STRATEGIC NETWORK FORMATION WITH MANY AGENTS

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Abstract. We derive asymptotic approximations for models of strategic network formation, where limits are taken as the number of nodes (agents) increases to infinity. Our framework assumes a random utility model where agents have heterogeneous tastes over links, and payoffs allow for anonymous and non-anonymous interaction effects, and the observed network is assumed to be pairwise stable. Our main results concern convergence of the link intensity from finite pairwise stable networks to the (many-player) limiting distribution. The set of possible limiting distributions is shown to have a fairly simple form and is characterized through aggregate equilibrium conditions, which may permit multiple solutions. We illustrate how these formal results can be used to analyze identification of link preferences and estimate or bound preference parameters. We also derive an analytical expression for agents' welfare (expected surplus) from the structure of the network. We apply our results to estimate a model of network formation with endogenous search effort in order to assess whether patterns of relative homophily, first documented by Currarini, Jackson, and Pin (2009), can be replicated in a richer empirical framework.

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1. Introduction

Network models can be used to describe systems of contracts, transactions, and other formal or informal relationships between economic agents. In many economic contexts, the incentives to form new network connections exhibit strategic interdependencies across links. In models of trust and social capital, risky exchanges may be secured through transactions with third parties (see e.g. Jackson, Rodriguez-Barraquer, and Tan (2012), Ambrus, Mobius, and Szeidl (2014), Gagnon and Goyal (2016)) which may help with screening, monitoring, and enforcement of an agreement. When networks provide access to information, link formation incentives depend crucially on how a signal is transmitted through that network (see Calvó-Armengol (2004) and Calvó-Armengol and Jackson (2004)). For example, an agent may obtain more widely sourced information through a more central nodes, but may at the same time have to compete with a larger number of network neighbors for access to that

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information. For theoretical or empirical models of peer effects or social coordination - for example in the decision to smoke or engage in other types of risky behavior among high school students - the friendship network often has to be regarded as endogenous with respect to the relevant outcome if that activity itself plays a significant role in agents' social life, or the decision depends on unobservables that may also influence friendship formation (see e.g. Goyal and Vega-Redondo (2005), Goldsmith-Pinkham and Imbens (2012), and Badev (2016)). Strategic incentives of this type may or may not lead to formation of the most beneficial links in terms of aggregate welfare or a social planner's objective. In either case, distinguishing "strategic" externalities from "intrinsic" preferences for forming social or economic relationships are of immediate policy relevance.

As an example, our empirical application revisits Currarini, Jackson, and Pin (2009)'s analysis of friendship formation where agents have potentially homophilous preferences for gender and race ("baseline homophily"), but also face a costly search for potential friends. With homophilous preferences the individually optimal effort choice depends crucially on the racial composition of the environment, with students in "homophilous" minorities having a lower incentive to socialize ("relative homophily"). Separating baseline from relative homophily is policy relevant - while the policy maker would likely regard homophilous preferences as given, the effect of a search friction on network segregation could be mitigated by creating additional opportunities for students to socialize across homophilous categories.

Contribution. This paper analyzes a random utility model of network formation which allows to translate premises and predictions of (typically more stylized) theoretical models into testable hypotheses. We consider settings where network links are undirected and discrete, and link preferences may depend on agents' exogenous attributes and (endogenous) position in the network.

To frame ideas, we could e.g. consider a parametric model which specifies a net payoff to agent i from establishing a link to agent j that depends on either node's exogenous attributes x_i, x_j, the respective number of agents s_i, s_j either node is directly connected to, and an indicator t_{ij} whether i and j have another network neighbor in common. In order to keep our focus on the main conceptual ideas, we also restrict our attention to the case in which the statistics s_i and $\mathbf{t}_{ij}$ take only finitely many values. The incremental benefit to i from forming such a link could then be of the form

$$U_{ij} = x_i'\beta_1 + x_j'\beta_2 + |x_i - x_j|'\beta_3 + \gamma_1 s_i + \gamma_2 s_j + \delta \mathbf{t}_{ij} + \varepsilon_{ij}$$

where ε_{ij} an idiosyncratic shock to i's preferences of forming a link to j. These random utilities could be regarded as continuation values or "reduced form" payoffs from economic activity on the resulting network, reflecting strategic motives of the kind discussed at the beginning of this introduction. Our framework allows for more general payoffs depending on exogenous and endogenous characteristics, and assumes that the observable network is pairwise stable (Jackson and Wolinsky (1996)), where a link ij forms if and only if the incremental benefit of that link to either node exceeds the cost of maintaining that link. Pairwise stability is the default solution concept for models strategic network formation in economics (see e.g. Jackson (2008)) and imposes only minimal requirements on agents' strategic sophistication. The main technical challenges in estimating a model of this form is that the variables s_i, s_j, t_{ij} are a function of the network graph, and therefore endogenous to the network formation model.

Our main theoretical result is a tractable approximation to the expectations network moments, assuming that the number of nodes (agents) in the network is large. Our analysis identifies the relevant aggregate state variables that characterize equilibrium and interdependence of individual link formation decisions, and shows how to use (many-agent) limiting approximations to simplify the representation of the network in terms of these variables. We derive a sharp characterization of the set of link distributions generated by pairwise stable networks. We do constrain the equilibrium selection mechanism in that while the set of pairwise stable networks depends on unobserved taste shocks, we do not allow agents to use them to coordinate on a particular outcome in that set. While we leave the selection rule otherwise unconstrained, this is an important restrictions which makes our framework a lot more tractable.

Based on this limiting approximation we then consider strategies for estimation and inference regarding the model parameters. In addition, we derive an analytical limiting expression for the agent's expected random payoff corresponding to the pairwise stable links, which can serve for welfare analysis regarding any policy interventions that affect the shape of the network. Our present results are complemented by a more recent paper by Menzel (2021) who obtains a law of large numbers and a central limit theorem for network moments that can be used to establish consistency and the asymptotic distribution of estimators, assuming data from a small number of large networks. In the empirical application we use that characterization of surplus to analyze incentives for participating in network formation or exerting effort searching for potential network connections (as in the model of Currarini, Jackson, and Pin (2009)).

The asymptotic approximation is obtained by embedding the finite-player network corresponding to the observable data into a sequence of network formation models with an increasing number of agents. Using statistical approximation techniques, we derive the limit for the distribution of links along that sequence. The primary motivation for many-agent asymptotics in the network model is to arrive at a tractable model that does not require an explicit account for certain interdependencies that are not of first order in the limiting experiment. In particular, the limiting sequence considered has the following qualitative features:

(1) each agent can choose from a large number of possible link formation opportunities, and (2) similar agents face similar choices, at least as measured by the inclusive values corresponding to link opportunity sets. (3) By construction, additional links become increasingly costly along the asymptotic sequence, so that the resulting network remains sparse. (4) The limiting distribution of links resulting from pairwise stable network formation need in general not be unique. Rather, a given realization of payoffs may support multiple pairwise stable networks that differ qualitatively both in terms of global, aggregate features, as well as locally in assigning nodes different roles under alternative equilibria. The limiting sequence does not impose any additional qualitative constraints on agents' incentives for forming network links.

Our approach incorporates some qualitative insights on many-agent limits of game-theoretic models and matching markets from Menzel (2016), Dagsvik (2000), and Menzel (2015). However the main new technical challenges in analyzing large networks cannot be addressed using the formal tools developed in these papers. Most importantly, many realistic models of strategic externalities in link formation need to allow for strong (statistical) dependence across the entire network. The limiting arguments developed in this paper (most importantly Lemma 3.2) relying on symmetric (exchangeable), rather than weak dependence are to my knowledge entirely new and may serve as a blueprint for limiting arguments in large games beyond the context of networks. Furthermore, in a network formation problem with link externalities, non-uniqueness of stable outcomes results in a non-singleton set of limiting distributions, adding conceptual difficulties in taking many-player limits. In contrast, the structure of the matching problem in Menzel (2015) was shown to imply weak dependence of matching outcomes and resulted in a unique limiting distribution.

Literature. A powerful and convenient formal framework for describing networks is the classical Erdős and Rényi (1959) random graph model (RGM). The RGM describes links as independent binary random variables and has been extended in various ways to allow for node- and edge-level heterogeneity and has been used in applied work (see e.g. Fafchamps and Gubert (2007)). For strategic models of network formation, strategic interdependence between link formation decisions typically leads to stochastic dependence between links and can therefore in general not be represented as RGM. Also, network data often exhibit clustering and degree heterogeneity in excess of levels compatible with that model. One approach to accommodate this empirical regularity into econometric models of link formation is to allow for preferential attachment and unobserved heterogeneity in the propensity of a node to form links (see Graham (2017) and Dzemski (2014)).

As an alternative, the model may directly incorporate (endogenous) network attributes - including degree centrality or network distance - as determinants of the link probability in a generalized exponential random graph model (ERGM). Mathematical properties of ERGM are by now fairly well understood, and some fairly general results on estimation and large-sample theory are already available.[1] Here Chandrasekhar and Jackson (2016) develop a flexible approach to match not only pairwise frequencies, but also subgraph counts involving three or more nodes. Our framework differs from these papers in that we characterize the network formation process using link preferences that may depend directly on endogenous network attributes. This introduces a strategic element into the model which in some cases produces interdependencies of link formation decisions between "distant" nodes, and typically yields a multiplicity of stable network outcomes. In particular, Chandrasekhar and Jackson (2016)'s assumption that subnetworks of certain types form independently is not generally consistent with pairwise stability under preferences that exhibit strategic interdependencies between different links. Lovász (2012) showed how to characterize a finite network graph as a sample from a continuous limiting object. However when the graph is the result of strategic decisions by the agents associated with the (finitely many) nodes, the relationship between features of the descriptive limiting "graphon" to stable, "structural" features of an underlying population is generally not transparent or even well-defined, especially if the network formation model admits multiple stable outcomes.

Most existing approaches to structural estimation rely heavily on simulation methods - this includes Hoff, Raftery, and Handcock (2002), Christakis, Fowler, Imbens, and Kalyanaraman (2010), Mele (2017), Sheng (2020), and Leung (2019) - whereas our approach focusses on analytic characterizations of pairwise stable networks. Instead of considering the joint distribution of the adjacency matrix or larger local "neighborhoods" within the network (as considered by Sheng (2020), de Paula, Richards-Shubik, and Tamer (2018) or Graham (2012)), we argue that it is typically sufficient for estimation to consider the frequencies of links between pairs of nodes (dyads) with a given combination of exogenous attributes and endogenous network characteristics. Our analysis differs from de Paula, Richards-Shubik, and Tamer (2018) in that our limiting model is constructed as a limiting approximation to a finite network, whereas their model assumes a continuum of players. Furthermore, we model link preferences as non-anonymous in the finite network, and therefore have to characterize explicitly how subnetworks interact with the full network through link availability and strategic interaction effects with neighboring nodes. Our asymptotic approximations allow to characterize that dependence using aggregate state variables that satisfy certain equilibrium conditions in order for the network to be pairwise stable. Boucher and Mourifié

[1]See e.g. Frank and Strauss (1986), Wasserman and Pattison (1996), Bickel, Chen, and Levina (2011), or Snijders (2011) for a survey. Jackson and Rogers (2007) analyze characteristics of large networks of homogeneous agents that result from a sequential random meeting process where links may be added "myopically" at each step.

(2012) give conditions for weak dependence of network links under increasing domain asymptotics, whereas our approach can be thought of as "infill" asymptotics where link frequencies between distant nodes are non-trivial under any metric on the space of node characteristics.

This work is most closely related to a concurrent paper by Leung (2019) and subsequent work by Leung and Moon (2019) and Menzel (2021). Menzel (2021) extends the formal ideas in this paper to obtain a central limit theory for network moments and to cover more general settings, including polyadic subnetwork counts and edge-specific interaction effects. Leung (2019) and Leung and Moon (2019) give weak laws, and a central limit theorem, respectively, that also cover static network formation. Their approach requires that strategic interaction effects are not too large in magnitude and that agents interact primarily locally in a (latent or observed) attribute space that grows in diameter along the asymptotic sequence. These conditions ensure that strategic interdependencies remain limited to subnetworks that are stochastically bounded in size even as the number of nodes grows large. Our approach holds the support of any payoff-relevant attributes fixed as we take limits and relies on symmetry and exchangeability arguments instead. This does allow for long-range stochastic dependence of arbitrary strength where even global features of the network may remain stochastic as we take limits. Either modeling approach may be more appropriate for a given application, depending on the nature of strategic interactions or the presence of node attributes that have a sufficient strong effect on link formation to produce the topology required for the framework in Leung (2019) and Leung and Moon (2019).[2] The other key difference is in the nature of our results, where our focus is on closed-form expressions or bounds for probabilities of network events, whereas their approach requires finding the pairwise stable subnetworks on strategic neighborhoods given simulated payoffs.

The remainder of the paper is organized as follows: we first describe the economic model, including alternative solution concepts. Section 3 defines the limiting model and gives formal results regarding convergence to that limit. Section 4 gives an outline of the main formal steps for the convergence argument. Section 5 discusses strategies for identification and estimation based on that representation, and gives an analytical characterization of agents' welfare (expected surplus) from the structure of the network. Section 6 gives an empirical application, estimating a model of network formation with endogenous search effort. The appendix presents additional Monte Carlo study illustrating the theoretical convergence results, and proofs of theoretical claims.

[2]Our empirical example uses the AddHealth data set, where networks consists of the students in each particular high school. While we find that residential location does have a nontrivial impact on link formation, it is of a magnitude comparable to, but not exceeding that of other relevant preference shifters, including idiosyncratic taste heterogeneity, so that we favor the framework put forward in the present paper for that particular setting.

2. Model Description

The network consists of a set of n agents ("nodes" or "vertices"), which we denote with $\mathcal{N} = \{1, \dots, n\}$. We assume that each agent is associated with a vector of exogenous attributes (types) $x_i \in \mathcal{X}$. Our network formation model regards exogenous attributes as random draws $\mathbf{x}_i$ from a distribution with marginal p.d.f. $w(x)$, and for the purposes of this paper the type space $\mathcal{X}$ is assumed to be finite. We also use $X = [x_1, \dots, x_n]'$ to denote the matrix containing the n nodes' exogenous attributes.

Using standard notation (see Jackson (2008)), we identify the network graph with the *adjacency matrix* $L = (L_{ij})_{ij}$, where the element

$$L_{ij} = \begin{cases} 1 & \text{if there is a direct link from node } i \text{ to node } j \\ 0 & \text{otherwise} \end{cases}$$

As a convention, we do not allow for any node i to be linked to itself, $L_{ii} = 0$. We furthermore assume that all links are *undirected*, so that the adjacency matrix L is symmetric, i.e.$L_{ij} = L_{ji}$. We also let $L - ij$ be the network resulting from deleting the link along the edge ij from L, that is from replacing the ijth entry of L with zero. Similarly, $L + ij$ denotes the network resulting from setting L_{ij} to one.

In an idealized application, the observed network data consists of X and L. However, the limiting approximations do not distinguish between observable and unobservable components of x_i and can therefore also be used in settings in which some relevant exogenous characteristics are unobserved. In a similar fashion a data set may consists of a subset of nodes and edges that were selected at random according to a known sampling rule. In that case, the asymptotic full-network likelihoods derived in this paper can be adjusted by the probability that a given edge or node would be selected into such a sample, as long as the resulting data set also contains sufficient information to evaluate the payoff-relevant network attributes for sampled nodes or edges.

2.1. Network Payoffs.

Player i's payoffs are of the form

$$\Pi_i(L) = B_i(L) - C_i(L)$$

where $B_i(L)$ denotes the gross benefit to i from the network structure, and $C_i(L)$ the cost of maintaining links. In economic terms the benefit function would typically represent a continuation value representing utility or profit from economic activity on the network once it has formed. Since the formal distinction between costs and benefits is arbitrary, we will take the cost $C_i(L)$ to be only a function of the number of direct links to player i, but not the identities or characteristics of the individuals that i is directly connected to under the network structure L.

We specify the model in terms of the incremental benefit of adding a link ij to the network L,

$$U_{ij}(L) := B_i(L + ij) - B_i(L - ij)$$

and the cost increment of adding that link,

$$MC_{ij}(L) := C_i(L + ij) - C_i(L - ij)$$

With a slight departure from common usage of those terms, we refer to $U_{ij}(L)$ and $MC_{ij}(L)$ as the *marginal benefit* and *marginal cost* (to player i), respectively, of adding the link ij to the network.

Throughout our analysis we specify the marginal benefit function as

$$U_{ij}(L) = U^*_{ij}(L) + \sigma\varepsilon_{ij} \tag{2.1}$$

where $U^*_{ij}(L)$ is a deterministic function of attributes $x_1, \dots, x_n$ and the adjacency matrix L, and will be referred to as the *systematic part* of the marginal benefit function. The idiosyncratic taste shifters ε_{ij} are assumed to be independent of x_i and x_j and distributed according to a continuous c.d.f. $G(\cdot)$, and $\sigma > 0$ is a scale parameter. Also, we assume that marginal costs to i for forming a link are given by

$$MC_{ij}(L) \equiv MC_i(L) := \max_{k=1,\dots,J} \sigma\varepsilon_{i0,k} \tag{2.2}$$

where $\varepsilon_{i0,k}$ are independent of x_i and across draws $k = 1, 2, \dots$, and the choice of the number of draws J will be discussed in Section 3. In particular, we let J to grow as n increases in order for the resulting network to be sparse.[3] In this formulation, marginal costs are assumed not to depend on the network structure or the identity of the target node, so that in the following we denote marginal cost of the link ij by MC_i without explicit reference to j or the network L. Note that in the absence of further restrictions on the systematic parts of benefits $U^*_{ij}(L)$, this is only a normalization.

The main application of our asymptotic results concerns identification of - parametric or nonparametric models for the function $U^*_{ij}(L)$, where the distribution of taste shocks $G(\cdot)$ need not be specified by the researcher as long as its upper tail is assumed to satisfy the shape restriction in Assumption 2.2 below. We find below that for some relevant aspects of the model, only the sum of the systematic part of marginal utilities between the two nodes constituting an edge matters, we also define the pseudo-surplus for the edge ij as

$$V^*_{ij}(L) := U^*_{ij}(L) + U^*_{ji}(L)$$

[3]Alternatively, it would be possible to model marginal costs $MC_i = \log J + \varepsilon_{i0,k}$ under the extreme-value arguments, however the specification in (2.2) is analytically more convenient for our derivations. Otherwise our specification of MC_i is analogous to the treatment of the outside option in the matching model in Menzel (2015), and we refer the reader to that paper for a further discussion.

Obviously $V_{ij}^*(L) = V_{ji}^*(L)$, so pseudo-surplus is symmetric with respect to the identities of the two nodes.

Our framework allows for various types of interaction effects on the marginal benefit function. The marginal benefit from adding the link from i to j may depend on agent i and j's exogenous attributes x_i and x_j, and the structure of the network through vector-valued statistics $s_i, s_j, \mathbf{t}_{ij}$ that summarize the payoff-relevant features,

$$U_{ij}^*(L) \equiv U^*(x_i, x_j; s_i, s_j, \mathbf{t}_{ij}) \tag{2.3}$$

Specifically, the marginal benefit of a link may directly depend on node i and j's exogenous attributes, x_i and x_j, respectively, as well as interaction effects between the two. $U_{ij}^*(L)$ may vary in x_i, e.g. because some node attributes may make i attach more value to any additional links. On the other hand, dependence on x_j allows for target nodes with certain attributes to be generally more attractive as partners. Finally, a non-zero cross-derivative between components of x_i and x_j could represent economic complementarities, or a preference for being linked to nodes with similar attributes (homophily).

In addition to preferences for exogenous attributes, the propensity of agent i to form an additional link, and the attractiveness of a link to agent j may depend on the absolute position of either node i and j in the network. To account for effects of this type, we can include node-specific network statistics of the form

$$s_i := S(L, X; i)$$

where we assume that the function $S(\cdot)$ is invariant to permutation of player indices.[4]

Example 2.1. ***(Degree and Composition)*** *Node specific network statistics include the network degree (number of direct neighbors),*

$$S_1(L, X; i) := \sum_{j \neq i} L_{ij}$$

Another statistic could measure the share of i's direct neighbors that are of a given exogenous type,

$$S_2(L, X; i) := \frac{\sum_{j \neq i} L_{ij} \mathbb{1}\{x_{jk} = \bar{x}_k\}}{\sum_{j \neq i} L_{ij}}$$

where the kth component of x_j may be e.g. gender or race, and $\bar{x}_k$ the value corresponding to the category in question (e.g. with respect to gender or race).

[4]Formally, we assume that for any $i = 1, \dots, n$ and one-to-one map $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ with $\pi(i) = i$, we have $S(L^\pi, X^\pi; i) = S(L, X; i)$, where the matrices X^π and L^π are obtained from X and L by permuting the rows (rows and columns, respectively) of the matrix according to π.

The network degree of a node plays a special role in the description of the link intensity. In the remainder of the paper, we therefore partition the vector of node i's network characteristics into $s_i = (s_{i1}, s_{i2}')'$, where $s_{i1} := \sum_{j=1}^n L_{ij}$ will always be understood to denote the network degree of node i, and s_{i2} a vector of the remaining payoff-relevant network statistics. In this paper, we restrict these node-specific statistics $S(L, X; i)$ to depend only on the links to i, i.e. the ith row of the adjacency matrix L, analogous results for the more general case are given in Menzel (2021).

