hence is underreacted to). For example, it does not tell us how long a sequence of earnings increases is required for its strength to cause significant overpricing. Nor does the evidence tell us the magnitude of the reaction (relative to a true Bayesian) to information that has high strength and weight, or low strength and weight (p. 317, emphasis added).

This assessment of empirical evidence stands in contrast to GS's assumption that the "distortion" caused by the heuristic arises *solely* from news. Barberis, *et al.* argue (p. 318) that the deviation of participants' forecasts from their REH counterpart arises from "the investor...using the wrong model [of earnings] to form expectations [of returns]." While an economist's model assumes that earnings evolve according to a random walk, the investor "thinks that the world moves between two 'states' or 'regimes' and that there is a different model governing earnings in each regime." Barberis, *et al.* formalize this assumption with the two-state stationary Markov chain.

Our alternative to GS's specification of DE adapts the key premise of Barberis, *et al.* and other pre-DE behavioral models. We assume that the participants' assessment of representativeness is implied by the "wrong" reference pdf, rather than being driven solely by news, and show that DE no longer implies the regularity of overreaction. Depending on the values of the parameters of both the REH and reference pdfs, as well as the realizations of x_t, DE overreacts in some periods and underreacts in other periods.

8.1 A Behavioral Markov (BM) Specification of DE

To facilitate comparison with GS's REH-like specification of the reference pdf, we use an AR(1) process, in (14), to characterize the 'objective" process driving x_t, which we restate here for convenience,

$$X_{t+1} = \rho X_t + \mu + \varepsilon_{t+1}, \tag{34}$$

However, adapting Barberis *et al.'s* assumption that participants "think that the world moves between two 'states,'" we specify the reference process that participants consider relevant as being based on the following "wrong" version of the "objective" process in (34):

$$X_{t+1} = \rho_t^{(b)} X_t + \mu_{t+1}^{(b)} + \varepsilon_{t+1}, \tag{35}$$

where "b" in the superscript denotes that $\rho_t^{(b)}$ and $\mu_t^{(b)}$ specify the BM model. Each of them evolves according to a Markov chain, which switches between two states, $\rho^{(b,i)}$, and $\mu^{(b,i)}$ $i = 1, 2$ with the transition probabilities, $p_{12,}$ and q_{12}, respectively and $\varepsilon_t \sim iidN(0, \sigma^2)$. To simplify the presentation, we assume that $\rho_t^{(b)}$ and $\mu_t^{(b)}$ are independent. It follows from (35) that, while x_t and μ_{t-i} for $i = 0, 1...$ are dependent, x_t and μ_{t+i} for $i = 1, 2....$are independent. We also assume that, while x_t and $(\mu_{t-i}^{(b)}, \rho_{t-1-i}^{(b)})$ for $i = 0, 1...$ are dependent, x_t and $(\mu_{t+i}^{(b)}, \rho_{t+1-i}^{(b)})$ for $i = 1, 2...$are independent. Analogously to the specification in (21), we also assume that $\rho_t^{(b)}$ and $\mu_t^{(b)}$ are stationary Markov chains.

8.1.1 A Mixture Characterization of the Behavioral Reference PDF

A proof analogous to that of Lemma 6 shows that the reference pdf implied by (35) is a mixture of the four normal pdfs:

$$g^{ref(b)}(x_{t+1}|x_t) = \sum_{i,j=1}^{2} \pi_\rho^{(j)} \pi_\mu^{(i)} f^{(b,i,j)}(x_{t+1}|x_t, \rho_t^{(b)} = \rho^{(b,j)}, \mu_{t+1}^{(b)} = \mu^{(b,i)}), \tag{36}$$

where $\pi_\rho^{(j)}$ and $\pi_\mu^{(i)}$, $j, i = 1, 2$, are components of the respective stationary distributions. Furthermore, the conditional mean and variance of $g^{ref(b)}(x_{t+1}|x_t)$

are given by

$$m^{ref(b)}_{t+1|t} = E(\rho^{(b)}_t)x_t + E(\mu^{(b)}_t) \tag{37}$$

$$= E(\rho^{(b)}_t)\rho x_{t-1} + \rho\mu + E(\mu^{(b)}_t) + \rho e_t, \tag{38}$$

$$\left(\sigma^{ref(b)}_{t+1|t}\right)^2 = \sigma^2 + V(\rho^{(b)}_t)E(x^2_t) + V\left(\mu_t\right) \tag{39}$$

where $E(\rho^{(b)}_t), E(\mu^{(b)}_t), V(\rho^{(b)}_t)$, and $V(\mu^{(b)}_t)$ are the means and variances, which are specified analogously to (22) and (23).

8.1.2 A Behavioral Markov DE May Overreact or Underreact

Because the mixture in (36) is a normal pdf, and (17) and (39) show that $\left(\sigma^{ref(b)}_{t+1|t}\right)^2 > \left(\sigma^{reh(gs)}_{t+1|t}\right)^2$, Proposition 2 holds. Thus, the diagnostic expectation implied by GS's specification of the time-invariant REH pdf, and the BM specification of the reference pdf is given by

$$m^{de(b)}_{t+1|t} = m^{reh(gs)}_{t+1|t} + \gamma^{(b)}\left(m^{reh(gs)}_{t+1|t} - m^{ref(b)}_{t+1|t}\right) \tag{40}$$

$$= m^{reh(gs)}_{t+1|t} + \gamma^{(b)}\left\{\left[\rho - E(\rho^{(b)}_t)\right]x_t + \mu - E(\mu^{(b)}_t)\right\} \tag{41}$$

where, from (11), (17) and (39), $\gamma^{(b)} = \theta\frac{\sigma^2}{\sigma^2+(1+\theta)\left[V(\rho^{(b)}_t)E(x_t)^2+V(\mu_t)\right]}$.

The expression in (41) shows that, according to the BM specification, whether DE overreacts, relative to its REH counterpart, depends on whether $\left[\rho - E(\rho^{(b)}_t)\right]x_t + \mu - E(\mu^{(b)}_t) > 0$. The following lemma states this point explicitly:

Lemma 8 *Suppose that the specification of DE based on (34) and (35) characterizes how the representativeness heuristic leads participants away from forecasting according to REH.*

Letting $\rho - E(\rho^{(b)}_t) > 0$ implies that if and only if

$$x_t > \frac{E(\mu^{(b)}_t) - \mu}{\rho - E(\rho^{(b)}_t)}.$$

holds, DE overreacts, that is, $m^{reh(gs)}_{t+1|t} > m^{ref(b)}_{t+1|t}$ holds. Conversely, letting

$\rho - E(\rho_t^{(b)}) < 0$ *implies that if and only if*

$$x_t < \frac{E(\mu_t^{(b)}) - \mu}{\rho - E(\rho_t^{(b)})}.$$

holds, DE underreacts, that is, $m_{t+1|t}^{reh(gs)} < m_{t+1|t}^{ref(b)}$ *holds.*

To be sure, representing change with a Markov chain, as Barberis, *et al.*'s (1998) behavioral specification of particiapnts' forecasts, is quite restrictive. However, relaxing GS's REH-like specification of the reference pdf illustrates a more general point. Once we acknowledge the behavioral economists' pre-DE evidence that participants' forecasts are not based on the "objective" process driving outcomes, as hypothesized by an economist's model, DE no longer implies the regularity of overreaction. Depending on the values of the model parameters of both the REH and reference pdfs, $\left(\rho, \mu, \sigma^2, \rho^{(b,i)}, \mu^{(b,i)}, \pi_\rho^{(i)}, \pi_\mu^{(i)}\right)$, $i = 1, 2$, and the realizations of x_t, DE overreacts in some periods and underreacts in others periods.

9 Extending the Applicability of Coibion and Gorodnichenko's Econometric Framework

Coibion and Gorodnichenko (CG, 2015) proposed a new regression-based framework for testing the predictions of sticky and noisy REH models based on survey data on participants' forecasts of macroeconomic variables. CG (pp. 2651, 2653) point out that the theoretical structure of these REH specifications "map" directly onto the following regression relationship between *ex post* forecast errors and forecast revisions:

$$x_{t+h} - F_t(x_{t+h}) = \alpha + \beta \left[F_t(x_{t+h}) - F_{t-1}(x_{t+h})\right] + \upsilon_{t+h}, \qquad (42)$$

where $F_t(x_{t+h})$ denotes participants' time-t forecast of a variable x_t at time $t + h$, and υ_{t+h} is the error term implied by a theoretical specification of participants' forecasts. We derived our mixture characterization of participants' forecasts of x_{t+1}, conditional on x_t. Consequently, in deriving predictions of alternative specifications of these forecasts for the CG regression, (42), we set $h = 1$.

CG point out that the FIRE specification of participants' forecasts, based on a time-invariant process for x_t predicts that

$$\alpha^{fire(gs)} = \beta^{fire(gs)} = 0. \tag{43}$$

where "$fire$" in the superscript "$fire(gs)$" denotes that FIRE's predictions for the coefficients in (42) are based on (14). CG also show that hypothesizing that the time-invariant model, in (14), represents the process driving an outcome and that the aggregate of participants' forecasts conforms to the sticky or noisy information alternatives to FIRE implies that $\alpha = 0$ and $\beta > 0$.

We extend the applicability of the CG regression to testing predictions of theoretical specifications of participants' forecasts that involve a stationary Markov component. Relying on our mixture characterization of the REH and reference pdfs, we derive predictions for α and β, in (42), of the REH-llike Markov and Behavioral specifications of DE in Sections 7 and 8.1, respectively. We also derive predictions of the early (pre-DE) behavioral specification of participants' forecasts, as exemplified by Barberis, *et al.* (1998), as well as that of FIRE based on a model involving a Markov component, as exemplified by (21)..

10 Predictions of DE's Specifications for the CG Regression-

Hypothesizing that DE represents participants' forecasts, expression (12) relates participants' forecast error, fe_{t+1}, to their forecast "distortion," fd_t, and REH's forecast error, fe^{reh}_{t+1}, which we restate as

$$fe_{t+1} = -\gamma fd_t + fe^{reh}_{t+1}, \tag{44}$$

where

$$fe_{t+1} = x_{t+1} - m^{de}_{t+1|t}, \tag{45}$$

$$fd_t = \left(m^{reh}_{t+1|t} - m^{ref}_{t+1|t}\right), \tag{46}$$

$$fe^{reh}_{t+1} = x_{t+1} - E(X_{t+1}|x_t). \tag{47}$$

On the other hand, the CG regression, in (42), involves the relationship between participants' forecast error, fe_{t+1}, and their forecast revision, fr_t:

$$fe_{t+1} = x_{t+1} - m^{de}_{t+1|t} = \alpha + \beta fr_t + \upsilon_{t+1}, \tag{48}$$

where

$$\begin{aligned} fr_t &= m^{de}_{t+1|t} - m^{de}_{t+1|t-1} \\ &= (1+\gamma)\left(m^{reh}_{t+1|t} - m^{reh}_{t+1|t-1}\right) - \gamma\left(m^{ref}_{t+1|t} - m^{ref}_{t+1|t-1}\right) \end{aligned} \tag{49}$$

We now derive the predictions for the coefficients α and β, in (42), that hold for *any* specification of DE satisfying Proposition 2, restated in (44). To this end, we note that the outcome x_{t+1} is the time-$t+1$ realization of X_{t+1}, in (14) (21), or (35). Moreover, from (45)-(47) and (49), the conditional means $m^{de}_{t+1|t-i}$, $m^{reh}_{t+1|t-i}$, $m^{ref}_{t+1|t-i}$, $i = 0, 1$, are the time-t realizations of random variables, which we denote with $M^{de}_{t+1|t-i}$, $M^{reh}_{t+1|t-i}$, and $M^{ref}_{t+1|t-i}$, $i = 1, 2$. respectively. Consequently, fe_{t+1}, fd_t, fe^{reh}_{t+1}, and fr_t are time-t realizations of the following random variables:

$$FE_{t+1} = X_{t+1} - M^{de}_{t+1|t},\ FD_t = M^{reh}_{t+1|t} - M^{ref}_{t+1|t}, \tag{50}$$

$$FE^{reh}_{t+1} = X_{t+1} - M^{reh}_{t+1|t},\ FR_t = M^{de}_{t+1|t} - M^{de}_{t+1|t-1}. \tag{51}$$

Constraining the "objective" process driving x_t to be time-invariant, as exemplified by (14), implies that $FE^{reh}_{t+1} = \varepsilon_{t+1}$. Moreover, from (20), GS's REH-like specification of the reference pdf implies that $FD_t = \rho\varepsilon_t$. Thus, from (44) and (49), $FE_{t+1} = -\gamma\rho\varepsilon_t + \varepsilon_{t+1}$and $FR_t = (1+\gamma)\rho\varepsilon_t - \gamma\rho\varepsilon_{t-1}$. Assuming $\varepsilon_t \sim iidN(0, \sigma^2)$ ensures that all variables in (50) and (51) are normally distributed.

When μ_t evolves according to a Markov chain, each of these variables involves a Markov component. However, an argument analogous to that in the proof of Lemmas 5 and 6 shows that for each of the behavioral specifications of participants forecasts considered in Sections 7 and 8.1, each of the variables, FE_{t+1}, FD_t, FE^{reh}_{t+1}, and FR_t, is a mixture of normal pdfs, and thus each is also normally distributed.

For example, consider FD_t implied by DE's specification involving a Markov component, formulated in 7. The following lemma shows that FD_t is a mixture with two normal components.