Payoffs may also depend on the relative position of the node i with respect to j in the network. Specifically, the researcher may also want to include edge-specific network statistics of the form

$$\mathbf{t}_{ij} := T(L, X; i, j)$$

where $T(\cdot)$ may again be vector-valued, and we assume that the function $T(\cdot)$ is invariant to permutations of player indices.[5]

Example 2.2. ***(Transitive Triads)*** *A preference for closure of transitive triads can be expressed using statistics of the form*

$$T_1(L, X; i, j) = \sum_{k \neq i,j} L_{ik} L_{jk}, \quad \text{or } T_2(L, X; i, j) = \max \{L_{ik} L_{jk} : k \neq i, j\}$$

Here, T_{1ij} counts the number of immediate neighbors that both i and j have in common, and T_{2ij} is an indicator whether i and j have any common neighbor. More generally, T_{ij} could include other measures of the distance between agents i and j in the absence of a direct link, or indicators for potential "cliques" of larger sizes.

Patterns of transitivity may emerge for example in economic models of social capital where supporting links to common neighbors may enhance the value or viability of a connection between an agent pair, see e.g. Jackson, Rodriguez-Barraquer, and Tan (2012) or Gagnon and Goyal (2016). Transitivity may also reflect a biased search process where agents may be more likely to "meet" through common neighbors. In this paper we will only consider dependence of payoffs on node-specific statistics, analogous results allowing for edge-specific statistics are developed in Menzel (2021).

In contrast to node attributes x_i, x_j, the variables s_i, s_j, and $\mathbf{t}_{ij}$ are endogenous to the network formation process, and the characterization of the limiting model therefore must include equilibrium conditions for the joint distribution of types x_i and network statistics s_i and $\mathbf{t}_{ij}$. We therefore refer to the payoff contribution of the exogenous attributes x_i, x_j as *exogenous interaction effects*, and the contribution of the endogenous network characteristics s_i, s_j, t_{ij} as *endogenous interaction effects*. In terms of this specification, we can also rewrite

[5] That is, we assume that for any $i, j = 1, \ldots, n$ and permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with $\pi(i) = i$ and $\pi(j) = j$, we have $T(L^\pi, X^\pi; i, j) = T(L, X; i, j)$.

the pseudo-surplus function as

$$V_{ij}^*(L) = V^*(x_i, x_j; s_i, s_j, \mathbf{t}_{ij}) := U^*(x_i, x_j; s_i, s_j, \mathbf{t}_{ij}) + U^*(x_j, x_i; s_j, s_i, \mathbf{t}_{ij})$$

This paper considers only the case of node-specific network statistics s_i, s_j entering payoffs for a link ij, that is

$$U_{ij}^*(L) \equiv U^*(x_i, x_j; s_i, s_j).$$

Results concerning the general case including edge-specific interaction effects are given in the more recent paper Menzel (2021). Furthermore, we maintain that the deterministic parts of random payoffs satisfy certain uniform bounds and smoothness restrictions. Specifically we assume the following

Assumption 2.1. ***(Systematic Part of Payoffs)*** *Payoffs do not depend on edge-specific network statistics, where (i) the systematic parts of payoffs are uniformly bounded in absolute value, $|U^*(x, x', s, s')| \le \bar{U} < \infty$. Furthermore, (ii) The supports of the payoff-relevant network statistics, $\mathcal{S}$ and the type space $\mathcal{X}$ are finite sets. (iii) The network statistics $S(L, X; i)$ do not depend on L_{kl} for any $k, l \neq i$.*

The uniform bound on systematic payoffs in part (i) serves primarily to simplify the formal argument, and might be replaced by bounds on other norms on the function $U^*(\cdot)$, a question we will leave for future research. Part (ii) is used to ensure that the limit of the game can be represented as an equilibrium in finite-dimensional state variables. This condition could in principle be relaxed under some additional regularity conditions, mainly to ensure compactness of the range of the corresponding fixed point mappings. This restriction applies only to the *payoff relevant* network statistics, for example in our setting network degree s_{i1} generally has unbounded support, but our setting would allow for preferences that depend on a transformation of s_{i1} that takes only finitely many values, e.g. a measure $\tilde{s}_i := \min\{s_{i1}, \bar{s}\}$ that censors degree at some finite value $\bar{s}$. Furthermore, while part (ii) implies part (i) of this assumption, we state this condition separately in order to define the bound $\bar{U}$ for later reference.

As discussed in the previous section, we also restrict the focus of this paper to the case of strategic effects from node-specific network statistics, where part (iii) requires that for any network L, the value of s_i is determined by the edges to/from i, i.e. the ith row and column of the adjacency matrix L. These restrictions are primarily to keep notation as simple as possible; the main theoretical ideas developed in this paper are extended to a less restrictive setup in Menzel (2021) who also derives and asymptotic distribution theory for generalized subgraph counts.

We next state our assumptions on the distribution of unobserved taste shifters. Most importantly, we impose sufficient conditions for the distribution of $\boldsymbol{\varepsilon}_{ij}$ to belong to the domain of attraction of the extreme-value type I (Gumbel) distribution. Following Resnick

(1987), we say that the upper tail of the distribution $G(\varepsilon)$ is of type I if there exists an auxiliary function $a(s) \geq 0$ such that the c.d.f. satisfies

$$\lim_{s\to\infty} \frac{1-G(s+a(s)v)}{1-G(s)} = e^{-v}$$

for all $v \in \mathbb{R}$. We are furthermore going to restrict our attention to distributions for which the auxiliary function can be chosen as $a(s) := \frac{1-G(s)}{g(s)}$, where $g(s)$ denotes the density associated with the c.d.f. $G(s)$. This property is shared for most standard specifications of discrete choice models, e.g. if $\boldsymbol{\varepsilon}_{ij}$ follows the extreme-value type I, normal, or Gamma distribution, see Resnick (1987). We can now state our main assumption on the distribution of the idiosyncratic part of payoffs:

Assumption 2.2. ***(Idiosyncratic Part of Payoffs)*** *$\boldsymbol{\varepsilon}_{ij}$ and $\boldsymbol{\varepsilon}_{i0,k}$ are i.i.d. draws from the distribution $G(s)$, and are independent of X, where (i) the c.d.f. $G(s)$ is absolutely continuous with density $g(s)$, and (ii) the upper tail of the distribution $G(s)$ is of type I with auxiliary function $a(s) := \frac{1-G(s)}{g(s)}$.*

2.2. **Solution Concept.** Our formal analysis assumes pairwise stability as a solution concept, which was first introduced by Jackson and Wolinsky (1996).

Definition 2.1. ***(Pairwise Stable Network)*** *The undirected graph L is a* ***pairwise stable network (PSN)*** *if for any link ij with $L_{ij} = 1$,*

$$U_{ij}(L) \geq MC_i, \quad \text{and } U_{ji}(L) \geq MC_j$$

and any link ij with $L_{ij} = 0$,

$$U_{ij}(L) < MC_i, \quad \text{or } U_{ji}(L) < MC_j$$

Pairwise stability as a solution concept only requires stability against deviations in which one single link is changed at a time. For a pairwise stable network there may well be an agent who can increase her payoff by reconfiguring two or more links unilaterally. While it is possible to consider stronger notions of individually optimal choice which require stability against more complex deviations, PSN gives a set of necessary conditions which have to be met by any such refinement.

Pairwise stability also does not necessarily impose particularly high demands on participating agents' knowledge and strategic sophistication: Jackson and Watts (2002) showed that pairwise stable networks can be achieved by tâtonnement dynamics in which agents form or destroy individual connections, taking the remaining network as given and not anticipating future adjustments. This makes PSN a plausible static solution concept for a decentralized network formation process even when agents have only a limited understanding of the network as a whole, and link decisions may in fact take place over time, where the exact sequence of adjustments is not known to the researcher.

A major limitation of PSN as a solution concept is that, without additional restrictions on payoffs a pairwise stable network is not guaranteed to exist. While to our knowledge there are no fully general existence results, there are some relevant special cases for which existence of a PSN is not problematic.[6]

A second challenge is that pairwise stability does not predict a unique outcome for the network formation game. Neither the static nor the tâtonnement interpretation of pairwise stability in a model of decentralized network formation appear to suggest a particular rule for selecting one stable outcome over another. In their most general version our results therefore do not constrain the mechanism for selecting among multiple pairwise stable matchings, but give sharp bounds on the distribution of network outcomes.

For a revealed-preference analysis it is useful to represent the pairwise stability conditions as a discrete choice problem, where preferences are given by the random utility model described above, and the set of available "alternatives" for links arises endogenously from the equilibrium outcome. Specifically, given the network L we define the *link opportunity set* $W_i(L) \subset \mathcal{N}$ as the set of nodes who would prefer to add a link to i,

$$W_i(L) := \{j \in \mathcal{N} \backslash \{i\} : U_{ji}(L) \geq MC_j\} \tag{2.4}$$

Using this notation, we can rewrite the pairwise stability condition in terms of individually optimal choices from the opportunity sets arising from a network L.

Lemma 2.1. *Assuming that all preferences are strict, a network* $\mathbf{L}^*$ *is pairwise stable if and only if for all* $i = 1, \ldots, n$,

$$L_{ij}^* = \begin{cases} 1 & \text{if } U_{ij}(L^*) \geq MC_i \\ 0 & \text{if } U_{ij}(L^*) < MC_i \end{cases} \tag{2.5}$$

for all $j \in W_i(L^*)$, *and* $L_{ij}^* = 0$ *for all* $j \notin W_i(L^*)$.

The proof for this lemma is similar to that of Lemma 2.1 in Menzel (2015) and is given in the appendix.

It is also instructive to contrast our use of a solution concept that is essentially static to the approaches in Christakis, Fowler, Imbens, and Kalyanaraman (2010) and Mele (2017) who consider link distributions resulting from myopic random revisions of past link formation decisions, where agents are not assumed to be forward-looking regarding future stages of the formation game. Christakis, Fowler, Imbens, and Kalyanaraman (2010) specify an initial condition and a stochastic revision process, so that (in the absence of further shocks to the process) further iterations of the tâtonnement process would generate a distribution over

[6]As an example, Miyauchi (2012) considers the case of non-negative link externalities, in which case pairwise stability can be represented as Nash equilibrium in a finite game with strategic complementarities. Hence, existence and achievability through a myopic tâtonnement mechanism follow from general results by Milgrom and Roberts (1990).

pairwise stable outcomes or cycles with mixing weights depending on that specification. The revision process in Mele (2017) is represented by a potential function, favoring formation of links that lead to larger cardinal utility improvements, and networks generating a large "systematic" surplus.

2.3. **Potential Values.** To verify whether a link on the edge ij is supported by a pairwise stable network, it suffices to check two separate conditions. For one, the network statistics $s_i, s_j, \mathbf{t}_{ij}$ must be supported by a pairwise stable network on the remaining edges after constraining the edge ij to the value $L^*_{ij} = l$ for $l \in \{0,1\}$. On the other hand, the value L^*_{ij} has to satisfy the payoff conditions $U(x_i, x_j; s_i, s_j, \mathbf{t}_{ij}) + \sigma\varepsilon_{ij} \geq MC_i$ and $U(x_j, x_i; s_j, s_i, \mathbf{t}_{ij}) + \sigma\varepsilon_{ji} \geq MC_j$ if and only if $L^*_{ij} = 1$, holding $s_i, s_j, \mathbf{t}_{ij}$ fixed at their respective values. In what follows we let $D_{ij} \in \{0,1\}$ denote an indicator whether i agrees to form a link with j, where we consider both the case of exogenously fixed, as well as pairwise stable link proposals $D^*_{ij} := \mathbb{1}\{U(x_i, x_j; s_i, s_j, \mathbf{t}_{ij}) + \sigma\varepsilon_{ij} \geq MC_i\}$.

After restricting the subnetwork on a selected subset of edges to a particular configuration, we refer to the values for a network statistic s_i that are supported by a network L^*_{-ij} that is pairwise stable on the remaining, unrestricted edges as the **potential values** for that statistic with respect to that subnetwork configuration. For the purposes of this paper it suffices to consider potential values with respect to a single edge L_{ij} together with the link proposals by nodes i and j, $D_i = (D_{ik})$ and $D_j = (D_{jk})$.

Definition 2.2. ***(Potential Values)*** *We say that for $l = 0, 1$ and proposals $D_i, D_j \in \{0,1\}^n$ the potential values $\mathbf{S}^*_{kij}(l; D_i, D_j)$ for s_k are supported by a pairwise stable network if given realized attributes and taste shocks there exists a network with nodes $\mathcal{N}$ and adjacency matrix $L^*_{-ij} = (L^*_{kl,-ij})_{k,l}$ such that (a) $L^*_{ij,-ij} = l$, (b) for each edge pq with $p, q \notin \{i,j\}$, $L^*_{pq,-ij} = 1$ if and only if $U_{pq}(L^*_{-ij}) \geq MC_p$ and $U_{qp}(L^*_{-ij}) \geq MC_q$, (c) for each $p \notin \{i,j\}$ and $q \in \{i,j\}$, $L^*_{pq,-ij} = 1$ if and only if $U_{pq}(L^*_{-ij}) \geq MC_p$ and $D_{qp} = 1$, and (d) $\mathbf{S}(L^*_{-ij} + ij, X; k) = \mathbf{S}^*_{kij}(l; D_i, D_j)$.*

These potential values can be interpreted as the set of possible structural responses to the link proposals by nodes i and j. That is, the difference between potential values $\mathbf{S}^*_{kij}(0; \cdot)$ and $\mathbf{S}^*_{kij}(1; \cdot)$ accounts for indirect, "general equilibrium" effects of exogenously setting the link L_{ij} to one or zero. These potential values are random variables, since stability of the network L^*_{-ij} given D_i, D_j depends on attributes and payoff shifters at the nodes in $\mathcal{N} \backslash \{i,j\}$. When the pairwise stable network is not unique, the realized potential values correspond to a selection from among those values that are supported according to this definition. The characterization of outcomes on subnetworks of more than two nodes requires a more general definition of potential outcomes, which is given in Menzel (2021).

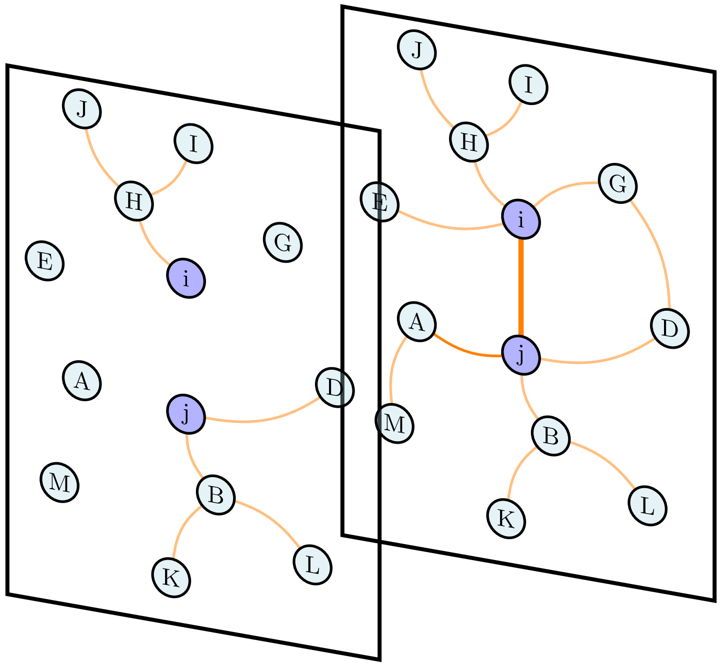

Figure 1. Constrained stable network configurations for $L_{ij} = 0$ and $L_{ij} = 1$, respectively. The respective potential values correspond to the network statistics for i and j, evaluated at either constrained stable network, which need not be unique for either value of L_{ij}.

Note also that we take both network statistics $\mathbf{S}^*_{kij}(0;\cdot), \mathbf{S}^*_{kij}(1;\cdot)$ to be evaluated at a network with the connection between i and j present. The potential outcomes with that link removed, i.e. evaluated under $L^*_{-ij} - (ij)$, do not appear in the payoff conditions for pairwise stability and we therefore do not need to refer to them in what follows. Note that D_i, D_j were defined to include the proposals $D_{ii}, D_{ij}, D_{ji}, D_{jj}$. This is entirely for notational convenience, and requirements (b) and (c) of this definition have been formulated in a way such that those proposals are redundant. Also, whenever a proposal D_{ip} is not reciprocated, the potential value $\mathbf{S}^*_{kij}(l; D_i + ip, D_j)$ is supported whenever $\mathbf{S}^*_{kij}(l; D_i - ip, D_j)$, where the notation $D_i + ij$ and $D_i - ij$ denote the proposal vector D after setting D_{ij} to one (zero, respectively). Since we only need to consider potential outcomes for proposals $D_{ip} = 1$ whenever for $\bar{U} + \sigma\varepsilon_{ip} \geq MC_i$ and $\bar{U} + \sigma\varepsilon_{pi} \geq MC_p$, the effective number of distinct potential outcomes is in fact bounded.

To illustrate this definition, we give a conceptual example in Figure 2: Given realized network payoffs, dashed arcs indicate links L_{ij} that may be pairwise stable for some values s_{i1}, s_{j1} of degree centrality for nodes i and j, whereas links between any other node pairs are not mutually agreeable regardless of the network positions of the end nodes. We then obtain the potential values for s_{i1}, s_{j1} that are supported given these payoffs by determining all pairwise stable networks after fixing either value $l \in \{0, 1\}$ for the edge L_{ij}. For $l = 0$,

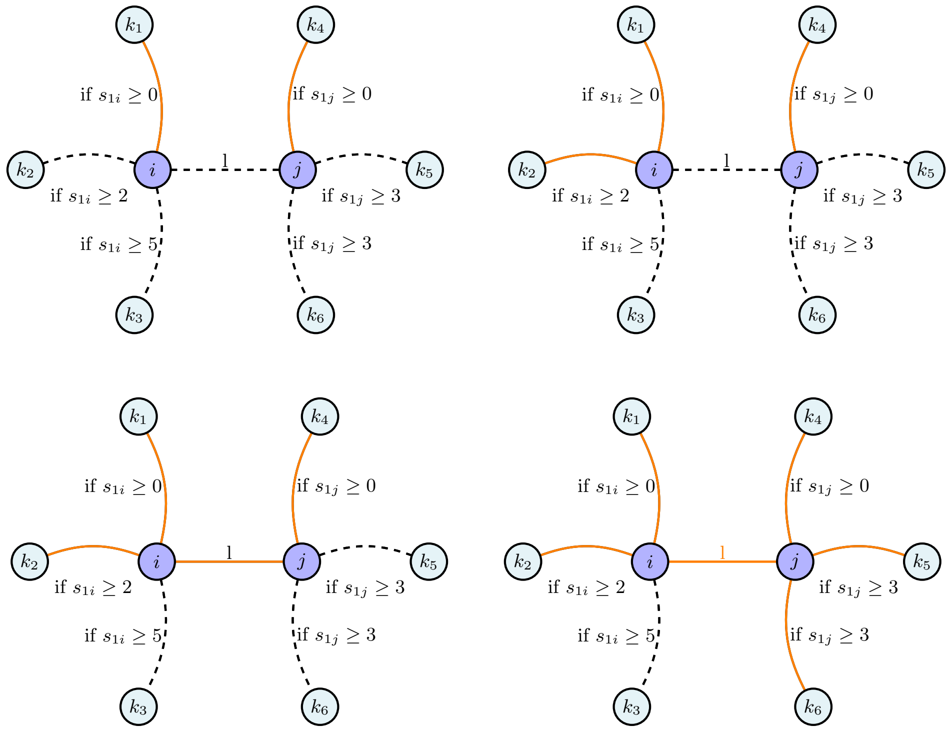

FIGURE 2. Potential values for s_{i1}, s_{j1} with respect to $L_{ij} = 0$ (top row) and $L_{ij} = 1$ (bottom row).

we can see that j can only form a pairwise stable link to k_4, but i can form a pairwise stable link either to k_1 alone, or to k_1 and k_2. Hence the supported potential values are $\{1, 2\}$ for s_{i1}, and $\{1\}$ for s_{j1}. Fixing L_{ij} at $l = 1$ instead, the supported potential values for s_{i1} are $\{3\}$, and $\{2, 4\}$ for s_{j1}. In general there may also be direct or indirect connections between the nodes $k_1, \ldots, k_6$ and pairwise stability may depend on the positions of those nodes as well. Hence solving for these potential values would be very cumbersome in realistically sized problems, and given our intermediate results - specifically Lemma 3.2 - an explicit calculation will not be necessary.

2.4. **Equilibrium Selection.** The number of distinct pairwise stable networks for a given realization of payoffs may be very large. In this paper we take a pragmatic approach to equilibrium selection which imposes an (in our view) mild constraint on stochastic equilibrium selection rules. Previous versions of this paper did not impose any conditions on equilibrium selection which are in fact not needed for the main approximation arguments, but greatly simplify the asymptotic representation and our approach towards estimation.

Assumption 2.3. ***Solution Concept*** *(i) The observed network L^* satisfies the payoff conditions for pairwise stability in Definition 2.1. (ii) In the presence of multiple pairwise stable networks, agents coordinate on a pairwise stable network via a public signal* $\mathbf{v}$ *that is* *measurable with respect to a sigma field $\mathcal{G}$ but need not be observable to the researcher. (iii) Unobserved taste shocks $(\mathbf{MC}_i, \varepsilon_{ij})_{i,j}$ are independent of $\mathcal{G}$.*

The selection rules permitted by this assumption can be viewed as mappings $\lambda : (v, \mathcal{L}^*) \mapsto \Delta(\mathcal{L}^*)$ where $\mathcal{L}^*$ denotes the random set of networks that are pairwise stable given attributes and payoff shocks, and $\Delta(\mathcal{L}^*)$ is the probability simplex of distributions over $\mathcal{L}^*$. One possible interpretation of such a selection rule with the properties in (ii) and (iii) is a myopic tâtonnement process from a network that is initialized at a random network L_0 that is selected independently of $(\mathbf{MC}_i, \varepsilon_{ij})_{i,j}$, and at each stage of the adjustment process the order in which edges L_{ij} are revised is determined independently of unobserved taste shocks as well.

There is no restriction to the dimension of the coordinating signal $\mathbf{v}$ which only enters the theory through the sigma field $\mathcal{G}$, and by Assumption 2.2 this assumption allows in particular for attributes x_i to be included with $\mathbf{v}$. Importantly, the set $\mathcal{L}^*$ of pairwise stable networks obviously depends on the realization of taste shocks $(\mathbf{MC}_i, \varepsilon_{ij})_{i,j}$, so part (ii) of this condition rules out the possibility of agents using those taste shocks to coordinate on a particular outcome in that set.

3. Asymptotic Representation of Network Moments

This section presents the limiting model for the network for the leading case in which the local structure of the network is uniquely determined by payoffs in a manner to be made more precise below. The main result in this section is contained in Theorem 3.2, an outline of the formal argument, including the main intermediate steps, is given in Section 3.5. This asymptotic approximation to the model can then be used for identification analysis, or to construct likelihoods and probability bounds for parametric estimation. We derive analytic characterizations for various specifications of the payoffs in (2.3) in Section 4.

3.1. **Dyadic Network Counts.** The focus of this paper is on generalized dyadic counts, which are moments of the form

$$\hat{m}_n(\theta) := n \binom{n}{2}^{-1} \sum_{i \neq j} L_{ij} h(x_i, x_j, s_i, s_j; \theta) \tag{3.1}$$

that are indexed by a finite-dimensional parameter $\theta \in \mathbb{R}^k$. In a typical application, θ parameterizes a random utility model for link preferences, and the researcher is interested in understanding what equality or inequality restrictions the link formation model implies on population analogs of these subgraph counts. In words, generalized dyadic counts are averages of functions $h(x_i, x_j, s_i, s_j; \theta)$ of node attributes over the connected edges ij of the network L. One important application of our results is estimation of structural payoff parameters based on moments of this form. The scaling factor in this definition is chosen to match the link formation rates that result from the sparse asymptotic sequence assumed below, under which the probability of a link among each of the $\binom{n}{2}$ dyads goes to zero at the rate n^{-1} as the network grows.

We consider the problem of evaluating the population expectation for a network moment of the form (3.1) under a random utility model for network payoffs when $L \equiv L^*$ is a pairwise stable network. Our results concern the joint probability of a link L_{ij}^* with values s_i^*, s_j^* for the (endogenous) network statistics given x_i, x_j. We provide an asymptotic approximation

$$f_0^*(s_1, s_2|x_1, x_2) := \lim_n n\mathbb{P}(\mathbf{L}_{ij}^* = 1, \mathbf{s}_i^* = s_1, \mathbf{s}_j^* = s_2|\mathbf{x}_i = x_1, \mathbf{x}_j = x_2) \tag{3.2}$$

which we refer to as the *link intensity.* Under the sparse asymptotic sequence considered in our analysis (see Assumption 3.1 below) it is more efficient to encode the information contained in the sparse adjacency matrix L^* as a labelled list of links, i.e. the collection of dyads (i, j) with $\mathbf{L}_{ij} = 1$. So while our approximation arguments could also be applied to the complementary event $\mathbf{L}_{ij} = 0$, $f^*(\cdot)$ assumes the role of a (Poisson-type) intensity that turns out to be sufficient to characterize the asymptotic likelihood.

For a given link intensity $f(s_1, s_2|x_1, x_2)$, the asymptotic analog to the moment in (3.1) is given by

$$m_0(\theta; f) := \int_{\mathcal{X}^2} h(x_1, x_2, s_1, s_2; \theta) f(s_1, s_2|x_1, x_2) w(x_1) w(x_2) dx_1 dx_2 \tag{3.3}$$

where $w(x)$ denotes the population distribution of attributes x. The main contribution of this paper is to show that $f_0^*(s_1, s_2|x_1, x_2)$ can be characterized analytically for a given stochastic model of link payoffs. This allows to derive asymptotic empirical restrictions on the population moment $m_0^*(\theta) := m_0(\theta; f_0^*)$ from a structural model for link payoffs, when it is not practically feasible to compute the exact expectation of the finite-network moment in (3.1) directly.

3.2. **Approximating Sequence.** We need to specify the approximating sequence of networks. Here it is important to emphasize that the main purpose of the asymptotic analysis is a reliable approximation to the (finite) n-agent version of the network rather than a factual description how network outcomes would change if nodes were added to an existing network. Hence our approach is to embed the n-agent model into an asymptotic sequence whose limit preserves the main qualitative features of the finite-agent model.

Specifically, we design the asymptotic sequence to match the following properties of a finite network: (1) the network should remain sparse in that degree distribution does not diverge as the size of the market grows. (2) The limiting conditional link formation frequencies given node-level attributes should be non-degenerate, and depend non-trivially on the systematic parts of payoffs. For the first requirement, it is necessary to increase the magnitude of marginal costs $\mathbf{MC}_i$ as the number of available alternatives grows, whereas to balance the relative scales of the systematic and idiosyncratic parts we have to choose the scale parameter

$\sigma \equiv \sigma_n$ at an appropriate rate. Specifically we are going to assume the following in the context of the reference model:

Assumption 3.1. ***(Sequence of DGP)*** *(i) The number n of agents in the network grows to infinity, and (ii) the random draws for marginal costs MC_i are governed by the sequence $J = \left[n^{1/2}\right]$, where $[x]$ denotes the value of x rounded to the closest integer. (iii) The scale parameter for the taste shifters $\sigma \equiv \sigma_n = \frac{1}{a(b_n)}$, where $b_n = G^{-1}\left(1 - \frac{1}{\sqrt{n}}\right)$, and $a(s)$ is the auxiliary function specified in Assumption 2.2 (ii).*

The rate conditions for marginal costs and the scale parameter in parts (i) and (ii) are analogous to the matching case and discussed in greater detail in Menzel (2015). Specifically, the rate for J in part (ii) is chosen to ensure that the degree distribution from a pairwise stable network will be non-degenerate and asymptotically tight as n grows. The construction of the sequence σ_n in part (iii) implies a scale normalization for the systematic parts $U_{ij}^* = U_{ij}^*(L)$, and is chosen as to balance the relative magnitude for the respective effects of observed and unobserved taste shifters on choices as n grows large. Rates for σ_n for specific distributions of taste shifters are also given in Menzel (2015).

The requirement that the sequence of networks remains sparse is primarily needed to obtain the limiting characterization of link opportunity sets with inclusive value functions, where the some of the arguments break down for a network that is more dense than what is implied by the asymptotic sequence in Assumption 3.1 (ii). However, asymptotic (conditional) independence of subnetworks across distinct network neighborhoods does not rely on sparsity but continues to hold for dense or semi-sparse network sequences.

3.3. **Limiting Model $\mathcal{F}_0^*$.** The limit of this sequence of network formation models will be given in terms of the set $\mathcal{F}_0^*$ of limits for the link intensity of network links. In general the limit of the link intensity is not uniquely defined, due to multiplicity of pairwise stable networks in the finite-n model. Instead, we can give a sharp characterization of the set $\mathcal{F}_0^*$ of distributions such that any empirical link intensity resulting from some pairwise stable network can be approximated by some element $f_0^* \in \mathcal{F}_0^*$. While decisions about whether to form (or eliminate) a link are interrelated across nodes, the asymptotic approximation developed in this paper allows to characterize the link intensity in terms of "global" aggregate states at the network level, and a "local" response to those aggregate states.

The model $\mathcal{F}_0^*$ characterizes the marginal probability of dyadic network outcomes, that is outcomes regarding the variables $\mathbf{L}_{ij}^*, \mathbf{s}_i^*, \mathbf{s}_j^*$ for any pair of nodes i, j. By Assumption 3.1, the network is sparse in the sense that for each node i a link with j is mutually acceptable only for a stochastically bounded number of nodes $j \in \{1, \dots, n\} \setminus \{i\}$.

The asymptotic representation then treats edges of the pre-network $\mathbf{L}_{ij0}$ as a random sample of node pairs with attributes and potential values $\mathbf{S}_{ij}^*(0)$ and $\mathbf{S}_{ij}^*(1)$ drawn at random from a common distribution. Specifically, for the case of discrete exogenous attributes x we define the **potential value distribution** as the conditional p.d.f.

$$M_l^*(s_1|x_1,x_2) := \mathbb{P}(\mathbf{S}_{ij}^*(l) = s|\mathbf{x}_i = x_1, \mathbf{x}_j = x_2, \mathbf{L}_{ij} = l) \tag{3.4}$$

for $l = 0, 1$ where $\mathbb{P}(\cdot)$ denotes an asymptotic equilibrium distribution for a node pair (i, j) sampled from $\mathcal{N}$ uniformly at random. The distribution pair $M^* := [M_0^*, M_l^*]$ is an equilibrium outcome which furthermore depends on which pairwise stable network was selected in the data.

Our results find that link acceptance probabilities on the edge ij given potential values are asymptotically of the form

$$\begin{aligned}\lim_n \quad n^{1/2} \quad & \mathbb{P}\left(U^*(x_1,x_2;s_1,s_2) + \sigma\boldsymbol{\varepsilon}_{ij} \geq \mathbf{MC}_i |\, \mathbf{S}_{ij}^*(l) = s_1, \mathbf{S}_{ji}^*(l) = s_2, \mathbf{x}_i = x_1, \mathbf{x}_j = x_2\right) \\ = \quad & \frac{s_{11}\exp\{U^*(x_1,x_2;s_1,s_2)\}}{1 + H^*(x_1;s_1)} \end{aligned} \tag{3.5}$$

for each $l = 0, 1$, where following our previous convention, $s_{i1} := \sum_{j=i}^n L_{ij}$ is degree centrality of node i. Here, $H^*(x;s)$ is a nonnegative function of the attributes x, s which is also determined in equilibrium across nodes in the network. This asymptotic probability can be interpreted as a generalization of Logit conditional choice probabilities, where $H^*(x;s)$ is the (properly scaled) average inclusive value of the set of link opportunities for a node with attributes x, s. We show below that the particular functional form of these probabilities results from extreme-value convergence, where Assumption 2.2 ensures that the distribution of idiosyncratic taste shocks belongs to the domain of attraction of the distribution of type I (see Resnick (1987)). It is also important to note that independence of taste shocks $\varepsilon_{ij}, \varepsilon_{ji}$ from potential values $\mathbf{S}_{ij}^*(l), \mathbf{S}_{ji}^*(l)$ is not assumed, but a result obtained under Lemmas 3.1 and 3.2.

To characterize pairwise stability of a link L_{ij}, we let

$$L_{ij}(s_1,s_2) := \mathbb{1}\{U^*(x_i,x_j;s_1,s_2) + \sigma\varepsilon_{ij} \geq MC_i, U^*(x_j,x_i;s_2,s_1) + \sigma\varepsilon_{ji} \geq MC_j\}$$

denote an indicator whether the payoff inequalities in Definition 2.1 hold at the edge ij given potential values $\mathbf{S}_{ij}^*(l) = s_1, \mathbf{S}_{ji}^*(l) = s_2$. We then define the edge response $\mathcal{L}_{ij}^*$ as the set

$$\mathcal{L}_{ij}^* := \left\{l \in \{0,1\} : L_{ij}(\mathbf{S}_{ij}^*(l), \mathbf{S}_{ji}^*(l)) = l\right\}$$

We say that the edge response is unique if $\mathcal{L}_{ij}^*$ is singleton with probability one.

By independence of draws for i, j, the probability that the outcomes $\mathbf{L}^*_{ij} = 1, \mathbf{s}^*_i = s_1, \mathbf{s}^*_j = s_2$ are supported by a pairwise stable network can be approximated by

$$\begin{aligned} \lim_n \quad n \quad & \mathbb{P}\left(1 \in \mathcal{L}^*_{ij}, \mathbf{S}^*_{ij}(1) = s_1, \mathbf{S}^*_{ji}(1) = s_2 \middle| \mathbf{x}_i = x_1, \mathbf{x}_j = x_2\right) \\ = \quad & \frac{s_{11}s_{21}\exp\left\{U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)\right\}}{(1 + H^*(x_1; s_1))(1 + H^*(x_2; s_2))} M_1(s_1|x_1, x_2)M_1(s_2|x_2, x_1) \end{aligned}$$

after conditioning on $\mathbf{x}_i, \mathbf{x}_j$, where potential values for the network statistics are evaluated at $l = 1$ and $H^*(x; s)$ is the same function as in 3.5. The normalization by the rate n - and $n^{1/2}$ in (3.5) respectively - reflects sparsity in the asymptotic sequence, where i would agree to a link to a given node j with probability of the order $n^{-1/2}$, and a link ij is pairwise stable only if both i and j agree to that link.

Limits for the respective probabilities of events $\left\{0 \in \mathcal{L}^*_{ij}, \mathbf{S}^*_{ij}(0) = s_1, \mathbf{S}^*_{ji}(0) = s_2\right\}$ and $\left\{\mathcal{L}^*_{ij} = \{0, 1\}, \mathbf{S}^*_{ij}(0) = s_{10}, \mathbf{S}^*_{ji}(0) = s_{20}, \mathbf{S}^*_{ij}(1) = s_{11}, \mathbf{S}^*_{ji}(1) = s_{21}\right\}$ are obtained in a similar fashion.

In general the **limiting model** $\mathcal{F}^*_0$ is a set of probabilities of events regarding $(\mathbf{L}^*_{ij}, \mathbf{s}^*_i, \mathbf{s}^*_j)$, together with fixed point (equilibrium) conditions for the potential value distribution $M^*_l(s_1|x_1, x_2)$ and inclusive value function $H^*(x, s)$.

This section gives a characterization of the limiting distribution only for the special case of a unique edge response: We say that the edge response for a node i is unique for a set of payoffs if there exists no other pairwise stable network L° such that $L^\circ_{kl} = L^*_{kl}$ for all $k, l \neq i, j$, but $L^\circ_{ik} \neq L^*_{ik}$ or $L^\circ_{jk} \neq L^*_{jk}$ for at least one $k \in \mathcal{N}$. That is, the edge response is unique given a network L^* if there exists a unique pairwise stable subnetwork on the edges connecting to i and j, holding all other edges fixed at their values under the network L^*. Examples with a unique edge response include models with payoffs that depend exclusively on exogenous attributes, as well as a many-to-many matching model with capacity constraints.

The basic approximation arguments continue to apply in the case of non-unique edge responses, but the limiting distribution can in general only be described in terms of bounds unless additional assumptions are placed on the equilibrium selection mechanism. Separate discussions on how these bounds can be implemented for estimation can be found in Section 4.3 in this paper, as well as in Menzel (2021).

The resulting limiting model can be characterized as follows:

- The link intensity is given by

$$\begin{aligned} f^*_0(s_1, s_2|x_1, x_2) \quad = \quad & \frac{s_{11}s_{12}\exp\{U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)\}}{(1 + H^*(x_1, s_1))(1 + H^*(x_2, s_2))} \\ & \times M^*_1(s_1|x_1, x_2)M^*_1(s_2|x_2, x_1) \end{aligned} \quad (3.6)$$

  where again $s_{i1} := \sum_{j=i}^n L_{ij}$ is degree centrality of node i. $f^*_0(\cdot)$ can therefore be characterized in closed form given the aggregate state variables H^*, M^*_1.

- The inclusive value function $H^*(x_1, s_1)$ is a nonnegative function satisfying the fixed-point condition

$$H^*(x; s) = \Psi_0[H^*, M^*](x; s) \tag{3.7}$$

where the fixed-point mapping Ψ_0 is defined as

$$\Psi_0[H, M](x; s) := \int \frac{s_{12} \exp\{U^*(x, x_2; s, s_2) + U^*(x_2, x; s_2, s)\}}{1 + H(x_2; s_2)} M_0(s_2|x_2, x_1) w(x_2) ds_2 dx_2$$

- The potential value distribution $M^*(s_1|x_1, x_2) := (M_0^*(s_1|x_1, x_2), M_1^*(s_1|x_1, x_2))$ must solve the equilibrium condition

$$M^*(s|x_1, x_2) = \Omega_0[H^*, M^*](s|x_1, x_2) \tag{3.8}$$

where Ω_0 maps H, M to conditional distributions for the network statistic $\mathbf{s}_i$ given $\mathbf{x}_i, \mathbf{x}_j$ resulting from the edge response $l = 0, 1$ in the cross section. Since $\mathcal{X}$ and $\mathcal{S}$ were assumed to be finite, H^* is of dimension $|\mathcal{S}||\mathcal{X}|$. and M^* is of dimension $2|\mathcal{S}||\mathcal{X}|$.

The set $\mathcal{F}_0^*$ then corresponds to the set of all distributions satisfying (3.6)-(3.8) for some inclusive value function H^* and potential value distribution M^*. We also denote the domains of the mappings Ψ_0 and Ω_0 with $\mathcal{H}$ and $\mathcal{M}$, respectively, which under Assumption 2.1 below can be taken to be compact subsets of Euclidean spaces. We show below that for any fixed potential value distribution $M^*(s_1|x_1, x_2)$, the fixed point of (3.7) is generally unique. However for a given value of H^*, the solution to the fixed-point condition (3.8) may admit multiple solutions, so that the resulting link intensity need not be uniquely defined even in the case of a unique edge response.

In the case of no endogenous interaction effects, the fixed-point mapping for the degree distribution is given by

$$\Omega_0[H, M](s_{11}|x_1, x_2) := \frac{H^*(x_1)^{s_{11}}}{(1 + H^*(x_1))^{s_{11}+1}}$$

where according to the convention introduced earlier in section 2.1, s_{11} denotes the network degree of node 1.

In general, Ω_0 has to be derived separately for each type of payoff-relevant network statistics, and we therefore only give high-level conditions on that mapping. Let $\hat{\Omega}_n[H, M]$ denote the empirical analog of the mapping $\Omega_0[H, M]$ in (3.8), where we take $\mathbf{x}_i$ to be distributed according to its empirical distribution in the cross-section across nodes.[7]

We can now formulate the main assumptions on the fixed-point mappings $\hat{\Omega}_n$ and Ω_0 for the potential value distributions in the finite network and the limiting economy, respectively:

[7]In the general case of set-valued edge responses, Ω_0 maps to a capacity rather than a single probability distribution, where we can represent its image as the subset of elements $\tilde{M}$ of the probability simplex $\Delta\mathcal{S}$ for distributions over $\mathcal{S}$ satisfying the constraints

$$\int_S \tilde{M}(s|x_1) ds \leq \Omega_0[H, M](S|x) \text{ for all } S \subset \mathcal{S}$$

Assumption 3.2. *(i) The mapping Ω_0 is compact and continuous in H, M for all $x \in \mathcal{X}$ and $S \subset \mathcal{S}$, and (ii) $\sup_{x,s} \left|\hat{\Omega}_n[H,M](s_1|x_1,x_2) - \Omega_0[H,M](s_1|x_1,x_2)\right| \to 0$ uniformly in H and distributions $M \in \mathcal{M}$.*

Uniform convergence of $\hat{\Omega}_n$ with respect to M in part (ii) is only stated only as a high-level condition in order to keep the result as general as possible. As the case of the conditional distribution of degree centrality s_{i1} in section 4 illustrates, for some cases of applied interest the mapping Ω_0 does in fact not depend on the sampling distribution of types, in which case uniform convergence as in part (ii) trivially holds.

Since the taste shifters $\boldsymbol{\varepsilon}_{ij}$ are independent across nodes $j = 1, \dots, n$, link formation decisions are also conditionally independent. As a result, for a given pairwise stable network the selected potential value distribution $M^*(s_1|x_1, x_2)$ coincides with the conditional distribution of s_1 given that node 1 is directly linked to at least one other node. This observation is useful for estimation since that conditional distribution can be estimated directly from the cross-sectional sample of nodes $i = 1, \dots, n$, without having to solve the fixed-point condition (3.8) and explicitly addressing the multiplicity of solutions to that problem.[8] Furthermore, under the maintained assumption of bounded systematic payoffs and unbounded taste shifters, the probability of a link between nodes with attributes x_1, x_2 is nonzero, so that this conditional distribution is always well-defined.