Lemma 9 *Suppose that the REH-like Markov specification of DE, in Section 7.1, represents participants' forecasts. Then, the pdf of* FD_t*, denoted with* $g^{(mk)}(fd_t)$*, is the mixture of the two conditional normal pdfs:*

$$g^{(mk)}(fd_t) = \pi f^{(1)}(fd_t|\mu_t = \mu^{(1)}) + (1-\pi) f^{(2)}(fd_t|\mu_t = \mu^{(2)}),$$

where the mean of each of the components, $f^{(i)}(fd_t|\mu_t = \mu^{(i)})$*,* $i = 1, 2$*, is given by*

$$m^{(i)}_{fd_t} = E(FD_t|\mu_t = \mu^{(i)}) = \rho\left[\mu^{(i)} - E(\mu_t)\right] \tag{52}$$

Proof in Online Appendix A.

The normality and stationarity of FE_{t+1}, FD_t, FE^{reh}_{t+1}, and FR provide a straightforward way to express the predictions of any DE specification satisfying Proposition 2 for the coefficients of the CG regression, (48). Using (44), the standard expression for the conditional mean of jointly normal variables expresses these predictions in terms of the moments of the variables in (50) and (51):

$$
\begin{aligned}
E(FE_{t+1}|FR_t) &= -\gamma E(FD_t|FR_t) + E(FE^{reh}_{t+1}|FR_t) \qquad (53)\\
&= -\gamma \left\{ E(FD_t) + \frac{Cov(FD_t, FR_t)}{V(FR_t)} [FR_t - E(FR_t)] \right\} \\
&\quad + E(FE^{reh}_{t+1}|FR_t)
\end{aligned}
$$

where $0 < \gamma < 1$ and $Cov(\cdot, \cdot)$ is the covariance.
While $E(FE^{reh}_{t+1}|FR_t) = 0$ for the time-invariant REH-like specification of DE, allowing for change with a Markov chain renders $E(FE^{reh}_{t+1}|FR_t) \neq 0$. Thus, the predictions of the Markov REH-like and behavioral specifications of DE for the coefficients in the CG regression in (48) must also take into account that $E(FE^{reh}_{t+1}|FR_t) = E(FE^{reh}_{t+1}) + \frac{Cov(FE^{reh}_{t+1}, FR_t)}{V(FR_t)} [FR_t - E(FR_t)]$. As we show in Sections 10.1 and 11.2, respectively, this point renders invalid Bordalo, *et al.*'s (2020, p. 2756) assertion that hypothesizing that DE represents individual forecasts unambiguously predicts $\beta < 0$, as well as the usual assertion that FIRE necessarily predicts $\beta = 0$.

Denoting with the superscript "*de*" the coefficients of the CG regression implied by a DE specification of participants' forecasts satisfying Proposition 2, we summarize the argument in this section with a proposition:

Proposition 10 *Suppose that the REH and reference pdfs can be represented with the mixtures of normal pdfs, which arise from stationary Markov chains or time-invariant processes representing x_t, and that DE represents participants" forecasts. The following expressions characterize the predations of any such DE specification for the coefficients in the CG regression (48):*

$$
\begin{aligned}
\alpha^{de} &= E(FE^{reh}_{t+1|t}) + \qquad (54)\\
&\quad -\gamma E(FD_t) - \frac{Cov(FE^{reh}_{t+1}, FR_t) - \gamma Cov(FD_t, FR_t)}{V(FR_t)} E(FR_t)\\
\beta^{de} &= \frac{Cov(FE^{reh}_{t+1}, FR_t) - \gamma Cov(FD_t, FR_t)}{V(FR_t)}. \qquad (55)
\end{aligned}
$$

Proposition 10 relates the predictions of the magnitudes and signs of the coefficients α^{de} and β^{de} to the moments of the variables FE_{t+1}, FD_t, FE^{reh}_{t+1}, and FR_t implied by a particular specification of the REH and reference pdfs. In the remainder of this section, we apply this proposition to derive the predictions of DE's REH-like and behavioral Markov specifications, formulated in Sections 7 and 8.1.

10.1 Predictions of the REH-like Specifications of DE

The following corollary to Proposition 10 derives the relevant moments of FE_{t+1}, FD_t, FE^{reh}_{t+1}, and FR_t implied by the REH-like Markov specification of the reference pdf, and states DE's prediction for the coefficients of the CG regression, (48), denoted with the superscript "$de(mk)$."

Corollary 11 *Suppose that DE's REH-like Markov specification, in Section 7.1, characterizes participants' forecasts. Then,*

1. $\alpha^{de(mk)} = 0$ *at all* t.
2. $\beta^{de(mk)} < 0$ *if* $1 < p_{12} + p_{21} < (\frac{1}{\gamma^{(mk)}\rho} - 1)\left[(1 + \gamma^{(mk)}(1 - \rho)\right]$.
    (a) *If this condition is not satisfied, there are values of the model parameters,* $(\mu^{(1)}, \mu^{(2)}, p_{12}, p_{21}, \rho, \sigma^2)$ *for which* $\beta^{de(mk)} > 0$
3. *However, the sign and the magnitude of* $\beta^{de(mk)}$ *are unchanging over time.*

Proof in Online Appendix A.

According to Proposition 7, the REH-like Markov specification implies overreaction, that is, $m^{de}_{t+1|t} > m^{reh}_{t+1|t}$. This has led Bordalo, *et al.* (2020) to assert that $\beta < 0$ indicates the regularity of overreaction. However, Corollary 11 reveals that, because $Cov(FE^{reh}_{t+1}, FR_t) \neq 0$, the slope coefficient in the CG regression implied by the that theoretical specification involving a Markov component may be either positive or negative. Nevertheless, the REH-like Markov specification implies an unambiguous testable prediction: $\alpha = 0$ and the sign and magnitude of β to remain unchanging over time, thereby suggesting either the regularity of overreaction or underreaction.

10.1.1 Predictions When the Markov Chain Persists in a Regime

According to Corollary 11, although the Markov specification of DE allows for change in how participants forecast outcomes, it nonetheless predicts that $\alpha = 0$ and it is unchanging over time. However, the persistence of a Markov chain in one state for a prolonged period of time might cause structural break(s) in α. For example, Engel and Hamilton (1990) formalized such persistence with a two-state Markov chain in which they constrained the probabilities of switching, p_{12} and p_{21}, to be small, implying that the process would be expected to stay in one state for a long time. During each of such subperiods, typically referred to as regimes, α would remain unchanged, but it would take a different sign and (generally) a different magnitude in one regime as compared with the other.

The following corollary states predictions of assuming regime persistence for the coefficients of the CG regression, denoted with $\alpha^{de(mp)}$ and $\beta^{de(mp)}$;

Corollary 12 *Suppose that the transition probabilities p_{12} and p_{21} are sufficiently small, so that the process driving μ_t, in (21), may stay in one of the regimes for a long period of time of time, and yet undergo intermittent structural breaks that can be detected by an econometric procedure. Then, the REH-like Markov specification of DE in Section 7.1 implies the following testable predictions for the coefficients of the CG regression, in (48), denoted with $\alpha^{de(mp)}$ and $\beta^{de(mp)}$:*

1. $\alpha^{de(mp)}$ *switches the sign (from positive to negative or vice versa) when the transition from* $\mu_t = \mu^{(i)}$ to $\mu_t = \mu^{(j)}$ occurs, $i, j = 1, 2$, $i \neq j$.

   (a) The magnitude of $\alpha^{de(mp)}$ is the same in each of the regimes of μ_t.

2. $\beta^{de(mp)} < 0$ *and its magnitude is unchanging over time*, *regardless of whether μ_t persists in one of the regimes.*

Proof in Online Appendix.

This corollary shows that while the regime persistence may help explain intermittent structural breaks in $\alpha^{de(mp)}$, the structural change in CG regression's constant term is constrained in a way that can easily be tested. For example, although $\left[\mu^{(1)} - E\left(\mu_t\right)\right] > 0$ and $\left[\mu^{(2)} - E\left(\mu_t\right)\right] < 0$, the magnitude of these terms is the same. As we show in the proof of this corollary, this implies that the magnitude of $\alpha^{de(mp)}$ is the same across the two regimes.

10.2 Predictions of the Behavioral Markov Specification of DE

The BM specification of DE in Section 8.1 defines the variables in (50) and (51). The following corollary derives the moments of these variables and uses Proposition 10 to state predictions for the coefficients of (48), denoted with $\alpha^{de(b)}$ and $\beta^{de(b)}$:

Corollary 13 *DE's Behavioral Markov specification, in Section 8.1, implies the following predictions for the coefficients of the CG regression, in (48):*

1. *Either* $\beta^{de(b)} < 0$ *or* $\beta^{de(b)} > 0$.

2. Either $\alpha^{de(b)} > 0$ or $\alpha^{de(b)} < 0$.

3. *However, the signs and magnitudes of* $\alpha^{de(b)}$ *and* $\beta^{de(b)}$ *are unchanging over time.*

Proof in Online Appendix A.[11]

11 Predictions of the Pre-DE Behavioral and REH Specifications of Participants' Forecasts

Here, we adopt the approach of the preceding section to derive predictions of the pre-DE behavioral specifications of participants' forecasts, as exemplified by Barberis *et al.* (1998), as well as predictions of FIRE involving a Markov component. To this end, we note that any specification of forecasts denoted

[11] The proof of this corollary shows that the signs and magnitudes of $\alpha^{de(b)}$ and $\beta^{de(b)}$ depend on the fixed model parameters, $(\rho, \mu, \gamma^{(b)})$, as well as on $E(\rho_t^{(b)})$ and $E(\mu_t^{(b)})$, which are also assumed to be unchanging over time.

with $m^{for}_{t+1|t}$ satisfies the following relationship:

$$x_{t+1} - m^{for}_{t+1|t} = \left(m^{reh}_{t+1|t} - m^{for}_{t+1|t}\right) + f\epsilon^{reh}_{t+1|t}, \tag{56}$$

Analogously to (50) and (51), we also define the random variables, the moments of which underpin the predictions of a specification of forecasts for the coefficients α and β in the CG regression, (48):

$$FE_{t+1} = X_{t+1} - M^{for}_{t+1|t},\ FD_t = M^{reh}_{t+1|t} - M^{for}_{t+1|t}, \tag{57}$$

$$FE^{reh}_{t+1} = X_{t+1} - M^{reh}_{t+1|t},\ FR_t = M^{for}_{t+1|t} - M^{for}_{t+1|t-1}.. \tag{58}$$

We assume that pdf implied by REH and a specification of participants forecasts are simply normal or can be characterized with a mixture of normal components. Using the standard expression for the conditional normal variables underpinning Proposition 10, the following proposition states predictions of any specification of participants' forecasts, denoted with α^{for} and β^{for}.

Proposition 14 *Suppose that the process driving outcomes and participants' forecasts can be represented with mixtures of normal pdfs arising from stationary Markov chains. The following expressions characterize the predictions of any such specification for the coefficients in the CG regression (48):*

$$\alpha^{for} = E(FE^{reh}_{t+1|t}) + \tag{59}$$

$$+E(FD_t) - \frac{Cov(FE^{reh}_{t+1}, FR_t) + Cov(FD_t, FR_t)}{V(FR_t)} E(FR_t)$$

$$\beta^{for} = \frac{Cov(FE^{reh}_{t+1}, FR_t) + Cov(FD_t, FR_t)}{V(FR_t)}. \tag{60}$$

11.1 Predictions of Barberis *et al.*'s (1998) pre-DE Behavioral Specification

The BM specification of DE, in Section 8.1, defines the variables in (57) and (58) implied by Barberis *et al.*'s (1998) pre-DE specification. Using the proof of Corollary 13 and Proposition 14, the following corollary states the predictions of this behavioral specification for the coefficients of (48), denoted with $\alpha^{de(b)}$ and $\beta^{de(b)}$:

Corollary 15 *Suppose that, while the "objective" process driving outcomes is time-invariant, in (34), participants's forecasts are based on process, in (35), the mean of which evolves according to a two-state Markov chain. Such non-REH specification of forecasts implies the following predictions for the coefficients of the CG regression, in (42):*

1. *Either* $\beta^{(beh)} < 0$ *or* $\beta^{(beh)} > 0$.

2. Either $\alpha^{(beh)} > 0$ or $\alpha^{(beh)} < 0$,

3. *However, the signs and magnitudes of* $\alpha^{(beh)}$ *and* $\beta^{(beh)}$ *are unchanging over time.*

11.2 Predictions of FIRE for the CG Regression

Consider the hypothesis that FIRE based on a model involving a Markov component, in (21), represents participants' forecasts. This defines $M^{for}_{t+1|t} = M^{fire}_{t+1|t}$, which sets $FD_t = 0$. The proof analogous to that of Lemma 9 implies the following expressions for the variables FE_{t+1} and FR_t:

$$FE_{t+1} = X_{t+1} - M^{fire(mk)}_{t+1|t} = \left[\mu_{t+1} - E\left(\mu_t\right)\right] + \varepsilon_{t+1}, \tag{61}$$

$$FR_1 = M^{fire(mk)}_{t+1|t} - M^{fire(mk)}_{t+1|t-1} = \rho\left[\mu_t - E\left(\mu_t\right)\right] + \rho\varepsilon_t \tag{62}$$

Using Proposition 14, the following corollary states the predictions of FIRE involving a Markov component for the coefficients of the CG regression, denoted with $\alpha^{fire(mk)}$ and $\beta^{fire(mk)}$:

Corollary 16 *Suppose that FIRE based on the model (21) represents participants' forecasts. Such specification implies the following predictions for the coefficients of he CG regression:*

1. $\alpha^{fire(mk)} = 0$ *at all* t.

2. *Either* $\beta^{fire(mk)} > 0$ *if* $p_{12} + p_{21} < 1$, *or* $\beta^{fire(mk)} < 0$ *if* $p_{12} + p_{21} > 1$.[12]

[12] $\beta^{fire(mk)} = 0$ if $p_{12} + p_{12} = 1$. Because this case does not affect our interpretation of empirical findings in Section 13, we omit it from the corrollary.

(a) *However, the sign and magnitude of* $\beta^{fire(mk)}$ *are unchanging over time.*

Proof in Online Appendix A.

A finding of $\beta > 0$ in the CG regression has traditionally been interpreted as an "underreaction," and $\beta < 0$ as an "overreaction." However, once change in how outcomes unfold over time is represented with a Markov chain, FIRE predicts either $\alpha = 0$ and $\beta < 0$, or $\alpha = 0$ and $\beta > 0$. Therefore, allowing for a Markov component in the process driving outcomes may render FIRE's prediction the same as that of GS's REH-like specification of DE, which has been thought to represent deviations from FIRE.

12 Summary of the Predictions

Table 1 summarizes predictions for the CG regression of the four behavioral specifications of participants' forecasts, including three implied by the DE approach, as well as the two versions of FIRE that we consider in this paper, and refers to the respective corollaries as well as CG (2015) and Bordalo, *et al.* (2020), for their derivations.

Table 1: Predictions of Theoretical Specifications of Participants' Forecasts

Model	Prediction for α	Prediction for β
REH-like Specifications of DE		
A: GS's Time-Invariant	$\alpha = 0$	$\beta < 0$
B: Involving a Markov Component	$\alpha = 0$	$\beta < 0$ or $\beta > 0$
C: Assuming Regime Persistence	$\alpha > 0$ and $\alpha < 0$	$\beta < 0$
D: Behavioral Markov Specification of DE	$\alpha > 0$ or $\alpha < 0$	$\beta < 0$ or $\beta > 0$
E: Pre-DE Behavioral Specification	$\alpha > 0$ or $\alpha < 0$	$\beta < 0$ or $\beta > 0$
REH-implied Specifications		
F: Time-Invariant FIRE	$\alpha = 0$	$\beta = 0$
G: FIRE Involving a Markov Component	$\alpha = 0$	$\beta < 0$ or $\beta > 0$

Caption: A: Bordalo, *et al* (2020), B: Corollary 11, C: Corollary 12,
D: Corollary 13, E: Corollary 15, F: CG (2015), G: Corollary 16..

CG also show that hypothesizing that the time-invariant model, in (14), represents the process driving an outcome, and that the aggregate of participants' forecasts conforms to the sticky or noisy information alternatives to FIRE, implies that $\alpha^{fire(ar)} = 0$ and $\beta^{fire(ar)} > 0$. However, as CG (p. 2652) emphasize, this prediction "should not be expected at the individual level." Because we confront predictions of specifications of participants' forecasts with the individual survey data, we leave a reexamination of the REH-implied limited information specifications for a companion paper.

The models listed in Table 1 differ in several important respects. However, regardless of the pdfs implied by their specification of the "objective" process driving outcomes and participants' forecasts of them, the Assumption 4 of these processes' stationarity implies that the moments of the variables underpinning the predictions in columns 2 and 3 are time-invariant. Thus, subjecting the coefficients α *and* β of the CG regression to tests of structural change, which we present in the following section, provides a hitherto unexplored way to confront behavioral specifications of participants' forecasts, including diagnostic expectations, with survey data.[13]

[13] Greenwood and Shleifer (2014) concluded, on the basis of a time-invariant specification for survey data, that participants' forecasts are largely unrelated to fundamental factors.

13 Empirical Findings

Here, we tests predictions in Table 1 by estimating individual CG regressions, based on survey data of inflation forecasts by 24 professionals.

13.1 Full-Sample Estimates of the CG Regression

We begin with the full-sample estimates, which the literature typically focuses on in assessing the empirical adequacy of theoretical specifications of expectations. Table B1 in Online Appendix B displays such estimates of the individual CG regressions. The estimates are based on data from 24 individuals in the Philadelphia Federal Reserve's Survey of Professional Forecasters with more than 50 observations of their one-quarter-ahead forecast revisions of the Gross Domestic Product's inflation. Table 2 presents the summary of the tests of the coefficients of these individual regressions grouped across forecasters.

Table 2: Grouping of Individuals Based on Tests of Full-Sample Regressions' Estimates

Individuals	α	β	Consistent with
13/24	$\alpha = 0$	$\beta = 0$	Model F in Table 1
5/24	$\alpha = 0$	$\beta < 0$	Models A, B and G
4/24	$\alpha \neq 0$	$\beta < 0$	Models D and E
1/24	$\alpha = 0$	$\beta > 0$	Models B and G
1/24	$\alpha \neq 0$	$\beta = 0$	No model in Table 1

Bordalo, *et al.* (2020) assert that "for individual forecasters the prevalent pattern is overreaction (p. 2779)." By contrast, we find that the estimates for only five of the 24 individual regressions are consistent with $\alpha = 0$ and $\beta < 0$.[14] Even more surprising in Table 2, is the number of individual CG regressions that fail to reject time-invariant FIRE (model F). For **13** of the 24

Frydman and Stillwagon (2018) reexamine this conclusion by testing for structural stability of participants' forecasts directly, rather than via the CG regression. They show that, in contrast to Shleifer and Greenwood's conclusion, both fundamental and psychological factors drive participants' forecasts. For a recent paper relating structural breaks in how participants forecast outcomes to market sentiment, see Frydman, *et al.* (2020).

[14] See Table B1 in Online Appendix B for the full-sample estimates and t-values for all 24 forecasters.

forecasters, we cannot reject at even 10% that $\alpha = 0$ and $\beta = 0$. However, these interpretations of the empirical findings presume the stationarity of the process driving outcomes and in how participants forecast them.

13.2 Time-Invariance of the Coefficients of the CG Regression

Models in Table 1 differ in a number of important respects. However, all of them rest on a common premise: the process driving outcomes and participants' revisions of their forecasting strategies can be represented with a stationary Markov chain. One of the central implications of our theoretical framework is that, although such representations do allow for *change* in the specification of individuals' forecasting strategies, they predict that the constant and slope coefficients in the CG regression *do not change* over time. Predictions of Models B, D, E, and G in Table 1 formalize this implication.

For example, according to Corollary 11, the REH-like specification of DE with a Markov component (Model B) predicts $\alpha = 0$ and either $\beta < 0$ or $\beta > 0$, for all t, depending on the values of transition probabilities and other parameters. However, under the stationarity assumption, Model B predicts that β must remain either positive or negative in both sign or magnitude for all t. Similarly, while the behavioral Markov specification (Model D) predicts that $\alpha > 0$ or $\alpha < 0$, and $\beta < 0$ or $\beta > 0$, it also predicts that the CG regression's coefficients are time-invariant both in sign and magnitude.

Model C is the only model among those in Table 1 predicting structural break(s) under the stationarity Assumption 4. These breaks could arise from intermittent switches between persistent regimes, as formalized with the low off diagonal transition probabilities. However, as Corollary 12 shows, while the regime persistence may cause the constant term α to undergo intermittent structural breaks, this change is constrained in a way that can easily be tested. Although α differs across the two regimes, its positive (or negative) sign and its magnitude are the same within each regime.

13.3 Structural Breaks in the Individual CG Regressions

According to Propositions 10 and 14, regardless of the specification of the "objective" and reference pdfs, the Assumption 4 of these processes' stationarity implies that the moments of the variables, in (50), (51), (57), and (58), that underpin these predictions are time-invariant. Thus, subjecting the coefficients of the CG regression to tests of structural change provides a hitherto unexplored way to confront alternative models of expectations, including diagnostic expectations, with survey data on participants' forecasts. Moreover, the predictions of any of the specifications in Table 1 for the constant and the slope in the CG regression depend on different moments of the variables, in (50), (51), (57), and (58). Whereas the prediction for α depends on the means and the covariances of these variables, the prediction for β depends only on the covariances. Thus, the tests for structural breaks require a procedure that allows the constant and slope of the CG regression to break at different times.

Consequently, we rely on the Multiplicative Indicator Saturation (MIS) procedure, which has been designed to detect breaks in the coefficients of the regression model at potentially different times. MIS is an extension of the Autometrics algorithm (Doornik, 2009) and the indicator saturation methods of Hendry, *et al.* (2008) and Castle, *et al.* (2015), whose consistency properties and appropriate size and power have already been demonstrated in previous studies under a range of conditions.[15] For an overview of the MIS methodology, see Online Appendix B.

By contrast, because the Bai-Perron (1998) procedure does not allow the constant and the slope to break at different times, on theoretical grounds, it is not suitable for testing the structural stability of predictions for α and β in the CG regression (42). Nonetheless, given its widespread use, we also report results of the Bai-Perron test.

[15] For recent applications of MIS and further references, see Castle, *et al.* (2017).

Table 3: Structural Breaks in the CG Regressions Detected by MIS and Bai-Perron Procedures

Full Sample	Individuals	α Break	β Break	α or β	Bai-Perron
$\alpha = \beta = 0$	13/24	12/13	6/13	13/13	7/13
$\alpha = 0; \beta < 0$	5/24	5/5	2/5	5/5	3/5
$\alpha \neq 0; \beta < 0$	4/24	4/4	3/4	4/4	1/4
$\alpha = 0; \beta > 0$	1/24	1/1	1/1	1/1	0/1
$\alpha \neq 0; \beta = 0$	1/24	1/1	0/1	1/1	1/1

Columns 1 and 2 of Table 3 restate the results of the tests of the CG regressions grouped by number of individuals. Row 2 and Column 2 restates the results of the full-sample estimates that 13 out of 24 individual CG regressions yielded a constant and the revision (slope) coefficient, not significantly different from zero, which is consistent with the time-invariant FIRE. Columns 3, 4, and 5 report the proportion of those 13 individual regressions that experience instability in the constant α, slope β, and either α or β. As column 5 row 2 shows, all of those 13 individual regressions experience a significant break in either the constant or the slope. The apparent prevalence of FIRE in the full-sample CG regressions is therefore completely overturned: the structural breaks detected by MIS in all of the individual regressions that did not reject time-invariant FIRE in the full sample are strongly inconsistent with that specification.

Of the 11 remaining individual regressions, all 11 experience a break in either α or β, as shown in column 5 of rows 3-5. This rejects all of our specifications in Table 1, with the possible exception of the persistent regime specification of DE (Model C).

In Table 3, column 4, there are 12 individual regressions with a time-invariant β. As shown in Figures B1-B4 in Appendix B, nine indicate $\beta < 0$. The constant term, α, in seven of these individual regressions experience breaks in magnitude, while maintaining the same sign. This renders the estimates of these seven regressions inconsistent with Model C.

However, for the other two individual regressions indicating a time-invariant $\beta < 0$, the constant term, α, undergoes two breaks within the sample, with its sign switching at each break. Thus, according to Corollary 12, the estimates of these two regressions would be consistent with Model C if the magnitude of α is the same in each regime. Whenever a Markov chain returns to and persists within one of the regimes, we can reject this hypothesis for one of the individual regressions at 5% and at 10% for the other. For the latter regression, the second regime consists of only five observations. Although this entails low power, its point estimate of α is six times larger in magnitude than that in the third regime.

In summary, MIS finds that all of our models in Table 1 are rejected based on the estimates of the 24 individual CG regressions.

The last column of Table 3 provides the summary for Bai-Perron tests relative to the full-sample estimates in columns 1 and 2. These results, like MIS, reveal that the full-sample estimates are misleading. More than half of the cases that cannot reject time-invariant FIRE in the full sample experience breaks, with sub-samples where either $\alpha \neq 0$ and/or $\beta \neq 0$ (see Table B2 in Online Appendix B for further details). Similarly, three of the five full-sample estimates apparently consistent with time-invariant DE are no longer robust after Bai-Perron tests: they experience either sub-periods with a statistically significant $\alpha \neq 0$, or β loses significance or changes sign.

It is clear from Table 3 that the Bai-Perron procedure detected breaks in fewer individual regressions than did MIS. What is again surprising, however, is the significant number of regressions that still cannot reject FIRE after the Bai-Perron tests (**six** of 24). When contrasted with MIS, which rejects FIRE for all 24 regressions, this indicates a problem with Bai-Perron constraining α and β to break only simultaneously.

In particular, we find that more individual regressions experience a break in the constant than in the slope (23 vs. 12, as shown in Table 3). Moreover, as Figures B1-B4 show, breaks in the constant typically outnumber breaks in the slope for most individual regressions. This is consistent with our argument for using MIS: because the predictions for the CG constant depend, in part, on different moments of the relevant variables, the slope and constant may break at different points in time.

13.4 Diversity of Forecasting Strategies

A number of papers have found significant diversity in how participants forecast outcomes.[16] A study that is of particular interest from the viewpoint of this paper is von Gaudecker and Wogrolly (2021) which documents significant diversity in households' beliefs about the stock market. They identify five separate groupings of forecasting strategies,[17] and then estimate panels of the CG regression for each group, based on the premise that individuals within these groups do not revise how they forecast outcomes.

Our results suggest that the diversity of participants' forecasting strategies is substantially compounded by their revision of how they forecast outcomes at times and in ways that cannot be characterized with a stationary process, such as a Markov chain. As can be seen in Figures B1-B4 in Online Appendix B, the timing, direction, frequency, and magnitude of the breaks across individual regressions differ vastly.

14 Concluding Remarks: A Way Forward

The empirical inadequacy of all five alternative behavioral specifications of participants' forecasts casts doubts on the behavioral approach's core premise that market participants commit systematic, predictable errors, and that an economist can specify these errors precisely with a probability measure. Our findings support an argument advanced by Lucas in the early 1970s. As Lucas (1995, pp. 254-255) pointed out in his criticism of adaptive expectations, macroeconomic and finance models that violate Muth's (1961) hypothesis, as DE and other behavioral-finance models do, suffer from a "glaring" inconsistency.[18] When an economist represents an individual's assessment of

[16]See Mankiw, *et al.* (2003), Reis (2020) and references therein.