As can be seen from this representation, our results achieve two important simplifications that, taken together, make the finite-player problem tractable:

- While link preferences are clearly non-anonymous, the asymptotic approximation converts the problem into one that is essentially anonymous, where link opportunities are random draws from a common equilibrium distribution. In other words, local interactions are entirely governed by global aggregate state variables that are shared by the entire network.
- The Logit form of asymptotic link acceptance probabilities is also crucial for our results, with location shifts in random payoffs resulting in a multiplicative adjustment to conditional probabilities, as well as using network degree and inclusive values as sufficient statistics for marginal costs and link opportunity sets, respectively.

In the more general setting considered in Menzel (2021), a local subnetwork among a set of nodes $\mathcal{N}_0 \subset \{1, \dots, n\}$ is consistent with pairwise stability if we can find a combination of

That subset is generally referred to as the *core* of Ω_0. Since this paper restricts attention to the case of unique edge responses for greater clarity, Ω_0 can be taken to be singleton-valued. The set-valued case was discussed in an earlier version of this paper.

[8]As pointed out in Menzel (2016), an approach of this form can be justified as conditional estimation or inference given a sufficient statistic for the selected equilibrium. The strategy of replacing equilibrium quantities with sample analogs in order to side-step a nested fixed-point problem has already been fruitfully applied in dynamic discrete choice Hotz and Miller (1993) and dynamic games Bajari, Benkard, and Levin (2007).

potential outcomes for the network variables $(s_i : i, j \in \mathcal{N}_0)$ that jointly satisfy the stability conditions given the realized types and payoff shocks. The fully general case has to deal with the added complication that such a combination need not be unique - we briefly discuss construction of bounds for that problem in Section 4, and more general results are also given in the companion paper Menzel (2021). Nevertheless the baseline case of a unique edge response already represents the main conceptual and theoretical ideas, while avoiding some additional technical challenges that are not central to this approach.

3.4. **Main Asymptotic Results.** We can now state the main formal results of this paper. One potential concern is that the limiting distribution in (3.6) may not be well defined if there exists no fixed point for the population problem (3.7) and (3.8). We find that the assumptions on the fixed-point mapping Ω_0 are sufficient to guarantee existence of an equilibrium inclusive value function and potential value distribution, as stated in the following proposition.

Theorem 3.1. ***(Fixed Point Existence)*** *Suppose that Assumptions 2.1 and 3.2 (i)-(ii) hold. Then the mapping $(H, M) \to (\Psi_0, \Omega_0)[H, M]$ has a fixed point.*

See the appendix for a proof. Theorem 3.1 ensures that the limiting model is always well-defined. We can now state our main asymptotic result, which establishes convergence to the limiting model described in Section 3.

Theorem 3.2. ***(Convergence)*** *Suppose that Assumptions 2.1-2.3 and 3.1-3.2 hold, and that the function $h(\cdot)$ in (3.1) is bounded. Then, for the set $\mathcal{F}_0^*$ of distributions characterized by (3.6)-(3.8), we have*

$$\inf_{f_0^* \in \mathcal{F}_0^*} \|\hat{m}_n(\theta) - m_0(\theta; f_0^*)\| \xrightarrow{p} 0$$

where convergence is uniform with respect to selection among pairwise stable networks.

We describe the main ideas behind this convergence result in Section 3.5, a formal proof of the theorem is given in the appendix. This limiting model gives a tractable characterization of the link distribution. By considering only dyadic averages rather than the full adjacency matrix, we do not need to characterize the structure of the full network explicitly, but the model is closed via equilibrium conditions on the aggregate state variables H^* and M^*. In contrast, the expressions in Chandrasekhar and Jackson (2011) and Mele (2017) can only be approximated by simulation over all possible networks, the number of which grows very fast as n increases.

Finally, we want to give conditions under which the characterization of the limiting model is sharp in the sense that all distributions satisfying the fixed-point conditions (3.7) and (3.8) can be achieved by a sequence of finite networks. To this end, we rely on the notion of regularity for the solutions of the fixed-point problem which correspond to standard local stability conditions in optimization theory.

To simplify notation, we define the mapping

$$\Upsilon_0 : \begin{bmatrix} H \\ M \end{bmatrix} \to \begin{bmatrix} \Psi_0[H,M] \\ \Omega_0[H,M] \end{bmatrix}$$

Using the notation $z = (H, M) \in \mathcal{Z} := \mathcal{H} \times \mathcal{M}$, the fixed-point conditions (3.7) and (3.8) can be written in the more compact form $z^* \in \Upsilon_0[z^*]$. We also define the sample fixed point mapping $\hat{\Upsilon}_n$ in a completely analogous manner.

The following theorem states that for equilibrium points that are regular in a specific sense, the characterization of the limiting model is sharp in that any solution of (3.7) and (3.8) can be achieved as the limit of a sequence of solutions to the finite-agent network.

Theorem 3.3. *Suppose that Assumptions 2.1-2.3 and 3.1-3.2 hold. Furthermore, suppose that for each point z^* satisfying $z^* = \Upsilon_0[z^*]$, the Jacobian of $\Upsilon_0[z]$ has full rank. Then for any $z_0^* := (H_0^*, M_0^*)$ solving $z^* = \Upsilon_0[z^*]$, there exists a sequence $\hat{z}_n := (\hat{H}_n^*, \hat{M}_n^*)$ solving $\hat{z}_n = \hat{\Upsilon}_n[\hat{z}_n]$ such that $d(\hat{z}_n, z_0^*) \stackrel{p}{\to} 0$.*

See the appendix for a proof.

3.5. **Outline of the Limiting Argument.** We now give a summary of the formal argument behind Theorem 3.2. One of the challenges in characterizing the exact model for the finite-player network is that the set of available link opportunities to each of the n nodes are unobserved and endogenous to that node's own choices. Furthermore, the pairwise stability conditions depend on the potential values for the relevant network attributes for each node (edge, respectively) under all possible configurations of the network. Clearly, the corresponding latent state space grows in dimension with the size of the network and contains both discrete and continuous components. Our argument relies on the inclusive value function and the potential value distribution as aggregate state variables that are asymptotically sufficient to represent that state space.

There are three additional aspects that complicate the formal argument: For one, strategic externalities across links may lead to long-range dependence of link decisions and result in simultaneity problems that do not exist in models without strategic interdependencies. Here we rely on a novel argument based on symmetric dependence which does not require any type of ergodicity or weak dependence. Furthermore, network formation allows for several, rather than just one direct connection to each node, so that not only the maximum, but other extremal order statistics of marginal benefits are relevant for link formation decisions. Finally, pairwise stability allows for multiple edge responses to a given set of link opportunities, so that even in the limit the link intensity need not be unique.

Independence. Given the potential values for the network statistics s_i^*, s_j^*, s_k^*, we also let

$$D_{kij}^*(l, D_i, D_j) := \mathbb{1}\left\{U^*(x_k, x_i; \mathbf{S}_{kij}^*(l, D_i, D_j), \mathbf{S}_{iij}^*(l, D_i, D_j)) + \sigma\varepsilon_{ki} \geq MC_k\right\}$$

and define $D^*_{kji}(l, D_i, D_j)$ in analogous fashion. In order to characterize a local response, we then stack the variables

$$\mathbf{Z}^*_{kij}(l, D_i, D_j) := \left(\mathbf{S}^*_{kij}(l, D_i, D_j), D^*_{kij}(l, D_i, D_j), D^*_{kji}(l, D_i, D_j)\right)$$

These potential values are constructed in a way such that support of the potential values $\mathbf{S}^*_{kij}(l; D_i, D_j)$ is determined by node attributes X and taste shocks for nodes $k \notin \{i, j\}$. We therefore have the following result:

Lemma 3.1. ***(Independence)*** *Suppose Assumption 2.2 holds. Then for any pair i, j and conditional on $\mathbf{X}$, the taste shocks $\{\boldsymbol{\varepsilon}_{ik}, \boldsymbol{\varepsilon}_{jk}, \mathbf{MC}_i, \mathbf{MC}_j : k \notin \{i, j\}\}$ are independent of whether the potential values $\left\{\mathbf{Z}^*_{kij}(l; D_i, D_j) : k \notin \{i, j\}\right\}$ are supported by a pairwise stable network for any fixed D_i, D_j and $l \in \{0, 1\}$.*

See the appendix for a proof. Most importantly, independence will allow us to evaluate stability of proposals in D_i, D_j conditional on the potential values for network statistics of adjacent nodes that are supported by a pairwise stable network. Note that this result relies crucially on the assumption that taste shocks are independent across nodes ex ante, and is therefore distinct from classical simultaneous equations models in econometrics in which the researcher also wants to allow for correlated unobservables in structural responses.

Invariance. As a second important step, we establish that potential values satisfy a certain symmetry or "anonymity" property, at least in a stochastic sense. In order to abstract from equilibrium selection among multiple pairwise stable networks, we characterize this property in terms of the probability that particular network events are supported by at least one pairwise stable network. Specifically, suppose that $A_{ijk} \equiv A_{ijk}(L, X)$ is a property of the adjacency matrix L and node attributes X that is invariant to any permutations of nodes other than i, j, k. We then define the probability that the event is *supported* by a pairwise stable network as

$$\bar{\mathbb{P}}\left(\mathbf{A}_{ijk}\right) := \mathbb{P}\left(\exists L^* \in \mathcal{L}^* : A_{ijk}(L^*, \mathbf{X}) \text{ is true}\right),$$

where $\mathcal{L}^*$ denotes the set of all networks for which all unconstrained edges satisfy the pairwise stability conditions in Definition 2.1. Whenever appropriate, we also define conditional probabilities accordingly, where for any events $\mathbf{A}_{ijk}, \mathbf{B}_{ikj}$, $\bar{\mathbb{P}}\left(\mathbf{A}_{ijk}|\mathbf{B}_{ijk}\right) := \bar{\mathbb{P}}\left(\mathbf{A}_{ijk} \cap \mathbf{B}_{ijk}\right) / \bar{\mathbb{P}}\left(\mathbf{B}_{ijk}\right)$ whenever $\bar{\mathbb{P}}\left(\mathbf{B}_{ijk}\right) > 0$.

We now consider permutations τ of the indices $\{1, \dots, n\}$ and show that after conditioning on "local" information, potential outcomes follow a common distribution, even if realizations generally differ between "comparable" nodes:

Lemma 3.2. ***(Invariance)*** *Suppose Assumptions 2.1-2.3 hold. Then for any $i, j = 1, \ldots, n$, l, D_i, D_j, and arbitrary permutation τ of $\{1, \ldots, n\}$,*

$$\bar{\mathbb{P}}\left(\mathbf{S}^*_{kij}(l, D_i, D_j) = s_k \middle| \mathbf{x}_i = x_1, \mathbf{x}_j = x_2, \mathbf{x}_k = x_3\right)$$
$$= \bar{\mathbb{P}}\left(\mathbf{S}^*_{\tau(k)ij}(l, D_i, D_j) = s_k \middle| \exists \tau' : D^*_{\tau'(i)\tau(k)} = D_{ik}, D^*_{\tau'(j)\tau(k)} = D_{jk}, \mathbf{x}_{\tau'(i)} = x_1, \mathbf{x}_{\tau'(j)} = x_2, \mathbf{x}_{\tau(k)} = x_3\right)$$

The analogous conclusion holds for the conditional probability that two potential values $\mathbf{S}^*_{kij}(l, D_i, D_j) = s_k$ and $\mathbf{S}^*_{lij}(l, D_i, D_j) = s_l$ are simultaneously supported by a pairwise stable network.

See the appendix for a proof. The main practical implication of this lemma for our purposes is that it will allow us to derive marginal probabilities for "local" events based on a conditional distribution for potential outcomes given "local" information alone. Most importantly, while proposals in D_i are directed at particular nodes that may in turn affect the network position for specific other nodes, this Lemma shows that those indirect, "equilibrium" effects of proposals by i and j equally affect potential values with respect to any other edges in the network. That symmetry then allows to capture dependence through strategic interaction effects using the potential value distribution as an aggregate state variable which affects all nodes in the network symmetrically.

Conditional Choice Probabilities. The next step of our asymptotic argument takes the limit of the conditional probability that agent i is willing to form a link to agent j given x_i, potential values for network statistics, and her other links. We find that given our specification the number of links "accepted" by agent i is substantially smaller than the number of "proposals" $j \in W_i(L^*)$, so that the conditional probability of proposing or accepting a link depends only on the upper tail of $G(\cdot)$, the distribution for the taste shifters ε_{ij}. The assumption that $G(\cdot)$ has tails of type I can then be used to establish that conditional choice probabilities can be approximated by those implied by the Logit model with taste shifters generated by an extreme-value type-I distribution. This finding is similar to the analysis of matching markets in Menzel (2015), where the chosen match corresponds to the most preferred available matching partner. Here a new complication arises from the fact that all links (and availability to j) are determined simultaneously, so that it is necessary to consider joint probabilities over multiple link proposals.

We then define

$$\Phi_n(i; j_1, \ldots, j_r) = \bar{\mathbb{P}}\left(\mathbf{D}^*_{ij} D_{ji} = \mathbb{1}_{j \in \{j_1, \ldots, j_r\}} \middle| \mathbf{Z}^*_{kij_1}(1, D_i, \mathbf{D}^*_{j_1}) = (s_k, D_{ki}, D_{kj}), k = 1 \ldots n\right)$$

where $D_i = \left(\mathbb{1}_{j \in \{j_1, \ldots, j_r\}}\right)$. We can interpret Φ_n as the conditional probability of link proposals by i to $j_1, \ldots, j_r$ given potential values for network statistics with respect to those proposals. Notice that marginal benefits U_{ij} depends on s_i and s_j, so that $\Phi_n(i; j_1, \ldots, j_r)$ cannot be directly interpreted as a conditional choice probability, but equals the probability that the configuration $L_{ij_1} = \cdots = L_{ij_r} = 1$ and $L_{ij'} = 0$ for all other $j' \in W_i$ satisfies the pairwise stability conditions regarding player i and j's payoffs. Such a configuration is not necessarily unique, but externalities among links emanating from i and j may support several stable outcomes for a given realization of random payoffs.

We find that under our assumptions, we can approximate the conditional probability $\Phi_n(i; j_1, \dots, j_r)$ with its analog under the assumption of independent extreme-value type-I taste shifters. To this end, we first prove the following general result about limits of non-exclusive multiple choice under our setup, where we treat options as exogenously given.

Lemma 3.3. *Suppose marginal benefits are generated according to* $\mathbf{U}_{ij} := \tilde{U}_{ij} + \sigma\boldsymbol{\varepsilon}_{ij}$, $j = 1, \dots, J$ *and* $\mathbf{MC}_i$ *where* $\tilde{U}_{ij}$ *are bounded constants and* $(\boldsymbol{\varepsilon}_{ij})_{j=1}^J$ *and* $\mathbf{MC}_i$ *are drawn independently according to Assumption 2.2.Then as* $\sum_{j\neq i} D_{ji} \to \infty$*, we have for any fixed* r *that*

$$\left| J^r\Phi_n(i, j_1, \dots, j_r) - \frac{r!\prod_{q=1}^r \exp\{\tilde{U}_{ij_q}\}}{\left(1 + \frac{1}{J}\sum_{q\geq r+1} D_{j_q i}\exp\{\tilde{U}_{ij_q}\}\right)^{r+1}}\right| \to 0 \tag{3.9}$$

if $D_{j_1 i} = \dots, = D_{j_r i} = 1$*, and* $J^r\Phi_n(i, j_1, \dots, j_r) \to 0$ *otherwise.*

It follows from the previous two steps that we can approximate the distribution of the edge response using the inclusive value of agent i's link opportunity set W, which we defined as

$$I_i[W] := \frac{1}{n^{1/2}}\sum_{j\in W}\exp\{U^*(x_i, x_j; s^*_{ij}(0), s^*_{ji}(0))\}$$

This means in particular that the composition and size of the set of link opportunities affects the conditional choice probabilities only through the inclusive value, which is a scalar parameter summarizing the systematic components of payoffs for the available options, see Luce (1959), McFadden (1974), and Dagsvik (1994).

Law of Large Numbers. The third step of the argument establishes a conditional law of large numbers for the inclusive values $\mathbf{I}^*_i := I_i[W_i(\mathbf{L}^*)]$ which are sample averages over the characteristics of agents in the link opportunity set $W_i(\mathbf{L}^*)$, where the size of the set $|W_i(\mathbf{L}^*)|$ grows at a rate $\sqrt{n}$ for any PSN.

Lemma 3.4. *Suppose Assumptions 2.1-2.3 and 3.1 hold. Then, (a) there exists a function* $\hat{H}^*_n(x; s)$ *such that for any pairwise stable network, the resulting inclusive values satisfy*

$$\mathbf{I}^*_i - \hat{H}^*_n(\mathbf{x}_i, \mathbf{s}_i) = o_p(1)$$

for each i drawn from a uniform distribution over $\{1,\dots,n\}$. Furthermore, (b), if the weight functions $\omega(x,x',s,s')\geq 0$ are bounded and form a Glivenko-Cantelli class in $(x;s)$, then

$$\sup_{x\in\mathcal{X},s\in\mathcal{S}}\frac{1}{n}\sum_{j=1}^{n}\omega(x,x_j,s,s_j)(\mathbf{I}_j^*-\hat{H}_n^*(\mathbf{x}_j,\mathbf{s}_j))=o_p(1)$$

See the appendix for a proof. The result implies that up to sampling error, for all but a vanishing share of nodes, inclusive values only depend on agents' own characteristics x_i, s_i, so that we do not need to account for the node-specific link opportunity sets separately as we take limits. In the following, we refer to $\hat{H}_n^*(x;s)$ as the inclusive value function in the finite network. Note also that part (a) still allows for some nodes to have inclusive values that differ substantially from the respective value of the inclusive value function even for large n, however their share among the n nodes vanishes as the network grows.

In the two-sided matching case an analogous result could be derived relying on bounds exploiting the ordinal structure of the set of stable matchings (see Lemma B.5 in Menzel (2015)), where the inclusive value was shown to converge to the inclusive value function for each agent. For pairwise stable networks with a non-unique edge response, this is in general not the case so that our argument relies on a different strategy: To illustrate the difficulty, suppose that there exists a stable network in which both values $s_j=\underline{s}$ and $s_j=\bar{s}\neq\underline{s}$ are supported by the edge response for a nontrivial share of nodes. Then switching between a network in which $s_j=\underline{s}$ to another in which $s_j=\bar{s}$ may make j more likely to be available to i, or increase the marginal benefit to i of forming a link with j. For a given realization of taste shifters ε_{ji} it may then be possible to construct a pairwise stable network in which nodes j with high values of ε_{ji} choose $s_j=\underline{s}$, whereas nodes with high values of ε_{jk} for another node k choose $s_j=\bar{s}$. Hence, if selection of pairwise stable networks is allowed to depend on the idiosyncratic taste shifters ε_{ji}, the inclusive values I_i^*, I_k^* could deviate substantially from the average for a small number of nodes. However, we find that for any pairwise stable network the share of nodes whose inclusive values differ substantially from the respective conditional average must vanish as the size of the network grows. In particular, we find that the problematic term in the characterization of the "worst-case" selection from edge responses can be bounded by the eigenvalue of a symmetric random matrix that is known to converge to a finite limit.

Fixed-Point Mapping for Inclusive Value Functions. Next, we derive an (approximate) fixed-point condition for the inclusive value function $H(x;s)$ resulting from the law of large numbers in the previous step. For any conditional distribution $M(s_1|x_1,x_2)$, define the mapping

$$\hat{\Psi}_n[H,M](x;s):=\frac{1}{n}\sum_{j=1}^{n}\int\frac{s_{j1}\exp\{U^*(x,x_j;s,s_j)+U^*(x_j,x;s_j,s)\}}{1+H(x_j;s_j)}M_0(s_j|x_j,x)ds_j\quad(3.10)$$

If we let $\hat{M}^*_{0n}(s_1|x_1, x_2)$ denote the empirical distribution of potential values given exogenous traits in the PSN, the next Lemma states that the inclusive value function is a fixed point of the mapping $\hat{\Psi}_n[\cdot, \hat{M}^*_n]$:

Lemma 3.5. *Suppose Assumptions 2.1-2.3 and 3.1 hold. The inclusive value function $\hat{H}^*_n(x; s)$ resulting from a PSN has to satisfy the approximate fixed-point condition*

$$\hat{H}^*_n(x; s) = \hat{\Psi}_n[\hat{H}^*_n, \hat{M}^*_n](x; s) + o_p(1) \tag{3.11}$$

where the remainder converges in probability uniformly in the arguments x, s.