[17]After determining the groupings, von Gaudecker and Wogrolly also examine differences across groups in terms of demographics, investment behavior, and response to returns and economic news.

[18]Lucas (1995, p. 255) recounts how the importance of ridding intertemporal models of such inconsistency persuaded macroeconomists to abandon the micro-founded models of the 1960s and embrace their REH counterparts. For an extensive discussion and formal illustration of this revolutionary development in macroeconomic theory, see Frydman and Goldberg (2007) and Frydman and Phelps (2013).

uncertainty about payoff-relevant outcomes in a way that is inconsistent with his own model's representation of this uncertainty, he contradicts his model's hypothesis: that it represents the actual uncertainty about these outcomes.

Lucas's argument that Muth's hypothesis should underpin the construction of logically coherent and empirically adequate macroeconomic and finance models appears persuasive. REH implements Muth's hypothesis in a model that represents the uncertainty about payoff-relevant outcomes with a stationary stochastic process. However, Muth's hypothesis neither presumes nor requires that an economist rely on such representations. Indeed, our finding that the two FIRE specifications based on a stationary stochastic process are inconsistent with the survey data points to the inadequacy of such representations.

There is another largely overlooked reason why economists should consider moving beyond models that represent outcomes with a stationary stochastic process. Imposing consistency within such models, as REH does, rules out, by design, the influence of non-fundamental factors, thereby rendering Muth's hypothesis inherently incompatible with behavioral economists' evidence that such factors play an important role in how participants form forecasts.

One way to move beyond the prevailing approach is to specify how outcomes unfold over time and how participants forecast them with non-stationary stochastic processes. Because such representations would not imply that the coefficients of the CG regression are time-invariant, they might be consistent with structural breaks in those coefficients. However, representing the change in the process driving outcomes with a non-stationary Markov chain and imposing Muth's hypothesis would also rule out the influence of non-fundamental factors on participants' forecasts.

Moreover, if the process driving outcomes and individuals' forecast revisions could be represented with a single non-stationary process, according to Muth's hypothesis, participants would revise their forecasts at approximately the same time and in similar ways. However, as we documented in the preceding section, there is substantial diversity in the estimates and timing of breaks across the individual CG regressions, which suggests that individuals revise their forecasts at different times and in widely diverse ways.

It is nonetheless possible to reconcile the representation of such diversity, as well as the influence of both fundamental and psychological factors (such as market sentiment) in a consistent model that *adheres* to Muth's hypothesis. This, however, requires recognizing, as Knight (1921) argued, that the process driving outcomes undergoes unforeseeable change, which by definition cannot be represented *ex ante* with a single stochastic process, regardless of whether it is stationary or non-stationary.

Acknowledging that participants recognize that they face so-called Knightian uncertainty as a result of unforeseeable change would enable economists to represent the role of psychological factors, such as market sentiment, in a model that is consistent with Muth's hypothesis. In a novel paper, Ilut and Schneider (2014) show that this enhances our understanding of business cycles. They introduce Knightian uncertainty into a standard New Keynesian Model and formalize how confidence in future total productivity drives fluctuations in aggregate outcomes. They show that changes in confidence arising from Knightian uncertainty are empirically significant in explaining these fluctuations.

More broadly, recognizing that market participants face Knightian uncertainty would allow consistent representations of the autonomous role their forecasts play in driving outcomes, as argued by Phelps (1970) in his seminal micro-foundations volume.[19] Thus, recognizing that the future is open to change that cannot be specified *ex ante* with probabilistic rules would enable a synthesis of major advances in macroeconomic theory since the 1970s. Building macroeconomic and finance models in accordance with Knight's seemingly uncontroversial yet profound insight promises to enhance substantially our understanding of market outcomes and the role of economic policy.

[19] For a formal demonstration, see Frydman, *et al.* (2021).

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15 Online Appendix A

Proof of Lemma 5

Using the law of total probability, we can express the pdf of x_{t+1}, conditional on x_t, as follows

$$g^{reh(mk)}(x_{t+1}|x_t) = \tag{63}$$
$$h^{(1)}(x_{t+1,}\mu_{t+1} = \mu^{(1)}|x_t) + h^{(2)}(x_{t+1,}\mu_{t+1} = \mu^{(2)}|x_t),$$

where $h^{(i)}$ $(\cdot|\cdot)$, $i = 1, 2$ denote the respective normal pdfs implied by (21). Furthermore, we rewrite the above as

$$\begin{aligned} g^{reh(mk)}(x_{t+1}|x_t) &= \frac{h^{(1)}(x_{t+1,}\mu_{t+1} = \mu^{(1)}, x_t)}{g(x_t)} \\ &+ \frac{h^{(2)}(x_{t+1,}\mu_{t+1} = \mu, x_t)}{g(x_t)} \\ &= \frac{h^{(1)}(x_{t+1}|\mu_{t+1} = \mu^{(1)}, x_t)P(\mu_{t+1} = \mu^{(1)}|x_t)g(x_t)}{g(x_t)} \\ &+ \frac{h^{(2)}(x_{t+1}|\mu_{t+1} = \mu^{(2)}, x_t)P(\mu_{t+1} = \mu^{(2)}|x_t)g(x_t)}{g(x_t)}, \end{aligned}$$

where $g(x_t)$ is the marginal pdf of x_t.Independence of μ_{t+1} and X_t and the stationarity of a Markov chain $\{\mu_t\}$ shows that the "objective" pdf is the mixture of the two normal pdfs, in (26):

$$g^{reh(mk)}(x_{t+1}|x_t) = \pi f^{(1,1)}(x_{t+1}|x_t, \mu_{t+1} = \mu^{(1)}) + (1-\pi) f^{(1,2)}(x_{t+1}|x_t, \mu_{t+1} = \mu^{(2)}). \tag{64}$$

where $\pi = P(\mu_{t+1} = \mu^{(1)})$ for all t.

Proof of Lemma 6

Using the law of total probability, we can express the pdf of x_{t+1}, conditional on x_{t-1}, as follows

$$g^{ref(mk)}(x_{t+1}|x_{t-1}) = \sum_{i,j=1}^{2} h^{(i,j)}(x_{t+1,}\mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}|x_{t-1}), \tag{65}$$

where $h^{(i,i)}\ (\cdot|\cdot)$, $i, j = 1, 2$ denote the respective pdfs implied by (21).

Analogously to the steps from (63) to (64) in the proof of Lemma 5, the above can be expressed as

$$\begin{aligned} &g^{ref(mk)}(x_{t+1}|x_{t-1}) \\ &= \sum_{i,j=1}^{2} f^{(i,j)}(x_{t+1}|\mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}, x_{t-1}) P\left(\mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}|x_{t-1}\right) \end{aligned}$$

Using the assumption that μ_{t+1} and μ_t are independent of x_{t-1}, we rewrite the above as

$$\begin{aligned} g^{ref(mk)}(x_{t+1}|x_{t-1}) &= \sum_{i,j=1}^{2} f^{(i,j)}(x_{t+1}|\mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}, x_{t-1}) \\ \times P\left(\mu_{t+1} = \mu^{(i)}|\mu_t = \mu^{(j)}\right) P(\mu_t = \mu^{(j)}). \end{aligned}$$

Noting that $P\left(\mu_{t+1} = \mu^{(i)}|\mu_t = \mu^{(j)}\right) = p_{ji}$ is the transition probability, this shows that the reference pdf is the mixture of the four pdfs, in (30):

$$g^{ref(mk)}(x_{t+1}|x_{t-1}) = \sum_{i,j=1}^{2} p_{ji}\pi_j f^{(i,j)}(x_{t+1}|x_{t-1}, \mu_{t+1} = \mu^{(i)}, \mu_t = \mu^{(j)}), \tag{66}$$

where $\pi_j = P(\mu_t = \mu^{(j)})$ for all t.

The mean of $g^{ref(mk)}(x_{t+1}|x_{t-1})$ is the weighted average of the means of the components pdfs, $m_{t+1|t}^{(mk,i,j)}$ in (29):

$$\begin{aligned} m_{t+1|t}^{ref(mk)} &= \rho^2 x_{t-1} + \sum_{i,j=1}^{2} p_{ji}\pi_j \left(\rho\mu^{(j)} + \mu^{(i)}\right) \\ &= \rho^2 x_{t-1} + \sum_{i=1}^{2} p_{1i}\pi_1 \left(\rho\mu^{(1)} + \mu^{(i)}\right) + \sum_{i=1}^{2} p_{2i}\pi_2 \left(\rho\mu^{(2)} + \mu^{(i)}\right), \end{aligned}$$

which, using $p_{ii} = (1 - p_{ij})$, $i, j = 1, 2, i \neq j$ and $\pi_2 = (1 - \pi_1)$, can be written as

$$\begin{aligned} m_{t+1|t}^{ref(mk)} &= E(X_{t+1}|x_{t-1}) = \rho^2 x_{t-1} + \pi_1(1+\rho)\mu^{(1)} - \pi_1 p_{12}(\mu^{(1)} - \mu^{(2)}) \\ &\quad + \pi_2(1+\rho)\mu^{(2)} + \pi_2 p_{21}(\mu^{(1)} - \mu^{(2)}) \\ &= \rho^2 x_{t-1} + (1+\rho)\mu^{(2)} + \{p_{21} + \pi\,[1 + \rho - (p_{12} + p_{21})]\}\,(\mu^{(1)} - \mu^{(2)}), \end{aligned}$$

where $\pi \equiv \pi_1 = P\left(\mu_t = \mu^{(1)}\right)$ for all t. Noting that $\pi = \frac{p_{21}}{p_{21}+p_{12}}$, implies that $p_{21} - \pi\,(p_{12} + p_{21}) = 0$. Thus, the conditional mean of the reference pdf is given by

$$\begin{aligned} m_{t+1|t}^{ref(mk)} &= \rho^2 x_{t-1} + (1+\rho)\left[\mu^{(2)} + \pi(\mu^{(1)} - \mu^{(2)})\right] \\ &= \rho^2 x_{t-1} + (1+\rho)E(\mu_t) \end{aligned} \tag{67}$$

To compute the variance of the reference pdf, in(66), we note that (21) and (67) imply that

$$\begin{aligned} \left(\sigma_{t+1|t}^{ref(mk)}\right)^2 &= E\left\{[X_{t+1} - E(X_{t+1}|x_{t-1})]^2\,||x_{t-1}\right\} \\ &= E\left\{\rho\,[\mu_t - E(\mu_t)] - + \left[\mu_{t+1} - E(\mu_t)\right]\right\}^2 \\ &= (1+\rho^2)\left[\sigma^2 + V(\mu_t)\right] \\ &\quad + 2\rho\left\{E\left(\mu_{t+1}\mu_t\right) - [E(\mu_t)]^2\right\} \end{aligned} \tag{68}$$

Finally, Proposition 2 requires that $(1+\theta)\left(\sigma_{t+1|t}^{ref(mk)}\right)^2 > \theta\left(\sigma_{t+1|t}^{reh(mk)}\right)^2$, which from (28) and (68) follows if

$$\left[1+\rho^2(1+\theta)\right]\sigma^2 + V(\mu_t) + 2(1+\theta)\rho E\left(\mu_{t+1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2 > 0 \tag{69}$$

We now show that

$$\left[1+\rho^2(1+\theta)\right]V(\mu_t) + 2(1+\theta)\rho E\left(\mu_{t+1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2 > 0 \tag{70}$$

holds for any values of the model parameters $\left(\theta, \rho, \mu^{(1)}, \mu^{(2)}, p_{12,}p_{21}\right)$. To this end we express $E\left(\mu_{t+1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2$ in terms of $V(\mu_t)$ and $\left(\mu^{(1)} - \mu^{(2)}\right)^2$:

$$\begin{aligned}
E(\mu_{t+1}\mu_t) &= E\left[\mu_t E(\mu_{t+1}|\mu_t)\right] \\
&= \mu^{(1)}\left(\mu^{(1)}(1-p_{12}) + \mu^{(2)}p_{12}\right)\pi \\
&\quad + \mu^{(2)}\left[\mu^{(2)}(1-p_{21}) + \mu^{(1)}p_{21}\right](1-\pi) \\
&= \left(\mu^{(1)}\right)^2\pi - \left(\mu^{(1)}\right)^2 p_{12}\pi + \mu^{(1)}\mu^{(2)}p_{12}\pi \\
&\quad + \left(\mu^{(2)}\right)^2(1-\pi) - \left(\mu^{(2)}\right)^2 p_{12}\pi + \mu^{(1)}\mu^{(2)}p_{12}\pi \\
&= E(\mu_t^2) - p_{12}\pi\left(\mu^{(1)} - \mu^{(2)}\right)^2, \qquad (71)
\end{aligned}$$

where we used $p_{21}(1-\pi) = p_{12}\pi$. This shows that

$$E\left(\mu_{t+1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2 = V\left(\mu_t\right) - p_{12}\pi\left(\mu^{(1)} - \mu^{(2)}\right)^2 \tag{72}$$

Furthermore, using $\pi = \frac{p_{21}}{p_{12}+p_{21}}$, we express $V\left(\mu_t\right)$ as follows:

$$\begin{aligned}
V\left(\mu_t\right) &= \left(\mu^{(1)}\right)^2\pi + \left(\mu^{(1)}\right)^2(1-\pi) - \left[\mu^{(1)}\pi + \mu^{(1)}(1-\pi)\right]^2 \\
&= \pi(1-\pi)\left(\mu^{(1)} - \mu^{(2)}\right)^2 \qquad (73) \\
&= p_{12}\pi\frac{1}{p_{12}+p_{21}}\left(\mu^{(1)} - \mu^{(2)}\right)^2, \qquad (74)
\end{aligned}$$

which implies that

$$E\left(\mu_{t+1}\mu_t\right)-\left[E\left(\mu_t\right)\right]^2 = V\left(\mu_t\right)-p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2$$
$$= p_{12}\pi\frac{1-p_{12}-p_{21}}{p_{12}+p_{21}}\left(\mu^{(1)}-\mu^{(2)}\right)^2 \tag{75}$$

Substituting (74) into (72) enables us to rewrite the condition (70) as follows

$$p_{12}\pi\frac{1}{p_{12}+p_{21}}\left(\mu^{(1)}-\mu^{(2)}\right)^2$$
$$\times\left[1+\rho^2(1+\theta)+2\rho(1+\theta)\left(1-p_{12}-p_{21}\right)\right]>0 \tag{76}$$

Finally, we note that both roots of the quadratic equation in the square brackets are negative, and thus (76) holds for any $\rho > 0$. Via (69), this shows that $(1+\theta)\left(\sigma_{t+1|t}^{ref(mk)}\right)^2 > \theta\left(\sigma_{t+1|t}^{reh(mk)}\right)^2$ holds for any values of the model parameters, $\left(\theta,\sigma^2\rho,\mu^{(1)},\mu^{(2)},p_{12,}p_{21}\right)$.

Proof of Proposition 7

The difference between the payoff-relevant variable, such as x_t, and its REH forecast,

$$\eta_t = X_t - E(X_t|x_{t-1}), \tag{77}$$

is usually referred to as news about x_t, where $E(X_t|x_{t-1})$ is a conditional expectation of the "objective" process driving x_t. The argument that participants overrreact to news for the time-invariant REH-like specification is presented in Section 6. Here we focus on DE's specification with a Markov component.

Recognizing that the process driving outcomes changes over time, substantially alters the specification and interpretation of the news variable, in (77), which using (27), is given by:

$$\eta_t^{(mk)} = X_t - E(X_t|x_{t-1}) = \mu_t - E(\mu_t) + \varepsilon_t. \tag{78}$$

This shows that the news comprises *both* of the realization of ε_t (e_t) and the value that μ_t takes, relative to its expectation, $E(\mu_t)$.Depending on the realized state at t, the realization of $\eta_t^{(mk)}$ can take one of the two values:

$$n^{(mk,j)} = \mu^{(j)} - E(\mu_t) + e_t,$$
$$= \left(\mu^{(j)} - \mu^{(2)}\right) - \pi(\mu^{(1)} - \mu^{(2)}) + e_t \quad j = 1, 2. \tag{79}$$

The expression for the news, in (79), implies that

$$\mu^{(j)} + e_t = n^{(mk,j)} + E(\mu_t) \quad j = 1, 2.$$

Substituting this into (27) implies that $m_{t+1|t}^{reh(mk,j)}$, expressed in terms of x_{t-1}, the realized state of μ_t and e_t, can take one of two values:

$$m_{t+1|t}^{reh(mk,j)} = \rho^2 x_{t-1} + \rho n^{(mk,j)} + (1+\rho)E(\mu_t) \quad j = 1, 2,$$

which, using

$$m_{t+1|t-1}^{reh(mk)} = \rho^2 x_{t-1} + (1+\rho)E(\mu_t), \tag{80}$$

implies that

$$m_{t+1|t}^{reh(mk,j)} - m_{t+1|t-1}^{ref(mk)} = \rho n^{(mk,j)},\ j = 1, 2.$$

Proof of Lemma 9

According Assumption 3,

$$FD_t = M_{t+1|t}^{reh} - M_{t+1|t}^{ref} = M_{t+1|t}^{reh} - M_{t+1|t-1}^{reh},$$

which, from (27), can be expressed as

$$FD_t = \rho\left[\mu_t - E\left(\mu_t\right)\right] + \rho\varepsilon_t.$$

An argument analogous to the proof of Lemma 5 shows that the pdf of FD_t is the mixture of the following two normal pdfs, conditional on the realization of μ_t:

$$g^{(mk)}(fd_t)$$
$$= \pi f^{(1)}(fd_t|\mu_t = \mu^{(1)}) + (1-\pi)f^{(2)}(fd_t|\mu_t = \mu^{(2)}),$$

where, the means and variances of the i's component, $i = 1, 2$, are given

$$m_{fd_t}^{(i)} = E(FD_t|\mu_t = \mu^{(i)}) = \rho\left[\mu^{(i)} - E(\mu_t)\right] \tag{81}$$

$$\left(\sigma_{fd_t}^{(i)}\right)^2 = V(FD_t|\mu_t = \mu^{(i)}) \tag{82}$$

$$= \rho^2\left[\sigma^2 \ + E(\mu_t^2) - 2E(\mu_t)\mu^{(i)} + \left(\mu^{(i)}\right)^2\right]$$

Proof of Corollary 11

We first consider

$$FD_t = M_{t+1|t}^{reh(mk)} - M_{t+1|t-1}^{reh(mk)}$$

It follows from (27) and (21) that

$$M_{t+1|t}^{reh(mk)} = E(X_{t+1}|X_t) = \rho X_t + E\left(\mu_t\right)$$

$$= \rho^2 X_{t-1} + \rho\mu_t + E\left(\mu_t\right) + \rho\varepsilon_t, \tag{83}$$

$$M_{t+1|t-1}^{reh(mk)} = E(X_{t+1}|X_{t-1}) = \rho E(X_{t+1}|X_t) + E\left(\mu_t\right)$$

$$= \rho^2 X_{t-1} + (1+\rho)\, E\left(\mu_t\right) \tag{84}$$

$$= \rho^3 X_{t-2} + \rho^2\mu_{t-1} + (1+\rho)\, E\left(\mu_t\right) + \rho^2\varepsilon_{t-1}, \tag{85}$$

which imply that

$$FD_t = \rho\left[\mu_t - E\left(\mu_t\right)\right] + \rho\varepsilon_t \tag{86}$$

We now consider $FR_t = M_{t+1|t}^{de} - M_{t+1|t-1}^{de}$:

$$FR_t = (1+\gamma^{(mk)})\left(M_{t+1|t}^{reh(mk)} - M_{t+1|t-1}^{reh(mk)}\right) \tag{87}$$

$$-\gamma^{(mk)}(M_{t+1|t-1}^{reh(mk)} - M_{t+1|t-2}^{reh(mk)}).$$

In order to relate $M_{t+1|t-1}^{reh(mk)} - M_{t+1|t-2}^{reh(mk)}$ to μ_t and ε_t, an argument analogous to those in Sections 7.2 and 7.3 shows that $g^{reh(mk)}(x_{t+1}|x_{t-2})$ is a mixture of eight normal pdfs with the mean

$$m_{t+1|t-2}^{reh(mk)} = \rho^3 x_{t-2} + (1+\rho+\rho^2)E\left(\mu_t\right),$$

which, from (80), shows that

$$M_{t+1|t-1}^{reh(mk)} - M_{t+1|t-2}^{reh(mk)} = \rho^2 \left[\mu_{t-1} - E\left(\mu_t\right)\right] + \rho^2 \varepsilon_{t-1}, \tag{88}$$

Substituting (89) and (88) into (87) yields

$$\begin{aligned} FR_t = &(1+\gamma^{(mk)})\rho \left[\mu_t - E\left(\mu_t\right)\right] - \gamma^{(mk)}\rho^2 \left[\mu_{t-1} - E\left(\mu_t\right)\right] \\ &+(1+\gamma^{(mk)})\rho\varepsilon_t - \gamma^{(mk)}\rho^2\varepsilon_{t-1}. \end{aligned} \tag{89}$$

Noting that

$$FE_{t+1|t}^{reh(mk)} = \left[\mu_{t+1} - E\left(\mu_t\right)\right] + \varepsilon_{t+1}, \tag{90}$$

and $V(FR_t) > 0$, the moments of FR_t, FD_t and $FE_{t+1}^{reh(mk)}$, which underpin the predictions in (54) and (55), are given by

$$E(FR_t) = 0,\ E(FD_t) = 0,\ E(FE_{t+1}^{reh(mk)}) = 0, \tag{91}$$

$$\begin{aligned} Cov(FR_t, FD_t) = &(1+\gamma^{(mk)})\rho^2\sigma^2 + (1+\gamma^{(mk)})\rho^2 V(\mu_t) \\ &-\gamma^{(mk)}\rho^3 \left\{E\left(\mu_{t-1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2\right\}, \end{aligned} \tag{92}$$

$$\begin{aligned} Cov(FR_t, FE_{t+1}^{reh(mk)}) = &(1+\gamma^{(mk)})\rho \left[E\left(\mu_{t+1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2\right] \\ &-\gamma^{(mk)}\rho^2 \left\{E\left(\mu_{t+1}\mu_{t-1}\right) - \left[E\left(\mu_t\right)\right]^2\right\} \end{aligned} \tag{93}$$

where $V(\mu_t) = E(\mu_t^2) - \left[E\left(\mu_t\right)\right]^2$.

According to Proposition 10,

$$sign(\beta) = sign\left[Cov(FR_t, FE_{t+1}^{reh(mk)}) - \gamma Cov(FR_t, FD_t)\right]. \tag{94}$$

We now show that whether $sign\left[Cov(FR_t, FE_{t+1}^{reh(mk)}) - \gamma Cov(FR_t, FD_t)\right] <$ 0 or > 0 depends on the values of the model parameters $\left(\mu^{(1)}, \mu^{(2)}, p_{12}, p_{21}, \rho, \gamma^{(mk)}, \sigma^2\right)$.

Substituting (72) into (92) yields

$$\begin{aligned} Cov(FR_t, FD_t) &= (1+\gamma^{(mk)})\rho^2\sigma^2 + \rho^2\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]V(\mu_t) \\ &\quad + \gamma^{(mk)}\rho^3 p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2 \\ &= \rho^2\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \quad (95) \\ &\quad + \rho^2(1+\gamma^{(mk)})\left[\sigma^2 + p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \quad (96) \end{aligned}$$

In order to derive an analogous expression for $Cov(Z_{1,t}, FE_{t+1|t}^{reh(mk)})$, (93), we consider

$$\begin{aligned} E\left(\mu_{t+1}\mu_t\right) - E\left(\mu_{t+1}\mu_{t-1}\right) &= E\left\{E\left[\mu_{t+1}(\mu_t - \mu_{t-1})|\left(\mu_t, \mu_{t-1}\right)\right]\right\} \\ &= E\left(\mu_t - \mu_{t-1}\right)E\left[\mu_{t+1}|\left(\mu_t, \mu_{t-1}\right)\right] \\ &= E\left\{\left(\mu_t - \mu_{t-1}\right)E\left[\mu_{t+1}|\mu_t\right]\right\} \end{aligned}$$

Noting that $\mu_t - \mu_{t-1}$ takes two non-zero values, $\mu^{(1)} - \mu^{(2)}$ with the probability $p_{21}(1-\pi)$ and $\mu^{(2)} - \mu^{(1)}$ with the probability $p_{12}\pi$ implies that

$$\begin{aligned} & E\left\{\left(\mu_t - \mu_{t-1}\right)E\left[\mu_{t+1}|\mu_t\right]\right\} \\ &= \left(\mu^{(1)} - \mu^{(2)}\right)E\left[\mu_{t+1}|\mu_t = \mu^{(1)}\right]p_{21}(1-\pi) \\ &\quad + \left(\mu^{(2)} - \mu^{(1)}\right)E\left[\mu_{t+1}|\mu_t = \mu^{(2)}\right]p_{12}\pi \\ &= \left(\mu^{(1)} - \mu^{(2)}\right)\left[\mu^{(2)}(1-p_{21}) + \mu^{(1)}p_{21}\right]p_{12}\pi \\ &\quad + \left(\mu^{(2)} - \mu^{(1)}\right)\left[\mu^{(1)}(1-p_{12}) + \mu^{(2)}p_{12}\right]p_{12}\pi \\ &= \{\left(\mu^{(1)} - \mu^{(2)}\right)\mu^{(2)} - \left(\mu^{(1)} - \mu^{(2)}\right)\mu^{(2)}p_{21} + \left(\mu^{(1)} - \mu^{(2)}\right)\mu^{(1)}p_{21} \\ &\quad + \left(\mu^{(2)} - \mu^{(1)}\right)\mu^{(1)} - \left(\mu^{(2)} - \mu^{(1)}\right)\mu^{(1)}p_{12} + \left(\mu^{(2)} - \mu^{(1)}\right)\mu^{(2)}p_{12}\}p_{12}\pi \\ &= \left\{-\left(\mu^{(1)} - \mu^{(2)}\right)^2 + \left(\mu^{(1)} - \mu^{(2)}\right)^2 p_{21} + \left(\mu^{(1)} - \mu^{(2)}\right)^2 p_{12}\right\}p_{12}\pi \\ &= \left(\mu^{(1)} - \mu^{(2)}\right)^2 p_{12}\pi\left(p_{12} + p_{21} - 1\right), \end{aligned}$$

which (via (71)) shows that

$$E(\mu_{t+1}\mu_{t-1}) - [E(\mu_t)]^2 = V(\mu_t) - p_{12}\pi(p_{12}+p_{21})\left(\mu^{(1)}-\mu^{(2)}\right)^2. \tag{97}$$