See the appendix for a proof. The analog of the fixed-point mapping in (3.10) for the limiting model is given by

$$\Psi_0[H, M^*](x; s) := \int \frac{s^*_{1j,+1} \exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1 + H(x_j; s^*_{1j,+1})} M^*_0(s^*_{1j,+1}|x_j, x)w(x_j)dx_j ds_j \tag{3.12}$$

for a potential value distribution M^* satisfying (3.8). Given that potential value distribution, we then let $H^*(x; s)$ be a solution of the fixed-point problem

$$H^* = \Psi_0[H^*, M^*]$$

We next give conditions under which for any given potential value distribution, the fixed point exists and is unique:

Proposition 3.1. *Suppose that Assumptions 2.1-2.3 and 3.1 hold. Then (i) for any given potential value distribution $M^*(s_1|x_1, x_2)$ for which the network degree s_{i1} satisfies $\mathbb{E}[s_{i1}|x_i] < B_s < \infty$ almost surely, the mapping $\log H \mapsto \log \Psi[H]$ is a contraction mapping with contraction constant $\lambda < \frac{B_s \exp\{2\bar{U}\}}{1+B_s \exp\{2\bar{U}\}}$. Moreover, (ii) the fixed points in (3.7) are continuous functions that have bounded partial derivatives at least up to order p.*

The formal argument for this result closely parallels the proof of Theorem 3.1 in Menzel (2015) with contraction constant equal to $\frac{B_s \exp\{2\bar{U}\}}{1+B_s \exp\{2\bar{U}\}}$, a separate proof is therefore omitted.

One case of particular interest for which the contraction property holds without additional assumptions is that of no endogenous interaction effects, as shown by the following corollary:

Corollary 3.1. *Suppose Assumptions 2.1-2.3 and 3.1 hold, and $U^*(x_1, x_2; s_1, s_2) = U^*(x_1, x_2)$. Then the solution $H^*(x; s) = H^*(x)$ to the fixed point problem (3.7) is unique.*

The proof of this corollary is given in the appendix.

Fixed Point Convergence. The cross-sectional distribution $\hat{M}^*_n(s_1|x_1, x_2)$ of potential values for s_i among nodes in the n-agent network can also be characterized with the equilibrium conditions

$$\hat{M}^*_n(s_1|x_1, x_2) = \hat{\Omega}_n[\hat{H}^*_n, \hat{M}^*_n](s_1|x_1, x_2) + o_p(1) \tag{3.13}$$

We can now combine the previous steps to show joint convergence for the potential value distribution $\hat{M}_n^*$ and the inclusive value function $\hat{H}_n^*(x; s)$ to solutions of the population fixed-point problem (3.7) and (3.8). Specifically, Lemmata 3.3 and 3.4 imply that link opportunity sets can be parameterized with the inclusive value functions, whereas the fixed-point mappings for the inclusive value function and potential value distribution converge to their respective population analogs. Finally, under our assumptions convergence of the fixed-point mappings also implies convergence of the (set of) fixed points, where we prove the following in the appendix:

Lemma 3.6. *Suppose that Assumptions 2.1-2.3 and 3.1-3.2 hold. Then for any stable network, the inclusive value function $\hat{H}_n^*(x; s)$ and potential value distribution $\hat{M}_n^*(s_1|x_1, x_2)$ satisfy the fixed-point conditions in (3.11) and (3.13). Moreover, there exist (H^*, M^*) jointly satisfying the population fixed-point conditions in (3.7) and (3.8) such that $\|\hat{H}_n^* - H^*\|_\infty = o_p(1)$ and $\|\hat{M}_n^* - M^*\|_\infty = o_p(1)$.*

Link Formation Probability Conditional on $\mathbf{S}_{ij}(l), \mathbf{S}_{ji}(l)$. Taken together, convergence of conditional link acceptance probabilities to a generalized Logit form, and convergence of node-specific inclusive values to their conditional expectation, imply that the conditional probability of a link can be expressed in closed form as in (3.6).

Lemma 3.7. *Suppose that Assumptions 2.1-2.3 and 3.1 hold. Then the conditional probability of a proposal $\mathbf{D}_{ij}^* = \mathbf{D}_{ji}^* = 1$ is approximated by*

$$\lim_n n\bar{\mathbb{P}}\left(\mathbf{D}_{ij}^* = \mathbf{D}_{ji}^* = 1 \middle| \mathbf{S}_{ij}^*(l) = s_1, \mathbf{S}_{ji}^*(l) = s_2, \mathbf{x}_i = x_1, \mathbf{x}_j = x_2\right) = \frac{(s_{11}+1)(s_{21}+1)\exp\{U_{12}^* + U_{21}^*\}}{(1 + H^*(x_1; s_1))(1 + H^*(x_2; s_2))}$$

where $U_{12}^* := U^*(x_1, x_2; s_1, s_2)$ and $U_{21}^* := U^*(x_2, x_1; s_2, s_1)$.

See the appendix for a proof. Note that the link proposals $\mathbf{D}_i^*$ are dependent conditional on attributes and potential values through the common shock $\mathbf{MC}_i$. One important implication of this lemma is that nevertheless, $\mathbf{D}_{ij}^*$ is asymptotically independent of potential values $\mathbf{S}_{ij}^*(l)$ after we condition on network degree. This is entirely due to the asymptotic Logit structure of the conditional link proposal distribution, where the conditional expectation of the factor $(s_{11} + 1)/(1 + H^*(x_1; s_1))$ given $\mathbf{x}_i = x_1, \mathbf{s}_i = s_1$ equals one.

Since in the limit the state variables $\hat{H}_n^*, \hat{M}_n^*$ fully capture the effect of "global" changes in the network on the conditional probability of link acceptance for the dyad (i, j), convergence of the fixed points in Lemma 3.6 together with the Logit representation of the link acceptance probabilities in Lemma 3.7 imply convergence of the link intensity as claimed in Theorem 3.2.

To summarize our results, interdependence of link formation decisions can be split into a "local" component at the level of a given pair of nodes which is characterized through the edge response, and a "global" component that acts on all units symmetrically. This global component is captured entirely by the inclusive value function H^* and potential value distribution M^* in our formulation, which serve as aggregate state variables. The same applies to multiplicity of stable outcomes, where "local" multiplicity is resolved by selecting from a multi-valued edge response corresponding to an individual potential link, and "global" multiplicity corresponds to selecting among multiple roots solutions for the equilibrium conditions for the inclusive value function and potential value distribution.

4. Identification, Estimation, and Welfare Analysis

We can now use the limiting approximation in Theorem 3.2 to analyze identification of the model and develop a strategy for estimating structural payoff parameters from network data. We also establish an analytic formula for agents' expected surplus from participating in the network. In the following we assume that all payoff-relevant attributes x_i and network characteristics s_i are observed for all n nodes in the network. The arguments below can be easily extended to various protocols for sampling nodes at random from the network, and certain cases in which some components of x_i are not directly observed but generated from a distribution that is known up to a parameter to be estimated. We first present an approach to parametric estimation for the case of a unique edge response. In the general case of a non-unique edge response, only inequality restrictions on the expectations of network moments may be available, which are discussed in Section 4.3. While our asymptotic representation greatly simplifies the construction of such bounds, a practical approach to set estimation in that case will be left for future research.

4.1. **Identification.** We first consider nonparametric identification of the payoff functions $U^*(x_1, x_2; s_1, s_2)$, where we assume that the researcher observes either the full network L, X, or a random sample of edges, i.e. $K \leq n(n-1)/2$ pairs i, j together with the variables $L_{ij}, x_i, x_j, s_i, s_j$. Note that in this case, the link intensity $f(s_i, s_j|x_i, x_j)$ is nonparametrically identified. These arguments can be adjusted for other sampling protocols with known sampling weights.[9] In the case of knowledge of the complete network $\mathcal{L}$ and perfectly observable attributes x_i, the network statistics S_i can also be computed from the available data.

Nonparametric Identification of the Reference Distribution. The potential value distribution $M_l^*(s_1|x_1, x_2)$ is the conditional distribution of potential values $\mathbf{S}_{ij}^*(l)$ conditional on $\mathbf{x}_i = x_1, \mathbf{x}_j = x_2$, and $\mathbf{L}_{ij} = l$. Conditional on $\mathbf{L}_{ij} = l$, the potential value $\mathbf{S}_{ij}^*(l)$ is realized, so that the potential value distribution $M_l^*(s_1|x_1, x_2)$ corresponding to the selected pairwise stable network is point-identified as long as the network data is sampled in a way that allows us to

[9]For example, the researcher may sample nodes at random and eliciting all links emanating from each node ("induced subgraph"), or only the links among the nodes included in the survey ("star subgraph"), see Chandrasekhar and Lewis (2011) for a discussion.

recover the network statistics $\mathbf{s}_i$ for a random sample of nodes $i \in \mathcal{N}$. Moreover, we can use the implied nonparametric estimator for the marginals of M^* for estimation which obviates the need to solve the fixed-point problem for $M^*(s_1|x_1, x_2)$ explicitly.

Nonparametric Identification with no Endogenous Interaction Effects. We first give a nonparametric identification result for the baseline case of no endogenous interaction effects between links, the marginal benefit of link ij is given by

$$U_{ij} \equiv U^*(x_i, x_j) + \sigma\varepsilon_{ij}$$

From our results in Section 3, it follows that we can fully characterize the limiting distribution of links in pairwise stable networks in terms of the pseudo-surplus function $V^*(x_1, x_2) := U^*(x_1, x_2) + U^*(x_2, x_1)$, so that the function $U^*(x_1, x_2)$ is not separately identified. Specifically, if we let s_{i1} denote the degree of node i, the density for the limiting distribution is given by

$$f_0^*(s_1, s_2|x_1, x_2) = \frac{s_{11}s_{12}\exp\{V^*(x_1, x_2)\}M^*(s_{11}|x_1, x_2)M^*(s_{12}|x_2, x_1)}{(1 + H^*(x_1))(1 + H^*(x_2))}$$

where the inclusive value function $H^*(x)$ satisfies the fixed-point condition

$$H^*(x) = \Psi_0[H^*, M^*](x) := \int_{\mathcal{X}\times\mathcal{S}} s\frac{\exp\{V^*(x, x_2)\}}{1 + H^*(x_2)}M^*(s|x_2, x)w(x_2)dsdx_2$$

and the degree distribution $M^*(s_1|x_1, x_2)$ is given by

$$M^*(s_1|x_1, x_2) = \frac{H(x_1)^{s_1}}{(1 + H^*(x_1))^{s_1+1}}$$

and does not depend on x_2. In particular, we have for any $r = 0, 1, \dots$ that

$$\mathbb{P}(s_{i1} \geq r|x_i = x) = \sum_{s=r}^{\infty}\frac{H^*(x)^s}{(1 + H^*(x))^{s+1}} = \left(\frac{H^*(x)}{1 + H^*(x)}\right)^r$$

resulting in the hazard rate

$$\frac{\mathbb{P}(s_{i1} = r|x_i = x)}{\mathbb{P}(s_{i1} \geq r|x_i = x)} = \frac{1}{1 + H^*(x)}$$

for any natural number r, including zero. Hence for any arbitrarily chosen $r = 0, 1, \dots,$ we can write the pseudo-surplus function in terms of log differences of link frequencies,

$$\begin{aligned} V^*(x_1, x_2) \;&=\; \log\frac{f_0^*(s_1, s_2|x_1, x_2)}{s_{11}s_{12}M^*(s_{11}|x_1, x_2)M^*(s_{12}|x_2, x_1)} \\ &\quad - \log\frac{\mathbb{P}(s_{i1} = r|x_i = x_1)}{\mathbb{P}(s_{i1} \geq r|x_i = x_1)} - \log\frac{\mathbb{P}(s_{j1} = r|x_j = x_2)}{\mathbb{P}(s_{j1} \geq r|x_j = x_2)} \end{aligned}$$

Note that all quantities on the right-hand side are nonparametrically identified from the observed network. Hence, the pseudo-surplus function $V^*(x_1, x_2)$ is nonparametrically identified for the "pure homophily" model. Note that the identification argument is constructive and does not require knowledge of the (unobserved) inclusive value function $H^*(x)$.

4.2. **Parametric Estimation.** We now turn to estimation of parametric models for link preferences when the edge response is unique. We assume that systematic utilities are specified as

$$U^*(x_i, x_j; s_i, s_j) \equiv U^*(x_i, x_j; s_i, s_j|\theta)$$

for a finite-dimensional parameter θ. We also define the resulting pseudo-surplus function

$$V^*(x_i, x_j; s_i, s_j|\theta) = U^*(x_i, x_j; s_i, s_j|\theta) + U^*(x_j, x_i; s_j, s_i|\theta)$$

Estimation and inference for θ in the network model are complicated by the presence of multiple stable outcomes. However, while the fixed-point conditions in (3.7) and (3.8), respectively, may admit multiple solutions, as discussed before the distribution $M^*(s_1|x_1, x_2)$ resulting from the equilibrium chosen in the data is identified from the observed network. Our approach is therefore conditional on the, possibly non-unique, equilibrium distribution M^*, which we replace by a consistent estimate. This strategy for dealing with multiple equilibria is analogous to Menzel (2016)'s approach for the case of discrete action games. A theoretical justification for this approach is given by Menzel (2021).

The other potential difficulty is that the limiting distribution in (3.6) depends on the (unobserved) inclusive value function. Following the approach in Menzel (2015) for the case of matching markets, we suggest to treat $H^*(x; s)$ as an auxiliary parameter in maximum likelihood estimation of the surplus function $V^*(x_1, x_2; s_1, s_2|\theta)$ satisfying the fixed-point condition (3.7).

We propose maximum likelihood estimation of θ, where the log-likelihood contribution for node i,

$$\ell_i(\theta, H) := \sum_{j=1}^{n} d_{ij} \log f^*(s_i, s_j|x_i, x_j; \theta, H)$$

is obtained from the limiting model. Hence, when the researcher observes the full network with n nodes, the log-likelihood function corresponding to the limiting distribution is

$$\mathcal{L}_n(\theta, H) := \sum_{i=1}^{n} \ell_i(\theta, H)$$

We also estimate the fixed-point mapping $\Psi_0[H, M^*]$ by its sample analog

$$\hat{\Psi}_n(\cdot) := \frac{1}{n} \sum_{i=1}^{n} \psi_i(\theta, H)$$

where the node-level contributions $\psi_i(\theta, H)$ are again derived from the limiting representation. We derive expressions for $\ell_i(\theta, H)$ and $\psi_i(\theta, H)$ for various examples below. When only a random sample of nodes or edges of the network is observed according to some known sampling protocol with uniformly bounded qualification probabilities, the formulae for $L_n(\cdot)$ and $\Psi_n(\cdot)$ can be adjusted using weights.

An estimator for θ is then obtained by maximizing the pseudo-log-likelihood, where $H^*(x; s)$ is treated as an auxiliary parameter that has to satisfy the sample analog of the fixed-point condition (3.7). That is, the estimator solves the constrained maximization problem

$$\max_{\theta, H} \mathcal{L}_n(\theta, H) \quad \text{s.t. } H = \hat{\Psi}_n(H) \tag{4.1}$$

The structure of this optimization problem, with a nuisance function that is defined by a fixed-point problem, is very similar to that of maximum likelihood estimation of dynamic discrete choice models where the value function has to be recomputed for each candidate value of the preference parameters. Popular approaches for estimating these models are nested fixed-point algorithms (Rust (1987), Ishakov, Lee, Rust, Schjerning, and Seo (2016)) and the MPEC algorithm (Su and Judd (2012)).

We conclude this section by giving expressions for the log-likelihood $\hat{L}_n$ and the fixed-point mapping $\hat{\Psi}_n$ for a few illustrative examples which form the basis for the Monte Carlo experiments in the last section of this paper. We only consider cases for which the edge response is unique, or we specify the equilibrium selection rule since the main purpose of these examples is to illustrate our convergence results, abstracting from potential issues with partial identification in the general case. In each case the likelihood function is derived from the corresponding limiting model $\mathcal{F}_0^*$, assuming that the researcher observes the relevant exogenous characteristics for all nodes, $x_1, \dots, x_n$, and the full adjacency matrix L.

4.2.1. *No Endogenous Interaction Effects.* We first consider the case of no endogenous interaction effects, with systematic marginal utility functions of the form $U^*(x_1, x_2) = U^*(x_1, x_2; \theta)$. For this case, the only relevant network variable is the network degree $s_{i1} := \sum_{j=1}^n L_{ij}$, and the inclusive value function does not depend on endogenous network characteristics, so that $H(x; s) = H(x)$ for all $s \in \mathcal{S}$.

Then the information in the sample can be summarized by the degree sequence $s_{11}, \dots, s_{1n}$ together with the non-zero link indicators, and the limiting model implies that the log-likelihood contribution of the ith node is given by

$$\begin{aligned} \ell_i(\theta, H) &= \frac{1}{2} \sum_{j=1}^n L_{ij} \left(V^*(x_i, x_j | \theta) - \log(1 + H(x_i)) - \log(1 + H(x_j)) \right) \\ &\quad + \log s_{i1} - \log(1 + H(x_i)) \end{aligned}$$

Note that the first term of the log-likelihood only receives weight one half to avoid double-counting of non-zero link indicators as we sum the log-likelihood contributions over the nodes $i = 1, \dots, n$. The constrained maximum likelihood estimator maximizes the network log-likelihood $\mathcal{L}_n(\theta, H) := \sum_{i=1}^n \ell_i(\theta, H)$ subject to the fixed-point condition $H(x) = \frac{1}{n}\sum_{i=1}^n \psi_i(\theta, H)$ with

$$\psi_i(\theta, H) = w_i s_{i1} \frac{\exp\{V(x, x_i|\theta)\}}{1 + H(x_i)}$$

where $w_j := \frac{\mathbb{1}\{s_{j1}>0\}}{\frac{1}{n}\sum_{k=1}^n \mathbb{1}\{s_{k1}>0\}}$. The importance weights w_j are used to obtain an unbiased estimator for Ψ_0 in (3.7), noting that the potential value distribution for the potential value of s_{j1} from setting $L_{ij} = 1$ is equal to the conditional distribution of s_{j1} given $s_{j1} > 0$ in the cross-section over nodes in the network.

4.2.2. *Many-to-Many Matching and Capacity Constraints.* Next, we state the likelihood for a many-to-many matching model that assumes the same preferences as in the previous case, but allows each node to form at most $\bar{s}$ direct links, i.e. capping the network degree at $\bar{s}$. Furthermore, in accordance with classical matching models, we modify the notion of pairwise stability for networks (PSN, Definition 2.1) to allow for deviations in which a node simultaneously severs one link and forms another:

Definition 4.1. ***(Pairwise Stability, PSN2)*** *The undirected network L is a* ***pairwise stable network according to PSN2*** *if for any link ij with $L_{ij} = 1$,*

$$U_{ij}(L) \geq \max\{MC_i(L), U_{ik}(L - ij)\}, \quad \text{and } U_{ji}(L) \geq \max\{MC_j(L), U_{jl}(L - ij)\}$$

and for any link ij with $L_{ij} = 0$,

$$U_{ij}(L) < \min\{MC_i(L), U_{ik}(L - ij)\}, , \quad \text{or } U_{ji}(L) < \min\{MC_j(L), U_{jl}(L - ij)\}$$

for any k such that $U_{ki}(L) \geq MC_k(L)$ and l such that $U_{lj}(L) \geq MC_l(L)$.

The log-likelihood contribution of the ith node resulting from the limiting model is then obtained as

$$\begin{aligned} \ell_i(\theta, H) &= \frac{1}{2}\sum_{j=1}^n L_{ij}\left(V^*(x_i, x_j|\theta) - \log(1 + H(x_i)) - \log(1 + H(x_j))\right) \\ &\quad + \log s_{i1} - \mathbb{1}\{s_{i1} < \bar{s}\}\log(1 + H(x_i)) \end{aligned}$$

and the fixed-point mapping for the inclusive value function is the average of contributions

$$\psi_i(\theta, H) := w_i s_{i1} \frac{\exp\{V(x, x_i|\theta)\}}{1 + H(x_i)}$$

where $w_j := \frac{\mathbb{1}\{s_{j1}>0\}}{\frac{1}{n}\sum_{k=1}^n \mathbb{1}\{s_{k1}>0\}}$ as in the previous case.

4.2.3. *Strategic Complementarities in Network Degree.* Next, we consider the case in which link preferences depend on the respective network degrees of nodes i and j, $s_i = \sum_{k=1}^n L_{ik} \equiv s_{i1}$ and $s_j = \sum_{k=1}^n L_{jk} \equiv s_{j1}$. For simplicity, we assume that s_i, s_j are strategic complements with L_{ij}, that is the systematic part $U^*(x_i, x_j; s_i, s_j|\theta)$ is nondecreasing in s_i and s_j.

With preferences of this form, the edge response is generally not unique. To illustrate how to perform the calculations under the limiting model, we first assume a particular rule for selecting among multiple pairwise stable networks and discuss bounds in Section 4.3. For the selection mechanism we assume that for any realization of payoffs, the observed network is selected as the maximal pairwise stable network under the partial order $L \geq L'$ if $L_{ij} \geq L'_{ij}$ for all i, j. It follows from standard arguments for monotone comparative statics (see Milgrom and Roberts (1990)) that the maximal stable network is well-defined and can be obtained from myopic best-response dynamics starting at the complete graph $L_{ij} = 1$ for all $i \neq j$.