Substituting (71) and (97) into (93) yields

$$\begin{aligned} Cov(Z_{1,t}, FE^{reh(mk)}_{t+1|t}) &= (1+\gamma^{(mk)})\rho\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \\ &\quad -\gamma^{(mk)}\rho^2\left[V(\mu_t) - p_{12}\pi(p_{12}+p_{21})\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \\ &= \rho\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \quad (98) \\ &\quad -\gamma^{(mk)}\rho^2 p_{12}\pi(1-p_{12}-p_{21})\left(\mu^{(1)}-\mu^{(2)}\right)^2 \quad (99) \end{aligned}$$

We are now ready to derive

$$\delta = Cov(FR_t, FE^{reh(mk)}_{t+1}) - \gamma Cov(FR_t, FD_t) \tag{100}$$

From (96) this difference includes one unambiguously negative term:

$$\delta_1 = -\gamma^{(mk)}\rho^2(1+\gamma^{(mk)})\left[\sigma^2 + p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] < 0 \tag{101}$$

Using (98) and (95) yields

$$\begin{aligned} &\rho\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \\ &-\gamma^{(mk)}\rho^2\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \\ =\ &\rho(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right], \end{aligned}$$

which, combined with (99), yields the second term of (100):

$$\begin{aligned} \delta_2 &= \rho(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)}) - \gamma^{(mk)}\rho\right]\left[V(\mu_t) - p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \quad (102) \\ &\quad -\gamma^{(mk)}\rho^2 p_{12}\pi(1-p_{12}-p_{21})\left(\mu^{(1)}-\mu^{(2)}\right)^2 \end{aligned}$$

We now show that there are values of the parameters $\gamma^{(mk)}, \rho, p_{12}$, and p_{21}

for which $\delta_2 < 0$, thereby implying (via (101) that

$$\delta = \delta_1 + \delta_2 = Cov(FR_t, FE_{t+1}^{reh(mk)}) - \gamma Cov(FR_t, FD_t) < 0.$$

Substituting (**74**) into (102) expresses δ_2 as

$$\begin{aligned}
\delta_2 =& \rho(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)})-\gamma^{(mk)}\rho\right]\left[V\left(\mu_t\right)-p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right] \\
&-\gamma^{(mk)}\rho^2 p_{12}\pi\left(1-p_{12}-p_{21}\right)\left(\mu^{(1)}-\mu^{(2)}\right)^2 \\
=& \rho(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)})-\gamma^{(mk)}\rho\right] p_{12}\pi\frac{1-p_{12}-p_{21}}{p_{12}+p_{21}}\left(\mu^{(1)}-\mu^{(2)}\right)^2 \\
&-\gamma^{(mk)}\rho^2 p_{12}\pi\left(1-p_{12}-p_{21}\right)\left(\mu^{(1)}-\mu^{(2)}\right)^2 \\
=& \rho p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\left(1-p_{12}-p_{21}\right)\left(\mu^{(1)}-\mu^{(2)}\right)^2 \\
&\times\left\{(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)})-\gamma^{(mk)}\rho\right]\frac{1}{p_{12}+p_{21}}-\gamma^{(mk)}\rho\right\} \\
=& \frac{\rho p_{12}\pi}{p_{12}+p_{21}}\left(\mu^{(1)}-\mu^{(2)}\right)^2\left(1-p_{12}-p_{21}\right) \qquad (103) \\
&\times\left\{(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)})-\gamma^{(mk)}\rho\right]-\gamma^{(mk)}\rho\left(p_{12}+p_{21}\right)\right\} \qquad (104)
\end{aligned}$$

In order to uncover the conditions under which $\delta_2 < 0$, we note that the term in (103) is negative if and only if

$$p_{12} + p_{21} > 1. \qquad (105)$$

Furthermore, the term in (104) is positive if

$$1 < p_{12} + p_{21} < (\frac{1}{\gamma^{(mk)}\rho} - 1)\left[(1 + \gamma^{(mk)}(1 - \rho)\right]. \qquad (106)$$

Thus, if $p_{12} + p_{21}$ satisfies (106), $\delta_2 < 0$..

However, although the right bound in (106) is greater than 1 for any values of $0 < \gamma^{(mk)}, \rho < 1$, if $\gamma^{(mk)}\rho. > 1/3$, there are values of $1 < p_{12} + p_{21} < 2$ such that

$$p_{12} + p_{21} > (\frac{1}{\gamma^{(mk)}\rho} - 1)\left[(1 + \gamma^{(mk)}) - \gamma^{(mk)}\rho\right] > 1, \qquad (107)$$

which implies that the term in (104) is negative, and thus $\delta_2 > 0$. We also note that if $p_{12}+p_{21}<1$, both (103) and (104) are positive, thus, $\delta_2>0$. This shows that the condition (106) is necessary and sufficient for

$$\delta=\delta_1+\delta_2=Cov(FR_t,FE_{t+1}^{reh(mk)})-\gamma Cov(FR_t,FD_t)<0$$

Finally if the condition (106) is not satisfied and

$$\begin{aligned}\delta_2 =& \frac{\rho p_{12}\pi}{p_{12}+p_{21}}\left(\mu^{(1)}-\mu^{(2)}\right)^2(1-p_{12}-p_{21})\\ &\times\left\{(1-\gamma^{(mk)}\rho)\left[(1+\gamma^{(mk)})-\gamma^{(mk)}\rho\right]-\gamma^{(mk)}\rho\left(p_{12}+p_{21}\right)\right\}\\ &>-\delta_1=\gamma^{(mk)}\rho^2(1+\gamma^{(mk)})\left[\sigma^2+p_{12}\pi\left(\mu^{(1)}-\mu^{(2)}\right)^2\right]\end{aligned}$$

then

$$\delta=\delta_1+\delta_2=Cov(FR_t,FE_{t+1}^{reh(mk)})-\gamma Cov(FR_t,FD_t)>0.$$

Proof of Corollary 12

We formalize regime persistence by constraining μ_t to take the same value from $t-2$ to $t+1$.Suppose (without a loss of generality) that

$$\mu_{t-2}=\mu_{t-1}=\mu_t=\mu_{t=1}=\mu^{(1)}$$

Imposing this constraint in (86), (89), and (90) specifies FR_t, FD_t, and $FE_{t+1}^{reh(mk)}$ as follows

$$\begin{aligned}FR_t =& (1+\gamma^{(mk)})\rho\left[\mu^{(1)}-E\left(\mu_t\right)\right]-\gamma^{(mk)}\rho^2\left[\mu^{(1)}-E\left(\mu_t\right)\right]\\ &+(1+\gamma^{(mk)})\rho\varepsilon_t-\gamma^{(mk)}\rho^2\varepsilon_{t-1},\\ FD_t =& \rho\left[\mu^{(1)}-E\left(\mu_t\right)\right]+\rho\varepsilon_t,\\ FE_{t+1}^{reh(mk)} =& \left[\mu^{(1)}-E\left(\mu_t\right)\right]+\varepsilon_{t+1}.\end{aligned}$$

This immediately implies that

$$E(FR_t) = \rho(1 + \gamma^{(mk)} - \gamma^{(mk)}\rho^2)\left[\mu^{(1)} - E(\mu_t)\right], \quad (108)$$

$$E(FD_t) = \rho\left[\mu^{(1)} - E(\mu_t)\right], \quad (109)$$

$$E(FE^{reh(mk)}_{t+1|t}) = \left[\mu^{(1)} - E(\mu_t)\right], \quad (110)$$

$$Cov(FR_t, FD_t) = (1 + \gamma^{(mk)})\rho^2\sigma^2, \quad (111)$$

$$Cov(FR_t, FE^{reh(mk)}_{t+1|t}) = 0, \quad (112)$$

which (via Proposition 10) indicates the following predictions for the coefficients of (48) in each of the regimes:

$$\alpha^{de(mp)} = \left[\mu^{(i)} - E(\mu_t)\right] \times \quad (113)$$

$$\times\left[1 - \gamma^{(mk)}\rho + \gamma^{(mk)}\frac{(1 + \gamma^{(mk)})\rho^2\sigma^2}{V(FR_t)} \times \rho(1 + \gamma^{(mk)} - \gamma^{(mk)}\rho^2)\right], i = 1, 2$$

$$\beta^{de(mp)} = -\gamma^{(mk)}(1 + \gamma^{(mk)})\rho^2\sigma^2 \quad (114)$$

Using $E(\mu_t) = \pi\mu^{(1)} + (1-\pi)\mu^{(2)}$, the magnitude of $\left[\mu^{(1)} - E(\mu_t)\right] = (1-\pi)\left(\mu^{(1)} - \mu^{(2)}\right)$ is the same as of $\left[\mu^{(2)} - E(\mu_t)\right] = (1-\pi)\left(\mu^{(2)} - \mu^{(1)}\right)$. Thus, (113) implies that although $\alpha^{de(mp)}$ switches sign between the two regimes, its magnitude remains the same.

Proof of Corollary 13

As in the GS time-invariant model, in Section 5,

$$FD_t = \rho\varepsilon_t,$$

$$FE^{reh}_{t+1} = \varepsilon_{t+1},$$

which, using (34), (37), (35), and (38, enables us to express FR_t, in (50), as follows:

$$FR_t = \left[1 + \gamma^{(b)}(1-\rho\right]\rho\varepsilon_t - \gamma^{(b)}\left[\rho - E(\rho_t^{(b)})\right]\varepsilon_t$$

$$-\gamma^{(b)}E\left(\rho_t^{(b)}\right)\left\{\left[\rho - E(\rho_t^{(b)})\right]\rho x_{t-1} + \left[\mu - E(\mu_t^{(b)})\right]\right\}.$$

These expressions imply that

$$\begin{aligned}
E\left(FD_t\right) &= 0, \\
E\left(FR_t\right) &= \gamma^{(b)} E\left(\rho_t^{(b)}\right) \left\{ \frac{\rho\mu}{1-\rho} \left[\rho - E(\rho_t^{(b)})\right] + \left[\mu - E(\mu_t^{(b)})\right] \right\}, \\
Cov(FR_t, FD_t) &= \left[\rho\left(1+\gamma^{(b)}\right) - \gamma^{(b)} E(\rho_t^{(b)})\right] \rho\sigma^2, \\
E\left(FE_{t+1}^{reh}\right) &= 0 \\
Cov(FR_t, FE_{t+1}^{reh}) &= 0,
\end{aligned}$$

where we used $E(x_{t-1}) = \frac{\mu}{1-\rho}$. Then, Proposition 10 implies that

1. Either $\beta^{de(b)} < 0$ if and only if $\frac{\rho}{E(\rho_t^{(b)})} > \frac{\gamma^{(b)}}{1+\gamma^{(b)}}$.
2. $\alpha^{de(b)} > 0$ if and only if $\left[\rho\left(1+\gamma^{(b)}\right) - \gamma^{(b)} E(\rho_t^{(b)})\right]$
   $\times \left\{ \frac{\rho\mu}{1-\rho} \left[\rho - E(\rho_t^{(b)})\right] + \left[\mu - E(\mu_t^{(b)})\right] \right\} > 0$
3. However, the signs and magnitudes of $\alpha^{de(b)}$ and $\beta^{de(b)}$ are unchanging over time.

Proof of Corollary 16

Expressions (61) and (62) imply that

$$\begin{aligned}
E\left(FE_{t+1}\right) &= 0, \\
E\left(FR_1\right) &= 0,
\end{aligned}$$

which, from (59), implies that

$$\alpha^{fire(mk)} = 0 \text{ at all } t$$

(61) and (62) also imply that

$$Cov(FE_{t+1}, FR_1) = \rho\left\{\left(E\mu_{t+1}\mu_t\right) - \left[E\left(\mu_t\right)\right]^2\right\},$$

which, using (75), enables us to express

$$Cov(FE_{t+1}, FR_1) = \frac{\rho p_{12} \pi}{p_{12} + p_{21}} \left(\mu^{(1)} - \mu^{(2)}\right)^2 (1 - p_{12} - p_{21}) \, .$$

Then, (60) implies predictions for $\beta^{fire(mk)}$ stated in the corollary.

16 Online Appendix B

16.1 Full-Sample Individual CG Regressions

Whereas the forecast error is always observed anytime a forecast is reported, the data point for the forecast revision requires two consecutive forecasts submitted by an individual. Table B1 is sorted by the number, N, of observations available for individuals in the survey with more than 50 observations for revisions. $\hat{\alpha}$ and $\hat{\beta}$ in the table are, respectively, the full-sample estimate of the constant and the slope in the CG regression, (42). These results are summarized in Table 2 in the body of the paper.

Table B1: Full-Sample Individual-Level Estimates

N	$\hat{\alpha}$	$\hat{\beta}$	N	$\hat{\alpha}$	$\hat{\beta}$	N	$\hat{\alpha}$	$\hat{\beta}$
100	0.002	−0.342	70	−0.001	−0.471	59	0.006	−0.540
	[2.19]	[−2.27]		[−0.69]	[−1.87]		[4.11]	[−1.99]
98	−0.002	−0.394	68	0.001	−0.353	56	0.001	−0.045
	[−3.15]	[−3.03]		[0.64]	[−1.21]		[1.45]	[−0.13]
90	−0.001	−0.460	65	−0.003	−0.550	55	0.001	−0.391
	[−1.38]	[−3.68]		[−3.77]	[−3.14]		[0.52]	[−2.87]
78	−0.001	−0.588	65	0.001	0.308	54	0.001	0.077
	[−1.71]	[−2.32]		[1.19]	[0.80]		[1.43]	[0.32]
78	−0.000	−0.217	64	0.001	0.139	53	0.000	0.744
	[−0.38]	[−1.60]		[0.88]	[0.67]		[0.15]	[1.74]
78	0.001	−0.004	63	0.001	−0.044	52	−0.003	−0.084
	[0.62]	[−0.02]		[0.63]	[−0.13]		[−3.99]	[−0.77]
78	−0.001	−0.388	62	0.000	0.067	52	0.001	−0.192
	[−1.22]	[−1.50]		[0.06]	[0.27]		[1.39]	[−0.67]
70	−0.002	−0.150	60	0.001	−0.460	51	0.001	−0.204
	[−1.64]	[−.55]		[0.49]	[−4.96]		[1.06]	[0.93]

Caption: t-values are displayed in brackets under the parameter estimates.