Under these assumptions the probability that a given network L is generated by this selection mechanism is equal to the probability that L is pairwise stable times the conditional probability that payoffs do not support any larger network $L' > L$ given that L is pairwise stable. After some standard calculations, we find that under $\mathcal{F}_0^*$, the probability that the values $s_0 < s_1, \cdots < s_r$ for s_{i1} are jointly supported is equal to

$$p(s_0, \ldots, s_r) = \frac{H(x; s_0)^{s_0} \prod_{q=1}^r (H(x; s_q) - H(x; s_{q-1}))^{(s_q - s_{q-1})}}{(1 + H(x; s_r))^{r+1}}$$

If we define

$$\begin{aligned} \pi^*(s_0; r) &:= \sum_{s_0 < .. < s_r} \frac{p(s_0, s_1 \ldots, s_r)}{p(s_0)} \\ &= 1 - \sum_{s_0 < .. < s_r} \frac{(1 + H(x; s_0))^{s_0+1} \prod_{q=1}^r (H(x; s_q) - H(x; s_{q-1}))^{(s_q - s_{q-1})}}{(1 + H(x; s_r))^{r+1}} \end{aligned}$$

the conditional probability that s_0 is the largest network degree for node i given that s_0 is supported by a pairwise stable network is given by

$$\pi^*(s_0) = 1 + \sum_{r=1}^{\infty} (-1)^r \pi^*(s_0; r)$$

For an implementation of the MLE in the Monte Carlo experiments in the appendix, we partially vectorize computation of $\pi^*(s_0)$. Specifically, if $H(x; s)$ only changes its value at a finite number r of values for s, then $\pi^*(s_0)$ can be computed by a double loop with a total of $2r$ iterations.

The log-likelihood contribution of the ith observation can then be written as

$$\ell_i(\theta, H) = \frac{1}{2}\sum_{j=1}^{n} L_{ij}\left(V^*(x_i, x_j; s_i, s_j|\theta) - \log(1 + H(x_i; s_i)) - \log(1 + H(x_j; s_j))\right) + \log s_{i1} - \log(1 + H(x_i; s_i)) + \log \pi^*(s_i)$$

and the fixed-point condition for the inclusive value function is obtained from the sample average of

$$\psi_i(\theta, H) := w_i s_{i1}\frac{\exp\{V(x, x_i; s, s_i|\theta)\}}{1 + H(x_i; s_i)}$$

where $w_j := \frac{\mathbb{1}\{s_{j1}>0\}}{\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}\{s_{k1}>0\}}$ as in the previous case.

4.3. **Set Estimation and Bounds.** In the general case of a non-unique edge response, the limiting model provides bounds on the link intensity. In complete analogy to estimation of discrete games, probability bounds of this type can then be used to construct moment inequalities to estimate identification regions for the payoff parameters, see Tamer (2003), Ciliberto and Tamer (2009), Beresteanu, Molchanov, and Molinari (2011), and Galichon and Henry (2011).

We consider a parametric specification for payoffs,

$$U(x_i, x_j; s_i, s_j) = U(x_i, x_j; s_i, s_j|\theta_0)$$

and will now focus on those cases in which the edge response is non-unique, so that the limiting model $\mathcal{F}_0^*$ consists of a non-trivial set of distributions even conditional on $M^*(s_1|x_1, x_2)$. This set can be described in terms of lower and upper bounds for probabilities of events $\mathbf{A}_{ij}$ in the variables $\mathbf{L}_{ij}, \mathbf{s}_i, \mathbf{s}_j$ for a dyad ij in the n-player network. That is, denoting for some set A the probability of the event $\mathbf{A}_{ij} := \{(\mathbf{L}_{ij}, \mathbf{s}_i, \mathbf{s}_j) \in A\}$ in the selected pairwise stable network with $P_n(\mathbf{A}_{ij}|\mathbf{x}_i, \mathbf{x}_j)$, we can derive functions $Q_L(\cdot), Q_U(\cdot)$ from the limiting model such that

$$Q_L(A|\mathbf{x}_i, \mathbf{x}_j; \theta_0, H^*) \le \lim_n P_n(\mathbf{A}_{ij}|\mathbf{x}_i, \mathbf{x}_j) \le Q_U(A|\mathbf{x}_i, \mathbf{x}_k; \theta_0, H^*) \tag{4.2}$$

As in the single-valued case, the bounds $Q_L(\cdot)$ and $Q_U(\cdot)$ depend on the aggregate states H^*, M^* where the inclusive-value function satisfies the fixed-point condition $H^* = \Psi[H^*, M^*]$, and the potential value distribution $M^* \in \Omega_0[H^*, M^*]$ with the mappings $\Psi_0[\cdot], \Omega_0[\cdot]$ are defined as before.

For singleton events $\mathbf{A}_{ij}$, the bounds (4.2) only depend on the marginal distributions of potential outcomes, $M_0^*(s_1|x_1, x_2), M_1^*(s_1|x_1, x_2)$ which are point-identified from full-network data as discussed before. Hence, the corresponding bounds $Q_L(\cdot), Q_U(\cdot)$ can be computed directly using a nonparametric estimator of $M_l^*(s_1|x_1, x_2)$.[10]

[10] As shown by Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011), sharp bounds for $\mathcal{F}_0^*$ typically also have to account for composite events $\mathbf{A}_{ij}$, i.e. events of the form $\mathbf{A}_{ij} = \{(L_{ij}, s_i, s_j) \in Z\}$ for certain non-singleton sets $Z \subset \{0, 1\} \times \mathcal{S}^2$. The asymptotic bounds (4.2) for composite events could

Given these probability bounds, the identified set for the payoff parameter θ is

$$\Theta_I := \Big\{\theta \in \Theta : Q_L(A^{(r)}|\mathbf{x}_i, \mathbf{x}_j; \theta_0, H^*) \le \lim_n P_n(\mathbf{A}^{(r)}|\mathbf{x}_i, \mathbf{x}_j) \le Q_U(A^{(r)}|\mathbf{x}_i, \mathbf{x}_k; \theta_0, H^*) \text{ a.s.}$$
$$\text{for each } r = 1, \dots, R \text{ and for some } H^* \text{ solving } H^* = \Psi_0[H^*, M^*](\theta)\Big\}$$

where $A^{(r)} := \{(d_{ij}, s_i, s_j) \in Z^{(r)}\}$ and $Z^{(1)}, \dots, Z^{(R)}$ denote the subsets of $\{0,1\} \times \mathcal{S}^2 \times \mathcal{T}$, and the event $\mathbf{A}^{(r)} := \big\{(d_{ij}, \mathbf{s}_i, \mathbf{s}_j) \in A^{(r)}\big\}$.

Estimation and inference regarding the identified set Θ_I can be implemented using moment functions of the form

$$\mathbf{m}(A^{(r)}|\theta, H) := \begin{pmatrix} \mathbb{1}\{\mathbf{A}_{ij}^{(r)}\} - Q_L(A^{(r)}|x_i, x_j; \theta, H) \\ Q_U(A^{(r)}|x_i, x_j; \theta, H) - \mathbb{1}\{A_{ij}^{(r)}\} \end{pmatrix}$$

From our convergence results and the definition of the probability bounds, we then have the asymptotic conditional moment restriction

$$\lim_n \mathbb{E}[\mathbf{m}(A^{(r)}; \theta_0)|x_i, x_j] \ge 0 \text{ a.s.}$$

These conditional restrictions can then be transformed into systems of unconditional moment equalities and inequalities for set estimation and inference, see e.g. Beresteanu, Molchanov, and Molinari (2011) for a description for the case of finite discrete games. Since the bounds in (4.2) are only satisfied as $n \to \infty$, these procedures can only be consistent (asymptotically valid, respectively) under the many-player limit. We conclude this section by giving an example for how to derive the probability bounds $Q_L(\cdot), Q_U(\cdot)$ from the limiting model $\mathcal{F}_0^*$.

4.3.1. *Strategic Complementarities in Network Degree.* Consider the payoffs from Example 4.2.3 with payoffs $U_{ij}(L)$ depending on $s_i := \sum_{j=1}^n L_{ij}$ and $s_j := \sum_{i=1}^n L_{ji}$. We now show how to construct probability bounds for dyad-level outcomes in (L_{ij}, s_i, s_j) which do not assume a particular selection mechanism.

Similar to the discussion for the case of a specific selection mechanism, let

$$p(s_1, \dots, s_r|x) := \frac{H(x; s_1)^{s_1} \prod_{q=1}^r (H(x; s_q) - H(x; s_{q-1}))^{(s_q - s_{q-1})}}{(1 + H(x; s_r))^{r+1}}$$

for any $s_1 < \dots < s_r$, and define

$$\tau^*(\bar{s}; r|x) := \sum_{s_1 < .. < \bar{s} < .. < s_r} \frac{p(s_1, \dots, \bar{s}, \dots, s_r|x)}{p(\bar{s}|x)}$$

also depend the joint distribution of potential values $\mathbf{S}_{ij}(0), \mathbf{S}_{ij}^*(1)$. That joint distribution is not directly observed but would have to be obtained from the fixed point mapping Ω instead, resulting in a finite but large system of equality and inequality restrictions. We are not aware of a practical method for solving that problem as part of an estimation algorithm, but note that bounds based on singleton events alone, while not sharp, sidestep that difficulty.

where the summation is over any ordered tuple of r values for s_{i1}, one component of which equals $\bar{s}$. Then the conditional probability that $\bar{s}$ is the unique pairwise stable value of s_i given that it is supported by a pairwise stable subnetwork is

$$\pi^*(\bar{s}|x) = 1 + \sum_{r=1}^{\infty} (-1)^r \tau^*(\bar{s}; r|x)$$

Since the sharp upper bound for the probability of the outcome L_{ij}, s_i, s_j corresponds to the probability that these values are supported by some pairwise stable subnetwork, we obtain

$$\begin{aligned} Q_U(s_1, s_2|x_1, x_2; \theta, H) &:= \lim_n n\mathbb{P}(\mathbf{L}_{ij} = 1, \mathbf{s}_i = s_1, \mathbf{s}_j = s_2 \text{ supported } |\mathbf{x}_i = x_1, \mathbf{x}_j = x_2) \\ &= \frac{s_1 s_2 \exp\{V(x_1, x_2; s_1, s_2)\} H(x_1; s_1)^{s_1} H(x_2; s_2)^{s_2}}{(1 + H(x_1; s_1))^{s_1+1}(1 + H(x_2; s_2))^{s_2}} \end{aligned}$$

Sharp lower bounds for specific values of these network outcomes correspond to the event that no other values of $\mathbf{L}_{ij}, \mathbf{s}_i, \mathbf{s}_j$ are supported by payoffs, and can be obtained by multiplying the upper bound with the conditional probability that the given pairwise stable outcome is unique. Specifically, we let

$$Q_U(s_1, s_2|x_1, x_2; \theta, H) := Q_U(s_1, s_2|x_1, x_2; \theta, H)\pi^*(s_1|x_1)\pi^*(s_2|x_2)$$

These bounds for singleton events are not sharp, but following Beresteanu, Molchanov, and Molinari (2011) and Galichon and Henry (2011), we can obtain additional constraints by considering composite events consisting of several distinct values of these network variables.

4.4. **Welfare and Surplus.** Our limiting framework also yields a straightforward analytic approximation to expected surplus from being connected to the network. Surplus calculations of this type are necessary e.g. to characterize ex ante incentives to participate in the network and exert search effort (as e.g. in the setting described by Currarini, Jackson, and Pin (2009)), or to evaluate welfare consequences of policies affecting the composition or structure of the network.

Focusing on the case of no edge-specific endogenous interaction effects, let $\mathbf{U}_{ij}(s) := U^*(\mathbf{x}_i, \mathbf{x}_j; s, \mathbf{s}_j(L^*)) + \sigma\boldsymbol{\varepsilon}_{ij}$ and let $\mathbf{U}_{i;r}(s)$ denote the rth (largest) order statistic of the sample $\{\mathbf{U}_{ij}(s) : j \in \mathbf{W}_i(\mathbf{L}^*)\}$ corresponding to the link opportunity set $\mathbf{W}_i(\mathbf{L}^*)$ defined in (2.4). Then if the sequence $s_1, \dots, s_{s_{i1}} = s_i$ of values for s_i results from successively adding links corresponding to the $1, \dots, r$th order statistics, agent is surplus can be obtained by integrating the marginal utilities,

$$\Pi_i(L^*) = \sum_{r=1}^{s_{i1}} (\mathbf{U}_{i;t}(s_r) - \mathbf{MC}_i) \tag{4.3}$$

Note that if marginal link utilities are indeed derived from a benefit function $B_i(L)$ as in Section 2.1, the expression for $\Pi_i(L^*)$ does not depend on the particular choice of such a sequence $s_1, \dots, s_r$.

For the Logit model it is known that the expected value of the first order statistic of such a sample is equal to a function of the inclusive value (see e.g. Train (2009)). We first show an analogous result for any other finite order statistic as $W_i(L)$ grows in size, and then derive limiting expressions for the expected net surplus in (4.3). In order to characterize the expectation of $\Pi_i(L^*)$, we also let $\mathbf{A}_i(r; s)$ denote the event that payoffs support $\mathbf{s}_i = s$ and network degree $\mathbf{s}_{i1} = r$. We can then derive the following limiting expressions for expected link surplus:

Proposition 4.1. *Suppose that the assumptions of Theorem 3.2 hold. Then for any $r' \geq r$,*

$$\begin{aligned}\lim_n \mathbb{E}[\mathbf{U}_{i;r}|\mathbf{A}_i(r'; s)] - \frac{1}{2}\log n &= \log(1 + H^*(x; s)) + \gamma - \sum_{q=1}^{r-1}\frac{1}{q} \\ \lim_n \mathbb{E}[\mathbf{MC}_i|\mathbf{A}_i(r'; s)] - \frac{1}{2}\log n &= \log(1 + H^*(x; s)) + \gamma - \sum_{q=1}^{r'}\frac{1}{q}\end{aligned}$$

where $H^*(x; s)$ is the inclusive value function and $\gamma \approx 0.5772$ denotes the Euler-Mascheroni constant.

See the appendix for the derivation of these expressions. Given this result, we can compute the expected surplus from being connected in the network. Conditional on $\mathbf{A}_i(s_{i1}, s)$, we have

$$\begin{aligned}\lim_n \mathbb{E}[\mathbf{\Pi}_i(\mathbf{L}^*)|\mathbf{A}_i(s_{i1}, s)] &= \lim_n \sum_{r=1}^{s_{i1}} (\mathbb{E}[\mathbf{U}_{i;r}|\mathbf{A}_i(s_{i1}; s)] - \mathbb{E}[\mathbf{MC}_i|\mathbf{A}_i(s_{i1}; s)]) \\ &= \sum_{r=1}^{s_{i1}} (\log(1 + H^*(x; s_r)) - \log(1 + H^*(x; s_{s_{i1}}))) + \sum_{r=1}^{s_{i1}} \left(\sum_{q=1}^{s_{i1}} \frac{1}{q} - \sum_{q=1}^{r-1}\frac{1}{q}\right) \\ &= \sum_{r=1}^{s_{i1}} (\log(1 + H^*(x; s_r)) - \log(1 + H^*(x; s_{s_{i1}}))) + s_{i1} \qquad (4.4)\end{aligned}$$

For the case where link preferences in exogenous attributes alone, $H^*(x; s) \equiv H^*(x)$, so that by the law of iterated expectations

$$\lim_n \mathbb{E}[\mathbf{\Pi}_i(\mathbf{L}^*)] = \mathbb{E}[\mathbf{s}_{i1}] = H^*(x)$$

where in the case of a non-unique edge response, the expectation is taken given the equilibrium selection rule. On the other hand, with preferences depending on network degree, the sum in (4.4) is taken along the sequence $s_1 = 1, \dots, s_{s_{i1}} = \mathbf{s}_{i1}$, so that

$$\lim_n \mathbb{E}[\mathbf{\Pi}_i(\mathbf{L}^*)] = \mathbb{E}\left[\sum_{r=1}^{\mathbf{s}_{i1}} (\log(1 + H^*(x; r)) - \log(1 + H^*(x; s_{i1}))) + \mathbf{s}_{i1}\right]$$

where the expectation with respect to $\mathbf{s}_{i1}$ also depends on the selection rule.

5. Empirical Application: Segregation in Friendship Networks with Endogenous Search Effort

In this section, we use the many-player approximations derived in this paper to estimate a model of friendship formation where agents can choose how much effort to spend on socializing and meeting potential friends. Currarini, Jackson, and Pin (2009) proposed a model for a matching process where individuals have homophilous preferences with respect to race, i.e. prefer to be friends with individuals of the same race ("baseline homophily"), but also choose how long to remain in a market in which they are matched at random to other available partners. They show that in combination, these two mechanisms may give rise to patterns of racial segregation in friendship networks which also depend on the relative population shares of different racial groups ("relative homophily"). Currarini, Jackson, and Pin (2010) found evidence supporting this theoretical model in social networks in the Add Health study sample of students in US high schools.

Since their empirical conclusions were obtained by calibrating a very stylized model without controlling for any other relevant node attributes, this leaves open the possibility that race may proxy for other omitted variables affecting the probability of friendship formation. For example, in the presence of residential segregation with respect to race, a higher likelihood of friendships among students of the same race may in fact be the result of a preference for forming friendships based on geographic proximity (e.g. based on after school activities) rather than race. Similarly, residential segregation may result in students of the same race being more likely to have attended the same primary or middle school prior to the high school that was surveyed under Add Health. In that event, a higher likelihood of friendships among students of the same race may simply reflect longer exposure rather than homophilous preferences.

If the patterns observed in the data are due to mechanical effects of residential segregation rather than racial preference, the policy implications are clearly different: while homophilous preferences can not be directly influenced through outside intervention, a search friction can be overcome more easily by creating more opportunities for students of different groups to socialize.

We show that our approach makes it straightforward to incorporate this problem into a model of pairwise stable network formation with endogenous search, allowing to control for rich covariate information. We show that our techniques allow us to make efficient use of more granular information on residential location and demographics in the restricted-use sample for the Add Health study. We also address measurement issues in constructing the social graph from incomplete "nomination" data, where the likelihood-based approach used in this paper is very versatile and can be easily adapted to meet those challenges. Otherwise we deliberately keep the problem as simple as possible so that we can mostly work with closed-form expressions and avoid additional numerical complications that would be incidental to the main contribution of this paper.

More broadly, this empirical application also illustrates the usefulness of our many-agent asymptotic framework for real-world network data. The Add Health sample contains high-quality network data for only 16 communities, which would not warrant asymptotics over the number of observed networks. Furthermore, while we find an effect for geographic distance on the likelihood of link formation, the observed friendship networks at the schools in the sample are relatively tightly knit. Arguably, students who commute from distant locations in the school district still have ample opportunities to socialize with most of their classmates in the classroom, sports teams, and other activities around the school. "Large domain" asymptotics which require that interactions between spatially distant components of the network are negligible to first order may therefore be less plausible for this particular context than "infill" asymptotics based on a fixed domain for relevant node attributes.

5.1. **Specification.** We introduce a search friction into the model by assuming a separate meeting stage before the friendship graph forms. That meeting stage consists of a process during which node pairs i, j meet independently at random. Each agent controls the intensity at which these meetings occur by choosing a level $e_i \geq 0$ of search effort. The meeting process is assumed to be unbiased with respect to types, and we assume that the probability that agents i and j meet is given by $\pi_{ij} := e_i e_j$. Search effort is chosen by the agent at the outset of the meeting process, at which stage the agent is assumed to know her own type x_i and the distribution $w(x)$ of types among the other agents in the network.

In the subsequent friendship formation stage, agents i and j can only form a link if they already met at the meeting stage. Our model assumes that conditional on a meeting, students have (ex post) link preferences depending on exogenous attributes x_i, x_j including gender, race, and residential location. We use a linear index specification,

$$U_{ij}(L) = x_i'\beta_{ego} + x_j'\beta_{alter} + |x_i - x_j|'\beta_{edge} + \varepsilon_{ij} \tag{5.1}$$

where we let $|x_i - x_j|$ denote a vector of absolute differences that are taken component by component. The parameters β_{ego} captures type-specific link productivity, i.e. general propensity for seeking to form links, β_{alter} type-specific link attractiveness, i.e. general desirability to be connected to, and β_{edge} captures homophily with respect to the attributes included in x_i and x_j. The timing structure of this model is similar to Lee and Schwarz (2012)'s model of interviews in two-sided matching markets but differs from Currarini, Jackson, and Pin (2009) in that friendships are not accepted/formed instantaneously as meetings occur, but that we require the observed network to be pairwise stable given payoffs and meeting outcomes. Also, their model does not allow agents to accept or reject specific friendship proposals based on homophilous preferences as meetings occur, but instead allow for biases in the meeting process to allow the model to reproduce homophilous network patterns for subgroups of any size.

This problem is strategic in that each agent chooses their effort level e_i^* optimally, and we can use the asymptotic approximation to expected network surplus to solve for the (unique) Nash equilibrium at the meeting stage.

5.2. **Model Solution.** The search friction results in a slight modification of (3.6), so that the p.d.f. of the link intensity becomes

$$\begin{aligned} f_0^*(s_1, s_2|x_1, x_2; e_1, e_2) &= \frac{s_{11}s_{12}\exp\{U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)\}e_1e_2}{(1 + H^*(x_1; e_1))(1 + H^*(x_2; e_2))} \\ &\quad \times M^*(e_1, s_1|x_1, x_2)M^*(e_2, s_2|x_2, x_1) \end{aligned} \tag{5.2}$$

where the inclusive value function and potential value distribution satisfy fixed-point conditions analogous to the case without a search friction, and the fixed-point mapping for $H^*(x; e)$ becomes

$$\begin{aligned} \Psi_0[H, M](x; e) &:= \int \frac{s_{12}\exp\{U^*(x, x_2) + U^*(x_2, x)\}e_1e_2}{1 + H(x_2; e_2)} \\ &\quad \times M^*(e_2, s_2|x_2, x_1)w(x_2)ds_2de_2dx_2 \end{aligned} \tag{5.3}$$

From this expression we can see immediately that $H^*(x; e) \equiv eH^*(x; 1)$, so that we can parametrize the fixed point conditions in terms of $H^*(x; 1)$ alone, where

$$\begin{aligned} \Psi_0[H, M](x; 1) &:= \int \frac{s_{12}\exp\{U^*(x, x_2) + U^*(x_2, x)\}e_2}{1 + H(x_2; 1)e_2} \\ &\quad \times M^*(e_2, s_2|x_2, x_1)w(x_2)ds_2de_2dx_2 \end{aligned}$$

Furthermore, if e_2 is stochastically bounded, this fixed-point mapping remains a contraction in logs, so that in the baseline case the equilibrium is unique.

To determine the optimal effort level e_i^*, we assume that cost is a power function of search effort,

$$c(e) = \left(1 + \frac{1}{\delta}\right)^{-1} e^{1+\frac{1}{\delta}}$$

for some $\delta > 0$. The return to search effort derives from the agent's expected surplus from the pairwise stable network. Specifically, from Proposition 4.1 and the expression in (4.4), it follows that the ex ante expected payoff for network links resulting from search effort level e is

$$W_i(e) = \lim_n \mathbb{E}\left[\Pi_i(L^*)|e_i = e\right] = H^*(x_i; e, s) \equiv eH^*(x_i; 1)$$

We can then solve the agent's maximization problem to obtain

$$e_i^* := \arg\max_e [W_i(e) - c(e)] = H^*(x_i; 1)^\delta$$

For the case of endogenous interaction effects, no closed-form solution for optimal effort exists, but instead we would need to impute e_i^* based on the general expression for $V_i(e)$. Hence we can estimate this model based on the following specification for pseudo-surplus:

$$\tilde{V}^*(x_i, x_j) := (x_i + x_j)'\tilde{\beta} + |x_i - x_j|'\tilde{\beta}_2 + \delta(\log H^*(x_i; 1) + \log H^*(x_j; 1))$$

where $\tilde{\beta}_1 := \beta_{ego} + \beta_{alter}$ and $\tilde{\beta}_2 := 2\beta_{edge}$.

Since the search friction is assumed to be static in that the meeting probability does not depend on the structure of the network L, an alternative version of the model could allow for meetings to continue taking place while the network is being formed without changing the resulting distribution of links. However we need to maintain that each agent commits to the search effort level e_i^* chosen at the outset of the process.

5.3. **Data.** Our empirical analysis is based on the National Longitudinal Study on Adolescent to Adult Health (Add Health, Harris (2009)) restricted-use data on friendship networks. The questionnaire and supplemental data also includes information about respondents' demographic characteristics, residential location, and educational history.

The data set covers 80 schools in communities across the US; 16 schools were selected for a saturation sample for which the entire relevant student population was included in the survey. This saturation sample includes 2 schools that are large, and 14 that are small. For each of the 64 remaining communities ("main sample"), only 200 students were selected at random to participate in the survey. For the main sample, most friendship nominations are students outside the the sample, so that no demographic or other information other than gender is available for most nominees in the main sample. While our results can also be used to derive the likelihood for those partially observed networks, we expect that data from those communities does not provide much additional information on network preferences. We therefore restrict our analysis to the 16 saturation samples that were specifically designed to obtain information about students' social networks.

We construct the social network within schools from friendship nominations in the data, where each student in the survey was asked to nominate up to five female and five male friends. Those nominations are not necessarily reciprocal, and a substantial fraction of students nominate the permissible maximum of five friends for at least one gender. This record of the friendship network is therefore likely incomplete. For our analysis we assume that a link between students i and j is present whenever i nominated j, j nominated i, or both (send and receive network). We furthermore allow for the possibility that additional friendship links may not be recorded if both i and j make all five nominations of the respective gender but fail to nominate each other. We operationalize this using the auxiliary assumption that the five recorded nominations correspond to the five most preferred friends of that gender.

In addition to students' demographic attributes, we also compute the geodesic distance between any pair of students from the GPS coordinates of a student's home relative to central reference point of community. Geographic distances within a community can be quite large, where the catchment areas of some schools in the sample consist of several clearly separated clusters which may be more than 20 miles apart.

5.4. **Results.** We estimated the model by maximizing the pseudo log-likelihood based on (5.2) using a nested fixed point algorithm. The school-specific inclusive value functions are treated as a nuisance parameter, where we solve for a fixed point in (5.3) by iterating the contraction mapping to convergence. We also adjust the log-likelihood contributions to account for the fact that recorded friendships are truncated at five friend nominations of each gender. That adjustment is obtained from the general approximation in (3.9) under the auxiliary assumption that student i's nominees of each gender are among her five most preferred friends of that gender.

We can apply the asymptotic theory in Menzel (2021) to the score equations defining the resulting maximum likelihood estimator and conclude that it is consistent and asymptotically normal (Theorems 5.1 and 5.2 in his paper). The estimator was computed using a standard constrained optimizer in Matlab (fmincon). Reported standard errors are based on the asymptotic distribution derived in Theorem 5.2 in Menzel (2021). According to that theory, point estimators are guaranteed to be consistent but should also expect to see an asymptotic bias of the order $O(n^{-1/2})$.[11]

We estimated four partially nested specifications - a baseline specification which only accounts for link productivity and homophily in demographic student attributes (gender and race), followed by a version of the endogenous search model controlling only for these discrete characteristics. The third specification again accounts only for exogenous student attributes but also controls for the geographic distance between student i and j's home, and the full specification includes endogenous search effort, controlling both for demographics and geographic distance. For the definition of the inclusive value function, we discretize each community into 100 representative "locations" based on the respective percentiles for geographic latitude and longitude within each community. Random utilities in (5.1) are evaluated directly in terms of the underlying, continuously distributed GPS coordinates.

[11]The proof of Lemma 5.1 in Menzel (2021) identifies two first-order bias contributions, the first of which can be eliminated by subtracting the contribution of realized links from the inclusive values. The second contribution derives from the curvature of a mapping of individual inclusive values, suggesting an analytical correction based on the conditional variance of I_i^*. However at present an asymptotically valid correction for that bias is not yet available.

	(1)	(2)	(3)	(4)
const	-0.5990	-2.6043	-0.8180	-0.7138
	(0.0516)	(0.0443)	(0.0114)	(0.0093)
white	-2.3222	-3.0019	-2.4068	-1.3075
	(0.0432)	(0.0354)	(0.0188)	(0.0194)
black	-2.0900	-2.3830	-2.0433	-1.3067
	(0.0132)	(0.0159)	(0.0087)	(0.0088)
hispanic	-1.4002	-1.1987	-1.3972	-1.0670
	(0.0109)	(0.0268)	(0.0100)	(0.0102)
female	0.0202	-0.6834	0.5779	0.9724
	(0.0238)	(0.0358)	(0.0159)	(0.0148)
both white	2.8905	4.4617	2.9774	1.7747
	(0.0258)	(0.0344)	(0.0191)	(0.0209)
both black	2.1667	2.2980	2.1788	1.5642
	(0.0149)	(0.0141)	(0.0084)	(0.0085)
both hispanic	1.4464	1.4374	1.4454	1.2159
	(0.0145)	(0.0254)	(0.0095)	(0.0099)
both female	0.2001	0.5774	0.0876	0.2254
	(0.0146)	(0.0271)	(0.0147)	(0.0145)
both male	0.3147	0.0269	0.5738	0.8201
	(0.0269)	(0.0243)	(0.0152)	(0.0152)
log distance			-0.1454	-0.1342
			(0.0498)	(0.0497)
log e_i^*		-0.5989		0.2506
		(0.0222)		(0.0428)

Table 1. Estimation results for the Network Formation Model (5.1) based on the 16 saturation samples from Add Health.

Estimation results are reported in Table 5.4. Across all specifications, results support strong homophilous patterns according to racial categories, with large positive coefficients for the indicators "both white," "both black," and "both hispanic." We can also see a less pronounced effect of gender that is nevertheless fairly consistent across specifications. As expected, the effect of log distance in specifications (3) and (4) is also negative.

The sign for $\log e_i^*$ predicted by our theory should be positive, so the result in specification (2), which fails to control for distance, is not consistent with that interpretation. However, after controlling for distance, we do obtain a positive coefficient corresponding to an effect that is of a comparable order of magnitude as the "baseline" patterns of homophily documented before. It is also interesting to note that after accounting for distance and the relative homophily effect in specification (4), the baseline link productivity parameters for the racial groups become more similar to each other in magnitude, suggesting that absolute and relative homophily may be the main drivers of degree heterogeneity between these groups.[12]

More broadly, this empirical example illustrates how we can incorporate additional covariate information into a parametric random utility model at little additional cost, where estimates of main effects may be sensitive to whether we succeed in controlling for relevant confounders. The estimation problem in (5.2) and (5.3) is isomorphic to parametric dynamic discrete choice problems, so the cost of including additional attribute information or allowing for certain types of unobserved heterogeneity will be comparable to those more familiar settings.

6. Conclusion

This paper develops an asymptotic representation of the link intensity resulting from a network formation game. In this limiting approximation, interdependence of link formation decisions can be split into a "local" component at the level of a given pair of nodes which is characterized through the edge response, and a "global" component, which is captured entirely by the inclusive value function H^* and potential value distribution M^* which serve as aggregate state variables. The same applies to multiplicity of stable outcomes, where "local" multiplicity is resolved by selecting from a multi-valued edge response corresponding to an individual potential link, and "global" multiplicity corresponds to selecting among multiple roots solutions for the equilibrium conditions for the inclusive value function and potential value distribution.

These results give us a simplified representation of the limiting moments for network moments, and we show how to use that representation for identification and estimation. A subsequent paper by Menzel (2021) provides a (many-agent) asymptotic distribution theory that allows for inference based on this approach. For greater clarity we restrict our attention to the case of models with a unique edge response - at least in principle the approximations provided by our approach could be used to construct asymptotic bounds in a similar fashion, however we leave this for future research.

Appendix A. Simulation Study

This section reports results from Monte Carlo experiments to illustrate the performance of the limiting approximations for the case of a unique best response. We focus on simulation designs with discrete types, where the only exogenous covariate $x_i \in \{0, 1\}$ (e.g. "red" nodes vs. "blue" nodes) is a Bernoulli random variable with success probability 0.4. The taste shifters ε_{ij} are i.i.d. draws from an extreme-value type-I distribution.

[12]Note that the model does not include a homophily effect for the omitted racial category "other," so that the link productivity parameter for the omitted category cannot be taken to be normalized to zero.

	Design 1				Design 2			
	$\mathbb{E}[s_i\|x_i = x]$		$\mathbb{E}[I_i\|x_i = x]$		$\mathbb{E}[s_i\|x_i = x]$		$\mathbb{E}[I_i\|x_i = x]$	
n	$x = 0$	$x = 1$	$x = 0$	$x = 1$	$x = 0$	$x = 1$	$x = 0$	$x = 1$
100	1.971	1.948	2.307	2.300	7.540	6.560	10.557	8.955
	(2.128)	(2.106)	(0.531)	(0.535)	(6.018)	(5.353)	(1.538)	(1.364)
500	2.435	2.439	2.613	2.610	11.082	9.334	13.087	10.965
	(2.740)	(2.726)	(0.413)	(0.412)	(9.945)	(8.560)	(1.334)	(1.175)
1000	2.403	2.395	2.523	2.524	11.530	9.812	13.005	10.920
	(2.745)	(2.767)	(0.344)	(0.344)	(10.725)	(9.332)	(1.153)	(1.022)
5000	2.579	2.577	2.637	2.636	13.294	11.128	14.072	11.721
	(2.989)	(2.993)	(0.241)	(0.241)	(13.088)	(11.069)	(0.854)	(0.753)
10000	2.632	2.637	2.675	2.675	13.836	11.580	14.406	12.008
	(3.055)	(3.061)	(0.206)	(0.206)	(13.815)	(11.661)	(0.738)	(0.651)
DGP	2.660	2.664	2.718	2.718	13.883	11.617	15.012	12.463

TABLE 2. Average degree (left) and average inclusive value (right).

A.1. **Convergence of the link intensity.** We first simulate pairwise stable networks without endogenous interaction effects ("pure homophily" case). Link preferences are given by

$$U_{ij} = \beta_0 + \beta_1 x_i + \beta_2 |x_i - x_j| + \varepsilon_{ij}$$

A nonzero coefficient for β_1 allows for the propensity to form links to vary between the two types, whereas β_2 can be interpreted as a complementarity between nodes of the same type. We use two different designs in our simulation experiments which set the preference parameters equal to $(\beta_0, \beta_1, \beta_2) = (0.5, 0, 0)$ and $(1.5, 0, -0.5)\}$, respectively. All simulation results were obtained using 200 Monte Carlo draws.

To illustrate the formal results on convergence of the link intensity, we compare summary statistics of the simulated distribution and their theoretical counterparts from the limiting distribution in Table 2: The first set of columns reports the conditional mean and standard deviation (in parenthesis) of the degree of a node $s_i := \sum_{j \neq i} L_{ij}$ given the covariate $x_i = 0, 1$, and the second set of columns the conditional mean and standard deviation of the inclusive value $I_i := n^{-1/2} \sum_{j \neq i} \mathbb{1}\{U_{ji} \geq MC_j\} \exp\{U^*(x_i, x_j)\}$. The DGP values in Table 2 correspond to the inclusive value function (left) and the expected degree conditional on x_i under the limiting distribution (right).

The first simulation design results in a very sparse network in which nodes have an average degree of around 2.6, whereas for the second design, the degree distribution is centered around 12-14 links per node, which may be more typical for real-world social networks. In the first design, types do not matter for agents' preferences since $\beta_1 = \beta_2 = 0$, so that, at least up to sampling and numerical errors, inclusive values and degree distributions do not differ across types $x_i = 0, 1$. For the second design, nodes with $x_i = 0$ have larger inclusive values and degree distributions than nodes with $x_i = 1$ since the complementarity β_2 is positive and the share of nodes with $x_i = 1$ was set to 0.4. This leaves nodes of the type $x_i = 0$ with a larger number of link opportunities within their own type category than nodes with $x_i = 1$.

The simulation results replicate by and large the theoretical predictions for large networks. In particular, the conditional means of I_i and s_i converge to their asymptotic counterparts, and the cross-sectional variance

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$
100	0.442	0.027	0.006	1.116	-0.018	-0.371
	(0.203)	(0.249)	(0.120)	(0.460)	(0.804)	(0.061)
500	0.564	0.002	0.004	1.413	-0.022	-0.432
	(0.077)	(0.099)	(0.046)	(0.229)	(0.444)	(0.022)
1000	0.542	0.004	0.003	1.451	-0.024	-0.450
	(0.053)	(0.071)	(0.030)	(0.177)	(0.364)	(0.016)
5000	0.535	0.001	0.000	1.512	0.003	-0.476
	(0.027)	(0.032)	(0.013)	(0.024)	(0.031)	(0.007)
10000	0.531	-0.002	-0.000	1.521	0.004	-0.483
	(0.016)	(0.022)	(0.009)	(0.016)	(0.022)	(0.004)
DGP	0.500	0.000	0.000	1.500	0.000	-0.500

Table 3. Model without capacity constraints - mean and standard deviation (in parentheses) of MLE

of I_i decreases, although at a fairly slow rate.[13] Note also that the conditional distribution of s_i given x_i remains non-degenerate in the limit.

A.2. **Parameter Estimation with no Endogenous Interaction Effects.** We next turn to estimation of the preference parameter $\beta := (\beta_0, \beta_1, \beta_2)'$. We estimate β via pseudo-maximum likelihood, using the asymptotic log-likelihood given in Section 4.2.1. Note also that in the absence of strategic interaction effects $\mathbb{E}[s_i|x_i = x] = H^*(x)$ in the limiting model, so that we can use any consistent nonparametric estimator for $\mathbb{E}[s_i|x_i = x]$ to obtain starting values for $H^*(x)$. We use the same design values for the parameter vector β, and results are for 200 Monte Carlo replications.

One source for small-sample bias in the likelihood results from the use of the inclusive value function $H^*(x)$ in the limiting representation for the distribution of the edge response when the node forms more than one link. The derivation for Lemma 3.2 suggests a (partial) bias correction in which we replace $H^*(x_i)$ with $\tilde{I}_i := H^*(x_i) - n^{-1/2} \sum_{j=1}^n L_{ij} \exp\{U^*(x_i, x_j)\}$. Since the degree distribution remains stochastically bounded as n increases, the correction term becomes negligible in a very large network. However our simulation results suggest that such a correction substantially reduces bias for networks of moderate size, especially in the second design for which the average degree is larger than 10.

The simulation results suggest that the estimators indeed converge to the population values of the parameter β, where both bias and standard deviation of the estimator decrease as n grows. However, in contrast to standard nonlinear estimators for i.i.d. samples from a fixed DGP, the bias of the MLE in our simulation results appears not to vanish at a rate faster than its standard deviation - in fact the simulation results are consistent with a root-n rate for both bias and standard error, similar to the findings for the two-sided matching model in Menzel (2015). This behavior is primarily a result of the slower convergence rate of the inclusive value functions.

A.3. **Parameter Estimation with Capacity Constraints.** For another set of simulation results we modify the previous design by adding a capacity constraint, where the degree of each node is not permitted to exceed $\bar{s} = 5$. We also impose the modified stability notion PSN2 introduced in Definition 4.1 rather than

[13]Based on the argument for Lemma 3.4, we conjecture that the rate of convergence is $n^{-1/4}$.

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$
100	0.375	0.033	0.011	1.455	0.088	-0.434
	(0.279)	(0.287)	(0.132)	(0.337)	(0.437)	(0.095)
500	0.526	-0.009	0.004	1.521	0.153	-0.471
	(0.116)	(0.115)	(0.053)	(0.185)	(0.276)	(0.037)
1000	0.491	0.013	0.002	1.517	0.079	-0.477
	(0.086)	(0.091)	(0.036)	(0.140)	(0.221)	(0.029)
5000	0.503	0.002	-0.000	1.517	0.026	-0.491
	(0.038)	(0.038)	(0.015)	(0.066)	(0.098)	(0.013)
10000	0.509	-0.003	-0.001	1.514	0.018	-0.493
	(0.024)	(0.028)	(0.011)	(0.045)	(0.062)	(0.009)
DGP	0.500	0.000	0.000	1.500	0.000	-0.500

TABLE 4. Model with capacity constraints - mean and standard deviation (in parentheses) of MLE

pairwise stability. This setup can be interpreted as a model of many-to-many matching where each node can be matched with at most 5 partners.

The constrained MLE maximizes the asymptotic log likelihood given in Section 4.2.2. Since in this design the degree of any node is capped at $\bar{s} = 5$, we omit the bias correction of inclusive values used in the first set of results, which produces less precise (higher-variance) estimates for networks of moderate sizes. The starting values for H^* were obtained by solving the fixed-point equations with the preference parameters β held fixed at their respective starting values. The simulation results for the MLE for the preference parameter β are reported in Table 4 and are by and large comparable to those for the baseline model.

A.4. **Endogenous Interactions based on Network Degree.** For the last simulation design, we allow for complementarities in network degree, where nodes with greater degree centrality are regarded as more "attractive" link prospects. Specifically we consider link preferences of the form

$$U_{ij} = \beta_0 + \beta_1 x_i + \beta_2 |x_i - x_j| + \beta_3 \min\{10, 2 * \lceil s_{j1}/2 \rceil\} + \varepsilon_{ij}$$

where $s_{j1} := \sum_{k=1}^{n} L_{jk}$ denotes the network degree of node j, and $\lceil x \rceil$ is the value of $x \in \mathbb{R}$ rounded up to the next integer. This specification groups agents into 6 discrete categories in terms of network degree, partitioning $\mathcal{S}$ into $\{\{0\}, \{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10,\dots\}\}$. This design follows the setup in Section 4.2.3, where the pairwise stable network is obtained from myopic best-response dynamics starting at the full network graph, $L_{ij} = 1$ for all $i \neq j$, in order to select the largest stable network.

We assume throughout that $\beta_3 \geq 0$ and choose the other design parameters $\beta_0, \beta_1, \beta_2$ in a way that generates a degree distribution with a reasonable amount of variation across these categories. Specifically, we use two different designs in our simulation experiments which set the preference parameters equal to $(\beta_0, \beta_1, \beta_2, \beta_3) = (1, -0.5, 0, 0.1)$ and $(0.5, 0, -0.5, 0.1)\}$, respectively.[14] All simulation results were obtained using 100 Monte Carlo draws.

[14]A back of the envelope calculation and simulation evidence suggest that both specifications exhibit cascading adjustments to small local changes to the network and do not meet the "subcriticality" condition of Assumption 6 in Leung (2019).