16.2 Overview of MIS

MIS was first proposed by Ericsson (2012) as an extension of robust estimation methods which, respectively, detect outliers and mean shifts: impulse indicator saturation (IIS), developed in Hendry, *et al.* (2008), and step indicator saturation (SIS), developed in Castle, *et al.* (2015). The general idea of MIS is to multiply each regressor by a step indicator for each observation that is equal to unity up until time j and zero thereafter. This allows for breaks in the regressors' coefficients separately and at any point in time. Combined with IIS and SIS, in the context of the CG regression yields:

$$\begin{aligned} x_{t+1} - F_t(x_{t+h}) = \alpha + \beta[F_t(x_{t+h}) - F_{t-1}(x_{t+h})] + \Sigma_{i=1}^{T-1}\beta^i \mu 1_{t<i}[F_t(x_{t+h}) - F_{t-1}(x_{t+h})] \\ + \Sigma_{i=1}^{T}\delta 1 + \Sigma_{i=2}^{T-1}\beta^i \mu 1_{t<i} + error. \end{aligned}$$

The impulse indicators $\Sigma_{i=1}^{T}\delta 1$ (one for each observation) allow for an outlier at any point in time. The step indicators $\Sigma_{i=2}^{T-1}\beta^i \mu 1_{t<i}$, allow for a differential shift in the constant, relative to the end-of-sample constant.[20] The multiplicative indicators $\Sigma_{i=1}^{T-1}\beta^i \mu 1_{t<i}[F_t(x_{t+h}) - F_{t-1}(x_{t+h})]$ allow for a differential slope coefficient at any point in time, relative to the end-of-sample estimate.

The significant multiplicative step and impulse indicators are selected by the Autometrics tree search algorithm (Doornik, 2009). After the multiplicative indicators have been determined by the algorithm, a model-selection bias correction is applied (Hendry and Krolzig, 2005). This correction eliminates the well-known bias originally documented by Lovell (1983).[21]

[20] The impulse indicator and step indicator, as specified, are identical in the first observation, so the latter is summed only from the second observation.

[21] The correction reduces the absolute value of the coefficients. The size of the adjustment depends inversely on the t-values and significance level used for selection. This reduces the selection bias, because this adjustment is larger when there is a greater probability of Type II error.

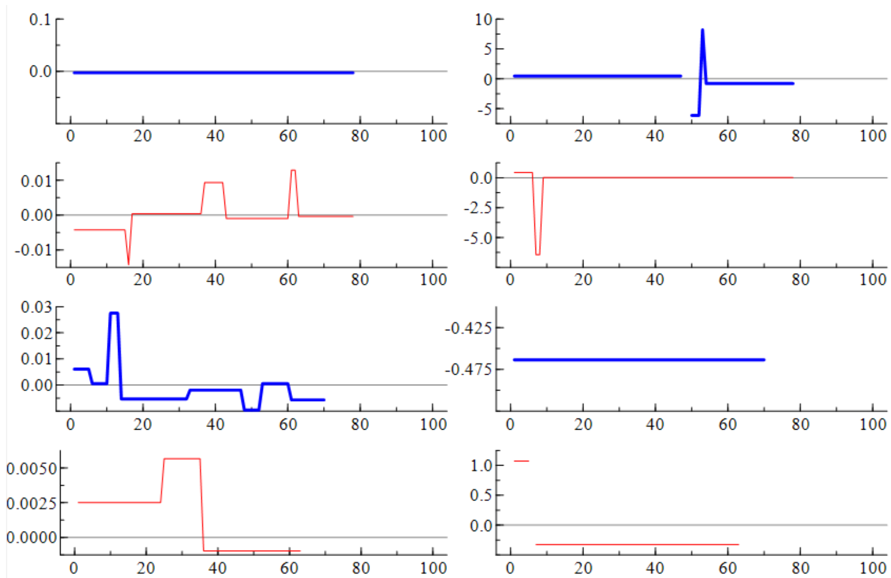
16.3 MIS Estimates of CG Regression

Figures B1-B4 below display MIS estimates of statistically significant breaks in the constant and revision coefficient of individual regressions based on survey data from the 24 forecasters. These results have been grouped in rows 2-5 and columns 3-5 of Table 3.

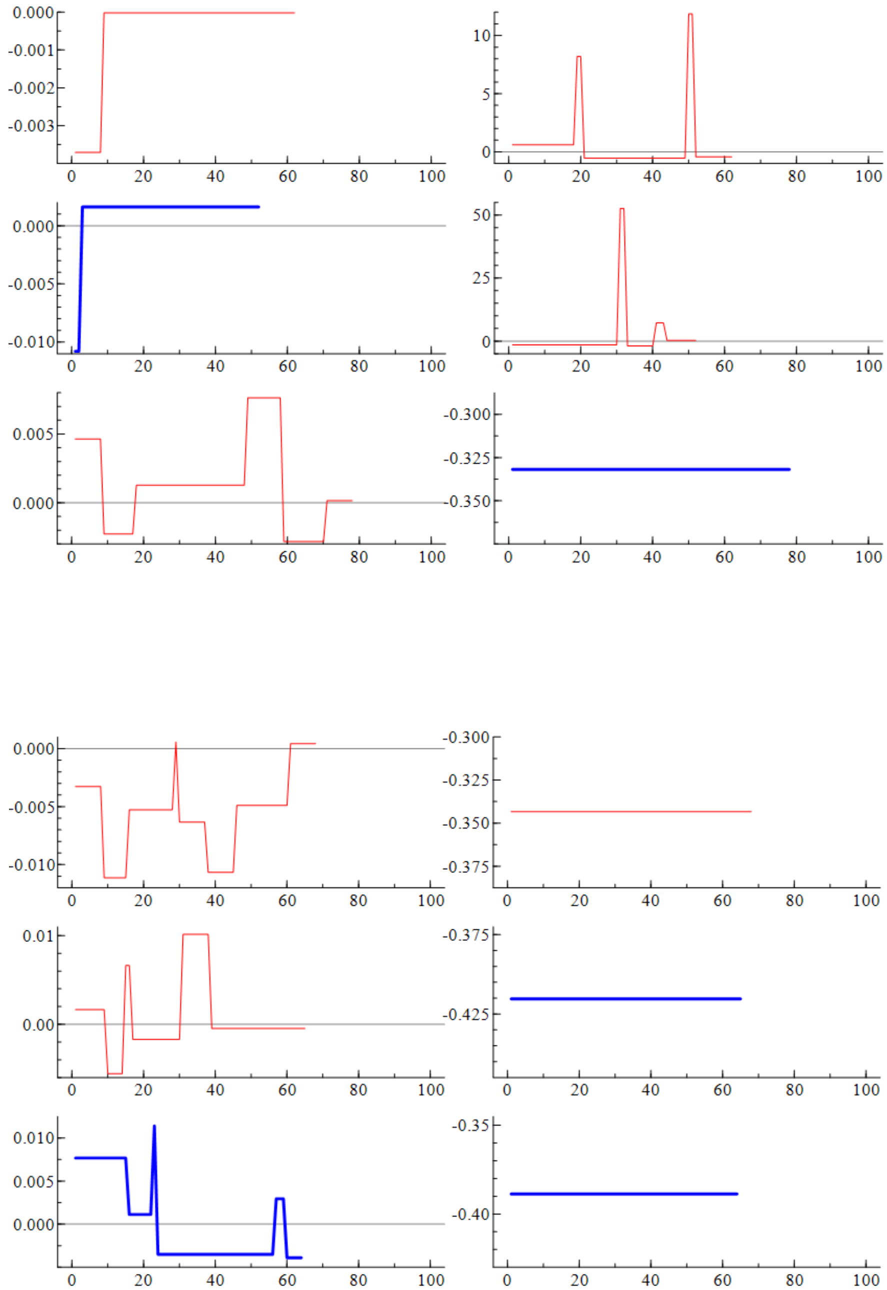
All of the individual regressions experience breaks in either the constant or revision coefficient, which are significant at 1%. We can see, however, that the constant and the slope break at different times, including frequent cases where the constant changes but the slope does not. As discussed in Section 13.3, this supports our argument for using MIS, rather than the Bai-Perron test, as a procedure for testing the stability of the individual coefficients in the CG regression. Each row represents an individual regression. Lines in thick blue indicate that the term for that individual regression is significant at the end of the sample. Breaks are relative to the end of sample and significant at 1%.

Figure B1: MIS Estimates of Significant Breaks in CG Regressions Not Rejecting FIRE in Full Sample

α β



α β



0.000
-0.001
-0.002
-0.003
0 20 40 60 80 100
10
5
0
0 20 40 60 80 100
0.000
-0.005
-0.010
0 20 40 60 80 100
50
25
0
0 20 40 60 80 100
0.005
0.000
0 20 40 60 80 100
-0.300
-0.325
-0.350
0 20 40 60 80 100
0.000
-0.005
-0.010
0 20 40 60 80 100
-0.300
-0.325
-0.350
-0.375
0 20 40 60 80 100
0.01
0.00
0 20 40 60 80 100
-0.375
-0.425
0 20 40 60 80 100
0.010
0.005
0.000
0 20 40 60 80 100
-0.35
-0.40
0 20 40 60 80 100

α β

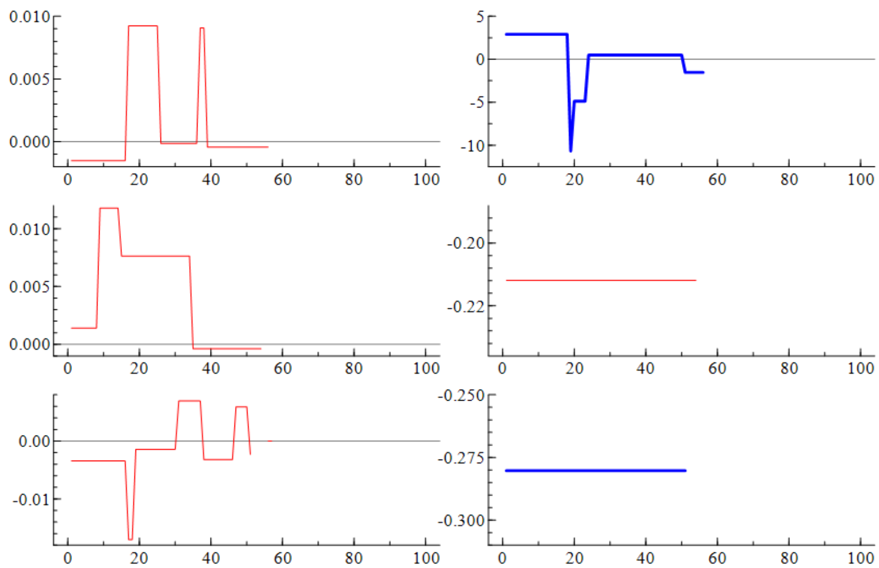


Figure B1 presents the MIS estimates of structural breaks in the 13 individual regressions that have not rejected $\alpha = \beta = 0$ based on the full sample. As summarized in row 2 of Table 2, all 13 of these CG regressions experience breaks in either the constant (12/13) or the slope (6/13).[22]

[22] For the revision coefficient in row 1 and 4 of Figure B1, we remove an outlier of two and one observations respectively, where the parameter spikes dramatically. Hence the discontinuity in those graphs. This does not alter our conclusion of the presence of breaks, but the scale adjustment allows one to see the other breaks more clearly.

Figure B2: MIS Estimates of Significant Breaks in CG Regressions Apparently Consistent with DE Based on the Full Sample

α β

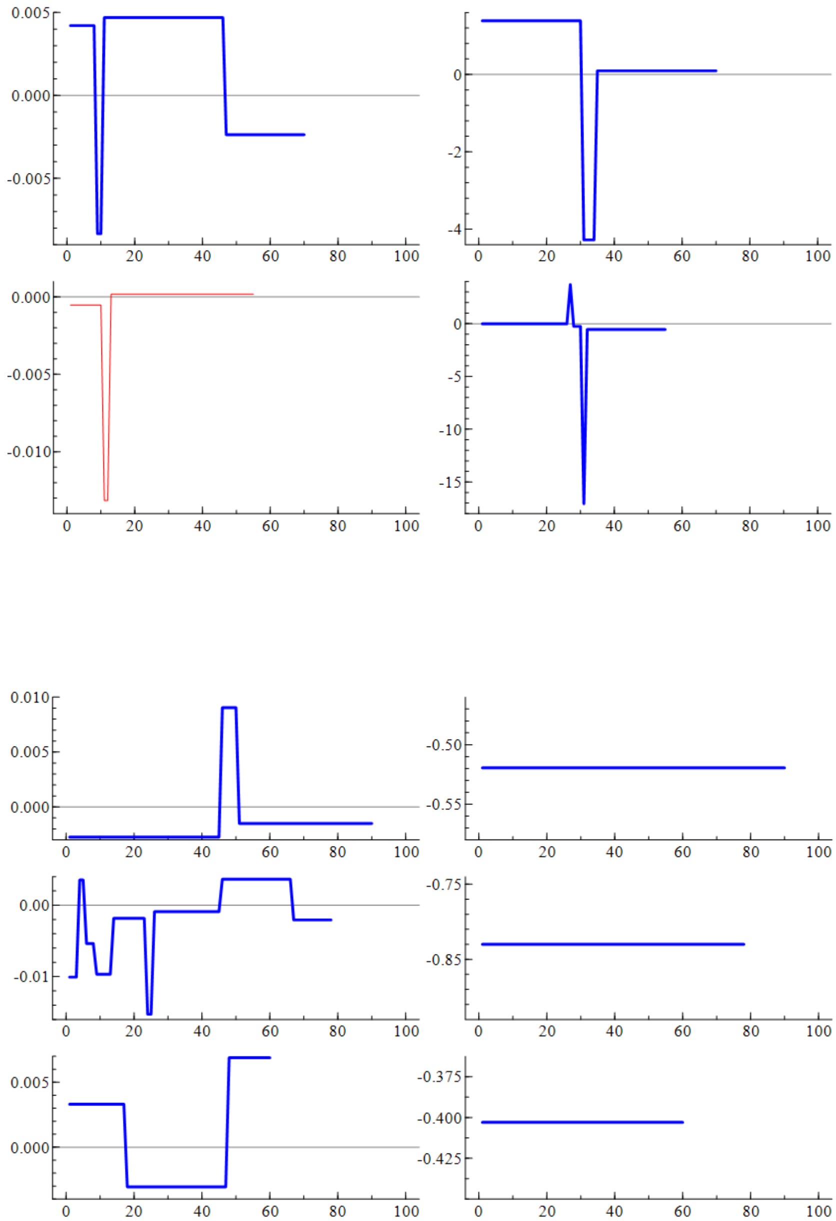


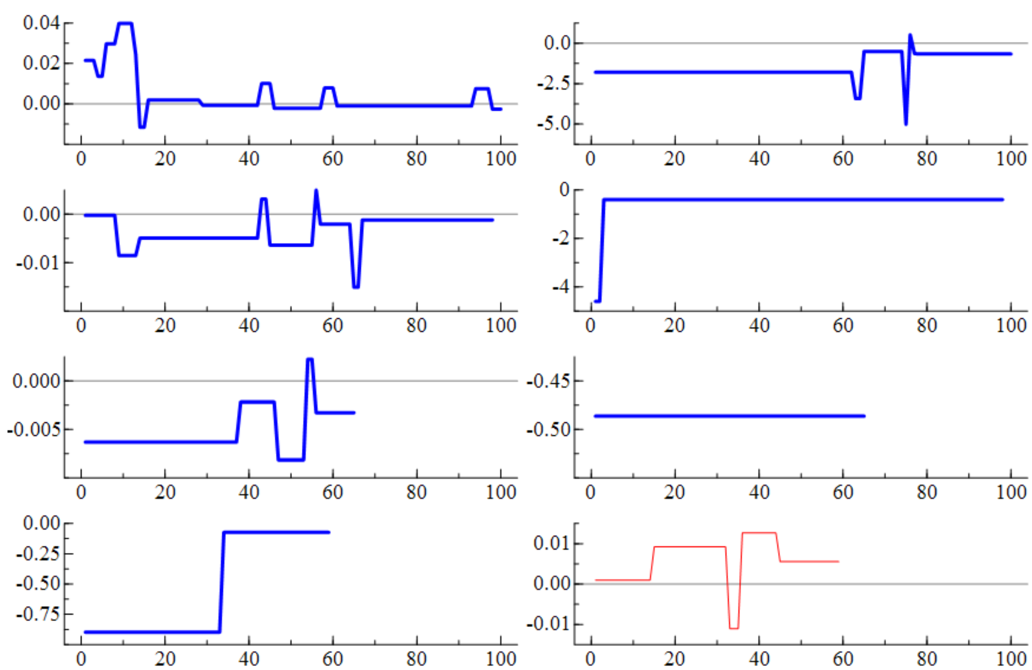
Figure B2 displays the MIS estimates of significant, at 1%, structural breaks in five individual regressions that are consistent with $\alpha = 0$ and $\beta < 0$, based on the full sample. As summarized in row 3 of Table 3, all regressions experience breaks in either α or β, which rejects time-invariant DE.

Among them are three cases where MIS estimates indicate a time-invariant $\beta < 0$, which, assuming regime persistence, could render it consistent with Model C in Table 1. However, one of these cases also experiences breaks in α that are inconsistent with Model C's predictions: α switches between two different values with the same sign.

For the other two, in the third and last rows in Figure B2, the regression experiences only two breaks, both of which involve sign reversals. For this to be consistent with model C, the constant term should return to the same magnitude. For the last row, we can reject at 5% that the constant at the end of the sample is equal to the magnitude at the beginning. For the case in the third row, we can reject this hypothesis at 10%. Model C also implies that the constant terms in successive regimes, while of opposite signs, remain of identical magnitudes. Although the second regime the third row consists of only five observations, which entails low power, its point estimate of α is six times larger in magnitude than that in the third regime.

Figure B3: MIS Estimates of Significant Breaks in Individual Regressions Indicating $\alpha \neq 0$ and $\beta < 0$ Based on the Full Sample

α β



As shown in Figure B3, none of the individual regressions are consistent with any model. Two have clear breaks in β. A third has a break very early in the sample, which could be viewed as an outlier. However, both that regression and the regression with a time-invariant β indicate breaks in the constant which maintain the same sign.

Figure B4: MIS Estimates of Significant Breaks in the Remaining Two Individuals

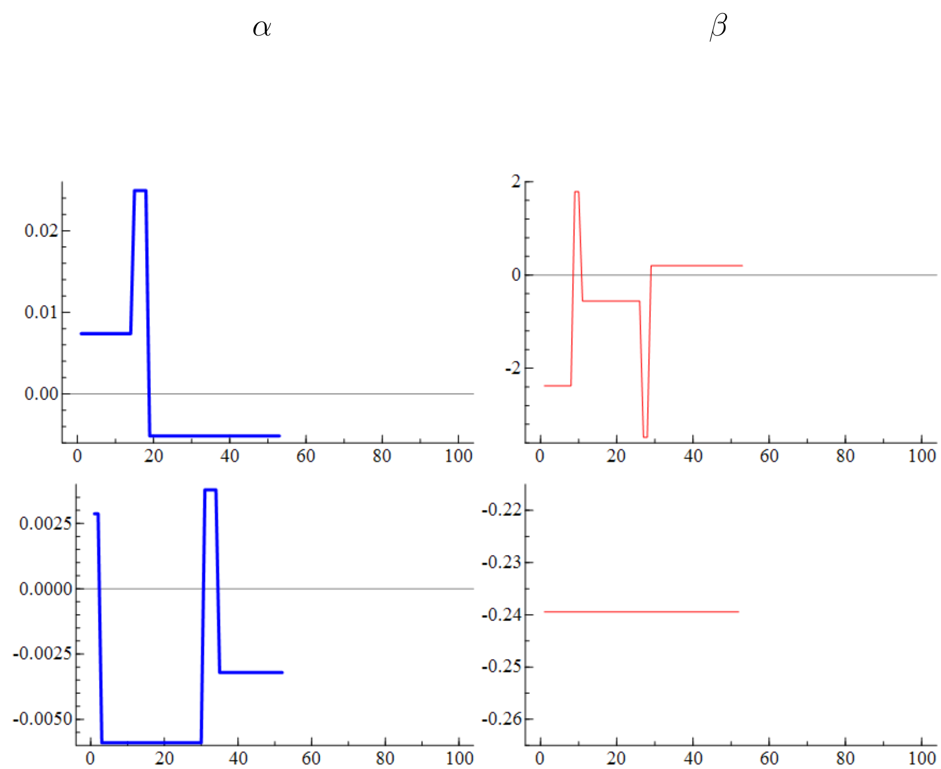


Figure B4 presents MIS estimates of the two remaining CG regressions. The first indicates $\beta > 0$ with $\alpha = 0$ based on full-sample estimates, which could be consistent with either FIRE or DE. However, the breaks in β render this individual regression inconsistent with all seven specifications in Table 1.

The second CG regression indicates $\beta = 0$ and $\alpha \neq 0$ based on the full-sample estimates. Although these inferences are not consistent with any of the models in Table 1, breaks in α with a time-invariant $\beta < 0$ could be consistent with Model C. As Figure B4 shows, however, β remains insignificant. Thus, this final individual regression is also inconsistent with all seven models in Table 1..

16.4 Bai-Perron Test for Structural Breaks in the CG Regression

The Bai-Perron Procedure determines the significant break dates subject to a user-input significance level and a trimming parameter which dictates the minimum duration and maximum number of breaks. A new regression is then estimated within those breaks.

MIS also detects breaks in each parameter at a user-input significance level. However, it does not constrain the constant and slope coefficient(s) to break simultaneously.[23]

We largely follow Bai-Perron (2003) with a 5% significance level, 15% trimming parameter (corresponding to a maximum of five breaks), and using the HAC standard errors with the Andrews' automatic kernel bandwidth estimator. We also use one-lag of pre-Whitening.

We recall that we used 1% significance for MIS. The 5% significance level used here for Bai-Perron should detect more breaks than if 1% were used for Bai-Perron. Nonetheless, Bai-Perron finds fewer individual regressions experiencing breaks than MIS (13 for Bai-Perron vs. all 24 for MIS).

Table B2 provides the estimates produced by the Bai-Perron procedure. For ease of interpretation, the model classifications are color coded. Red indicates a full sample or sub-sample estimates consistent with time-invariant FIRE ($\alpha = \beta = 0$), green indicates DE ($\alpha = 0$ and $\beta < 0$), and blue underreaction ($\alpha = 0$ and $\beta > 0$). The yellow indicates $\alpha \neq 0$ and $\beta < 0$, light blue is $\alpha \neq 0$ and $\beta = 0$, and the white $\alpha \neq 0$ and $\beta = 0$.

[23] MIS also has the advantage of controlling for outliers using the impulse indicator saturation of Hendry, *et al.* (2008).

Table B2: Bai-Perron Estimates of Structural Breaks in the Individual CG Regressions

N	Regime 1 α	Regime 1 β	Reg 2 α	Reg 2 β	Reg 3 α	Reg 3 β	Reg 4 α	Reg 4 β	Reg 5 α	Reg 5 β	Reg 6 α	Reg 6 β
100	0.002 (2.19)	-0.342 (-2.27)										
98	-0.002(-3.15)	-0.394 (-3.03)										
90	-0.003 (-2.15)	-0.274 (-1.24)	.007 (5.42)	-2.401 (-11.27)	-0.002 (-1.84)	-0.777 (-5.47)						
78	-0.001 (-1.22)	-0.388 (-1.50)										
78	-0.004 (-2.19)	-0.255 (-0.52)	.001 (0.53)	-0.915 (-5.79)								
78	-0.001 (-0.96	-0.236 (-1.72)	0.007 (4.77)	-1.672 (-4.56)	-0.003 (-3.03)	-0.726 (-3.44)						
78	0.001 (.62)	-0.004 (-0.02)										
70	-0.002 (-1.64)	-0.150 (-.55)										
70	-0.001 (-.69)	-0.471 (-1.87)										
68	.003 (2.02)	0.328 (0.65)	-0.002 (-2.03)	-0.985 (-5.44)								
65	-0.003 (-3.77)	-0.550 (-3.14)										
65	0.001 (0.32)	-0.924 (-2.61)	-0.003 (-0.73)	0.369 (0.53)	0.010 (4.05)	-1.134 (-1.22)	-0.001 (-0.94)	-0.599 (-7.50)	0.001 (2.65)	1.351 (4.37)		
64	.008 (3.98)	-0.106 (-0.25)	-0.001 (-1.18)	0.216 (-1.53)								
63	0.001 (0.63)	-0.044 (-0.13)										
62	0.000 (0.06)	0.067 (0.27)										
60	0.003 (1.85)	-0.525 (-9.09)	-0.004 (-2.08)	-0.289 (-2.86)	0.007 (7.94)	-0.426 (-0.70)						
59	0.002 (0.56)	-0.443 (-0.96)	0.009 (5.88)	-1.580 (-3.03)	0.008 (4.27)	0.352 (1.44)						
56	-0.000 (-0.03)	0.187 (0.27)	-0.003 (-7.07)	-0.701 (-3.32)	0.006 (3.49)	-2.109 (-7.39)	0.001 (0.90)	1.193 (2.12)	-0.002 (-3.64)	0.523 (0.57)	0.002 (4.32)	0.359 (0.61)
55	0.001 (0.52)	-0.391 (-2.87)										
54	-0.003 (-2.41)	-0.490 (-1.40)	0.006 (4.56)	0.008 (0.28)	0.001 (-1.09)	-0.194 (-5.22)	0.004 (3.42)	0.054 (0.22)	0.002 (-1.90)	-0.177 (-0.32)		
53	0.0003 (0.15)	0.744 (1.74)										
52	-0.005 (-6.08)	-0.007 (-0.06)	0.001 (0.23)	-1.110 (-0.95)	-0.003 (-7.31)	2.097 (6.49)	-0.003 (-1.41)	-1.202 (-2.51)				
52	0.001 (1.39)	-0.192 (-0.67)										
51	-0.001 (-0.65)	0.010 (0.02)	0.009 (9.71)	-0.528 (-2.27)	-0.000 (-0.22)	-0.361 (-3.27)	0.004 (2.67)	0.807 (3.03)				

The Bai-Perron test did not detect breaks for **six** individual regressions that did not reject FIRE in the full sample (the first six individual regressions displayed in Figure B1). Therefore, it was unable to reject time-invariant FIRE for one-quarter of the regressions. Similarly, the Bai-Perron procedure did not detect breaks for two of the five individual regressions consistent with $\alpha = 0$ and $\beta < 0$, as well as for the one consistent with $\alpha \neq 0$ and $\beta = 0$, and three of four with $\alpha \neq 0$ and $\beta < 0$ based on the full-sample estimates. By contrast, MIS finds breaks in either α or β for all individual regressions.

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