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_3^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_3^{ML}$
200	0.820	-0.009	-0.387	0.045	0.956	-0.399	0.009	0.079
	(0.357)	(0.173)	(0.040)	(0.039)	(0.405)	(0.124)	(0.024)	(0.041)
500	0.711	-0.008	-0.424	0.071	0.965	-0.426	0.003	0.093
	(0.256)	(0.101)	(0.023)	(0.028)	(0.264)	(0.089)	(0.014)	(0.026)
1000	0.640	0.008	-0.447	0.080	0.973	-0.462	0.001	0.098
	(0.203)	(0.077)	(0.017)	(0.021)	(0.112)	(0.069)	(0.009)	(0.010)
5000	0.528	-0.002	-0.482	0.097	1.042	-0.536	-0.001	0.100
	(0.059)	(0.031)	(0.007)	(0.006)	(0.096)	(0.074)	(0.004)	(0.007)
DGP	0.500	0.000	-0.500	0.100	1.000	-0.500	0.000	0.100

Table 5. Model with degree externalities - mean and standard deviation (in parentheses) of MLE

Simulation results are reported in Table 5. Bias and dispersion of the MLE appear to be of a comparable order of magnitude as the previous cases, where the bias on the constant β_0 is particularly large for smaller networks. Separate maximization over $(\beta_0, \beta_1, \beta_2)$ and β_3, respectively, constraining the remaining parameters to DGP values yield much more accurate partial estimates (not reported here), suggesting that for small n the likelihood may be fairly flat in the direction of some linear combination of the two parameters.

Appendix B. Proofs

We first give the proofs for the results from Sections 2 and 3. The proof of Theorem 3.2 relies on the auxiliary results from Section 3.5 which are proven separately in Appendix B.5 below.

B.1. **Proof of Lemma 2.1.** To verify that the statement in Lemma 2.1 is indeed equivalent to the usual definition of pairwise stability, notice that if L^* is not pairwise stable, there exists two nodes i, j with $L^*_{ij} = 0$ such that $U_{ij}(L^*) > MC_{ij}(L^*)$ and $U_{ji}(L^*) > MC_{ji}(L^*)$. In particular, j is available to i under L^*, i.e. $j \in W_i(L^*)$, violating (2.5). Conversely, if (2.5) does not hold for node i, then there exists $j \in W_i(L^*)$ such that $U_{ij}(L^*) \geq MC_{ij}(L^*)$. On the other hand, $j \in W_i(L^*)$ implies that $U_{ji}(L^*) \geq MC_{ji}(L^*)$, where all inequalities are strict in the absence of ties. □

B.2. **Proof of Theorem 3.1.** We give a proof for a more general version of the result which allows for set-valued edge responses, in particular the fixed-point mapping Ω_0 may be set-valued. Specifically Ω_0 maps H^*, M^* to the subset of the probability simplex $\Delta\mathcal{S}$ for distributions over $\mathcal{S}$ satisfying the constraints

$$\int_S \tilde{M}(s|x_1)ds \leq \Omega_0[H, M](S|x) \text{ for all } S \subset \mathcal{S}$$

Formally, the images of Ω_0 are Choquet capacities and the set of distributions $\tilde{M}$ satisfying these inequality constraints is called the *core* of $\Omega_0[H, M](\cdot|x_1, x_2)$ (see Molchanov (2005) for definitions).

Note first that the conditions of Proposition 3.1 ensure that $\Psi_0[H, M]$ is a continuous, single-valued compact mapping. Next, notice that for any two distributions $M_1(s_1|x_1, x_2), M_2(s_1|x_1, x_2)$ satisfying $\int_S M_j(s_1|x_1, x_2)ds \leq \Omega_0(\mathbf{S}|x_1, x_2)$ for all core-determining sets $S \subset \mathcal{S} \times \mathcal{T}^{d_\cap}$, the convex combination $\lambda M_1 + (1-\lambda)M_2$ satisfies the same inequality constraints. Hence, the core of Ω_0 is a convex subset of the probability simplex. Furthermore, if M_3 is in the complement of the core, there exists at least one set $S \in \mathcal{S}^\circ$ such that

$\int_S M_3(s_1|x_1,x_2)ds > \Omega_0(\mathbf{S}|x_1,x_2)+\varepsilon$, where $\varepsilon>0$. Then for any distribution M' with $\|M'-M_3\|_\infty \le \varepsilon/2$, we have $\int_S M'(s|x)ds > \Omega_0(\mathbf{S}|x_1,x_2)+\varepsilon/2$. Hence the complement of the core is open, implying that the core is also a closed subset of the relevant probability simplex with respect to the L_∞-norm. Hence, given the conditions on Ω_0 in Assumption 3.2 (i)-(ii), existence of a fixed point is a direct consequence of the Kakutani-Fan fixed point theorem for Banach spaces (Theorem 3.2.3 in Aubin and Frankowska (1990)) □

B.3. **Proof of Theorem 3.2.** We start by deriving the conditional probability of a link $\mathbf{L}_{ij}^*$ for an arbitrarily chosen dyad ij given $\mathbf{x}_i,\mathbf{x}_j,\mathbf{s}_i,\mathbf{s}_j$. By Lemma 3.2, for an any permutation π the potential values for the network statistics $\mathbf{S}_{ij}(l) := \mathbf{S}_{iij}(l;D_i^*,D_j^*), \mathbf{S}_{ji}(l) := \mathbf{S}_{jij}(l;D_i^*,D_j^*)$ follow the same distribution as $\mathbf{S}_{\pi(i)ij}(l;D_i^*,D_j^*), \mathbf{S}_{\pi(j)ij}(l;D_i^*,D_j^*)$ conditional on $\mathbf{x}_i,\mathbf{x}_j$. In particular, the lemma implies that potential values $\mathbf{S}_{ij}(l),\mathbf{S}_{ji}(l)$ are finitely exchangeable. Therefore using Theorem 3.1 in Kallenberg (2005) probabilities of events in $\mathbf{S}_{ij}(l),\mathbf{S}_{ji}(l)$ can be approximated to order n^{-1} with conditionally independent draws from a common marginal distribution.

From the definition of the potential value distribution, it follows that node l's attributes, including the potential values for $\mathbf{s}_l$, are distributed according to $\hat{M}_l^*(s_l|x_{ijl})w(x_l)$, where $\hat{M}_l^*$ satisfies (3.13). By Lemma 3.6, $d(\hat{M}_l, M_0^*) = o_P(1)$ for some M_0^* satisfying the condition (3.8), and $d(\hat{H}_n^*, H^*) = o_P(1)$ for an inclusive value function H^* satisfying condition (3.7). Finally, by Lemma 3.7, the conditional link formation probability given potential values is given by (3.5). This establishes the stochastic representation of $\mathcal{F}_0^*$.

In order to establish convergence in probability for $\hat{m}_n(\theta)$, we first show that the conditional variance of the appropriately centered moment converges to zero. To this end, let

$$m_n(\theta, f_0^*) := \binom{n}{2}^{-1} \sum_{i<j} h(\mathbf{x}_i,\mathbf{x}_j;\mathbf{s}_i,\mathbf{s}_j) f_0^*(\mathbf{s}_i,\mathbf{s}_j|\mathbf{x}_i,\mathbf{x}_j)$$

and

$$\xi_{ij} := (n\mathbf{L}_{ij}^* - f_0^*(\mathbf{s}_i,\mathbf{s}_j|\mathbf{x}_i,\mathbf{x}_j))h(\mathbf{x}_i,\mathbf{x}_j;\mathbf{s}_i,\mathbf{s}_j)$$

Using this notation, $\hat{m}_n(\theta) - m_n(\theta,f_0^*) = \binom{n}{2}^{-1}\sum_{i<j}\xi_{ij}$. We now show that the conditional variance of $\hat{m}_n(\theta) - m_n(\theta,f_0^*)$ converges to zero.

Using the formula for the variance of a sum,

$$\begin{aligned}
\mathrm{Var}\left(\hat{m}_n(\theta) - m_n(\theta,f_0^*)\right) &= \binom{n}{2}^{-2} \mathrm{Var}\left(\sum_{i<j}\xi_{ij}\right) \\
&= \binom{n}{2}^{-2} \left\{ \sum_{i<j} \mathrm{Var}(\xi_{ij}) + \sum_{i<j<k<l} \mathrm{Cov}(\xi_{ij},\xi_{kl}) \right. \\
&\quad \left. + \sum_{i<j<k} \left(\mathrm{Cov}(\xi_{ij},\xi_{ik}) + \mathrm{Cov}(\xi_{ij},\xi_{jk}) + \mathrm{Cov}(\xi_{ij},\xi_{kj}) + \mathrm{Cov}(\xi_{ij},\xi_{ki})\right) \right\} \\
&= \binom{n}{2}^{-1} \mathrm{Var}(\xi_{12}) + \binom{n}{2}^{-2}\binom{n}{4}\mathrm{Cov}(\xi_{12},\xi_{34}) \\
&\quad + 4\binom{n}{2}^{-2}\binom{n}{3}\mathrm{Cov}(\xi_{12},\xi_{13})
\end{aligned} \tag{B.1}$$

We next determine the first two moments of ξ_{ij},ξ_{kl} for any index pairs (i,j), (k,l). By the stochastic representation of $\mathcal{F}_0^*$, first note that $n\mathbb{P}(\mathbf{L}_{ij}^*=1,\mathbf{s}_i,\mathbf{s}_j|\mathbf{x}_i,\mathbf{x}_j) - f_0^*(\mathbf{s}_i,\mathbf{s}_j|\mathbf{x}_i,\mathbf{x}_j) \stackrel{p}{\to} 0$ for some $f_0^* \in \mathcal{F}_0^*$ as established before. So in particular $\mathbb{E}[\xi_{ij}] \to 0$ by the law of iterated expectations.

Furthermore, by Assumption 2.1 the systematic part of payoffs is bounded by $\bar{U}$, and by Assumption 3.1 (ii), $J = \left[n^{1/2}\right]$. It then follows from Assumption 2.2 that

$$f_0^*(\mathbf{s}_i, \mathbf{s}_j | \mathbf{x}_i, \mathbf{x}_j) \leq \bar{f} := \frac{\exp\{2\bar{U}\}}{(1+\exp\{\bar{U}\})^2} < \infty$$

We can therefore bound

$$\binom{n}{2}^{-1} \mathrm{Var}(\xi_{12}) \quad \leq \quad \binom{n}{2}^{-1} n^2 \bar{f}/n(1-\bar{f}/n)\bar{h}^2 \leq \frac{2\bar{f}\bar{h}^2}{n-1}$$

where $\bar{h} := \sup_{x_1, x_2, s_1, s_2} h(x_1, x_2; s_1, s_2) < \infty$.

By Corollary B.1, we furthermore have that for any edges $(ij), (kl)$,

$$n^2 \left(\mathbb{E}[\mathbf{L}_{ij}^* \mathbf{L}_{kl}^* | \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_l; \mathbf{s}_i, \mathbf{s}_j, \mathbf{s}_k, \mathbf{s}_l] - \mathbb{E}[\mathbf{L}_{ij}^* | \mathbf{x}_i, \mathbf{x}_j; \mathbf{s}_i, \mathbf{s}_j] \mathbb{E}[\mathbf{L}_{kl}^* | \mathbf{x}_k, \mathbf{x}_l; \mathbf{s}_k, \mathbf{s}_l]\right) \rightarrow 0$$

almost surely whenever $\{i,j\} \neq \{k,l\}$. In particular this last statement also holds when the two edges have one node in common. Since $h(\cdot)$ is bounded, $\mathrm{Cov}(\xi_{12}, \xi_{13}) \rightarrow 0$, and $\mathrm{Cov}(\xi_{12}, \xi_{34}) \rightarrow 0$. This rate is not sharp, but Menzel (2021) finds that $\mathrm{Cov}(\xi_{12}, \xi_{34}) = O(n^{-1})$. Since the present result does not make claims regarding the convergence rate, we do not replicate that argument here.

We therefore have

$$\begin{aligned}
\binom{n}{2}^{-2}\binom{n}{3} |\mathrm{Cov}(\xi_{12}, \xi_{13})| &= \frac{2(n-2)}{3n(n-1)} |\mathrm{Cov}(\xi_{12}, \xi_{13})| = o(n^{-1}) \\
\binom{n}{2}^{-2}\binom{n}{4} |\mathrm{Cov}(\xi_{12}, \xi_{34})| &= \frac{(n-2)(n-3)}{6n(n-1)} |\mathrm{Cov}(\xi_{12}, \xi_{34})| = o(1)
\end{aligned}$$

Substituting these rates into (B.1), $\mathrm{Var}\left(\hat{m}_n(\theta) - m_n(\theta, f_0^*)\right) = o(1)$, so that from Chebyshev's inequality, $\hat{m}_n(\theta) - m_n(\theta, f_0^*) \stackrel{p}{\rightarrow} 0$. Since $x_1, \ldots, x_n$ are assumed to be i.i.d., we also have $m_n(\theta, f_0^*) - m_0(\theta, f_0^*) \stackrel{p}{\rightarrow} 0$, given the bounds on $h(\cdot)$ and f_0^* stated before ∎

We next give a proof for a more general version of Theorem 3.3 which allows for set-valued edge responses, in particular the fixed-point mapping Ω_0 may be set-valued. In that scenario, the fixed point mapping $\Omega_0[H, M]$ will generally be set valued, so we first introduce the notion of a Choquet capacity and its core to describe its formal properties:

A mapping $\bar{Q} : 2^{\mathcal{S}} \rightarrow [0,1]$ is called a *Choquet capacity* (upper probability) on the set $\mathcal{S}$ if (a) $\bar{Q}(\emptyset) = 0$, $\bar{Q}(\mathcal{S}) = 1$, (b) $\bar{Q}$ is monotone with respect to set inclusion, i.e. $\bar{Q}(S') \leq \bar{Q}(S)$ whenever $S' \subset S \subset \mathcal{S}$, and (c) for any increasing sequence of subsets $(S_n)_{n \geq 0}$ of $\mathcal{S}$, $\lim_n \bar{Q}(S_n) = \bar{Q}\left(\bigcup_{n \geq 0} S_n\right)$, whereas for any decreasing sequence of subsets $(S_n)_{n \geq 0}$, $\lim_n \bar{Q}(S_n) = \bar{Q}\left(\bigcap_{n \geq 0} S_n\right)$. The *core* of the capacity $\bar{Q}$ is then defined as the set of all probability distributions $Q(s)$ over $\mathcal{S}$ such that

$$\int_S Q(s) ds \leq \bar{Q}(S) \qquad \text{for all subsets } S \subset \mathcal{S}$$

B.4. **Proof of Theorem 3.3.** The joint mapping $\Upsilon_0 : (\mathcal{H} \times \mathcal{M}) \rightrightarrows (\mathcal{H} \times \mathcal{M})$ then generalizes to

$$\Upsilon_0 : \begin{bmatrix} H \\ M \end{bmatrix} \rightarrow \begin{bmatrix} \Psi_0[H, M] \\ \text{core } \Omega_0[H, M] \end{bmatrix}$$

Following Aubin and Frankowska (1990), the *contingent derivative* of Υ_0 at $(z_0', y_0)' \in \text{gph}\,\Phi$ is defined as the set-valued mapping $D\Upsilon_0(z_0, y_0) : \mathcal{Z} \rightrightarrows \mathcal{Z}$ such that for any $u \in \mathcal{Z}$

$$v \in D\Upsilon_0(z,y)(u) \Leftrightarrow \liminf_{h\downarrow 0, u' \to u} d\left(v, \frac{\Upsilon_0(z_0 + hu') - y}{h}\right) = 0$$

where $d(a, B)$ is taken to be the distance of a point a to a set B (see their Definition 5.1.1 and Proposition 5.1.4 in Aubin and Frankowska (1990)). For the special case of a unique edge response, note that if the correspondence Υ_0 is singleton-valued and differentiable, the contingent derivative is also single-valued and equal to the derivative of the function $\Upsilon_0(z)$. The contingent derivative of Υ_0 is *surjective* at z_0 if the range of $D\Upsilon_0(z_0, y_0)$ is equal to $\mathcal{Z}$.

Furthermore, recall that the *tangent cone* to a set $K \subset \mathcal{Z}$ (say) is defined as the set $T_K(z) := \limsup_{h\downarrow 0} \frac{1}{h}(K - z)$ where $K - z := \{(y - z) : y \in K\}$. In particular, the tangent cone at a point z in the interior of K relative to $\mathcal{Z}$ is all of $\mathcal{Z}$. The proof of the Theorem then relies on a fixed point theorem for inward mappings, where the mapping Υ_0 is said to be *inward* on a convex set $K \subset \mathcal{Z}$ if $\Upsilon_0[z] \cap (z + T_K(z)) \neq \emptyset$ for any $z \in K$ and $T_K(z)$ denotes the tangent cone to K in $\mathcal{Z}$. Since the mapping $\Upsilon_0[H, M]$ is well-defined for any non-negative function H and distribution M, we can furthermore $\mathcal{Z}$ take to be convex without loss of generality.

Since the contingent derivative of the mapping $\Upsilon_0[\mathbf{z}] - \mathbf{z}$ is surjective by assumption, we can use Lemma C.1 in Menzel (2016) to conclude that Υ_0 is an inward mapping when restricted to a neighborhood of any of its fixed points. Furthermore, Υ_0 and $\hat{\Upsilon}_n$ are also convex-valued mappings since the sets $\hat{\Psi}_n$ and Ψ_0 and core Ω_0 are convex by standard properties of the core. Finally, $\hat{\Upsilon}_n$ converges uniformly to Υ_0 by Lemma B.2, so that w.p.a.1 $\hat{\Upsilon}_n$ is also locally inward. In complete analogy to the proof for Theorem 3.1 part (b) in Menzel (2016), local existence of a fixed point then follows by Theorem 3.2.5 in Aubin and Frankowska (1990), noting that this fixed point result applies to general Banach spaces □

B.5. **Proofs for Results in Section 3.5.** We next prove the Lemmas from section 3.5, which are then used to establish the conclusion of Theorem 3.2.

Proof of Lemma 3.1. By construction, support of the potential values $\mathbf{Z}^*_{kij}(l; D_i, D_j)$ for a fixed proposal network D_i, D_j and $l \in \{0, 1\}$ is fully determined by node attributes $\mathbf{X}$ and taste shocks for nodes $k \notin \{i, j\}$. Since there are no common components of these variables with $\{\boldsymbol{\varepsilon}_{ik}, \boldsymbol{\varepsilon}_{jk}, \mathbf{MC}_i, \mathbf{MC}_j : k \notin \{i, j\}\}$, independence follows from the fact that taste shocks were assumed to be i.i.d. conditional on $\mathbf{X}$ by Assumption 2.2 □

Proof of Lemma 3.2. We first prove the assertion for a pairwise permutation, where $\tau(k) = h$, and then argue that the pairwise argument extends to arbitrary permutations of indices. For this pairwise permutation, we now compare the probability that potential values $\mathbf{S}^*_{kij}(l, D_i, D_j) = s_k$ are supported by a pairwise stable network to that of the permuted analog $\mathbf{S}^*_{\tau(k)ij}(l, D_i, D_j) = s_k$ being supported.

We establish the conclusion by induction, where we start from a pairwise stable network supporting proposals D_i^*, and then restrict elements of D_i, one by one. For the start of induction, we can immediately see that the conclusion of the Lemma holds for $D_i = D_i^*$ since node attributes and taste shocks are identically distributed, and payoffs are therefore jointly exchangeable. Therefore we only need to establish the inductive step where for D_i differing in at most $r - 1$ components from D_i^*, we alter an additional proposal $D_{ij_r} := 1 - D^*_{ij_r}$.

Specifically, suppose that the assertion of the lemma holds for any D_i such that there exists a set of proposals D_i^* that is supported by a pairwise stable network and $\|D_i - D_i^*\| \leq r - 1$. For such a pair D_i, D_i^*, we now change $D_{ij_r} := 1 - D^*_{ij_r}$, while leaving all other entries unchanged. Without loss of generality, we assume that the proposal is changed from zero to $D_{ij_r} = 1$. Given the inductive hypothesis, it now suffices to show that the effect of this change is the same for the probability that the potential values $\mathbf{S}^*_{kij}(l, D_i, D_j) = s_k$, or $\mathbf{S}^*_{\tau(k)ij}(l, D_i, D_j) = s_k$ respectively, are supported by a pairwise stable network.

To this end, we need to distinguish whether or not there may be indirect "interference" effects from a changing D_{ij_r} on the potential values for s_k that are supported by a pairwise stable network holding fixed l and the other entries of D_i, D_j. Adapting the terminology from Leung (2019), we say that a link ij is *not robust* if there exist values of s_1, s_2 and s'_1, s'_2 such that $U^*(x_i, x_j; s_1, s_2) + \sigma\varepsilon_{ij} \geq MC_i$ and $U^*(x_j, x_i; s_2, s_1) + \sigma\varepsilon_{ji} \geq MC_j$, but also either $U^*(x_i, x_j; s'_1, s'_2) + \sigma\varepsilon_{ij} < MC_i$ or $U^*(x_j, x_i; s_2, s_1) + \sigma\varepsilon_{ji} < MC_j$. That is, a link is not robust given realized payoffs if there exist one configuration of values of s_i, s_j such that $L_{ij} = 1$ is pairwise stable, and another such that it is not.

To make this argument precise, we now characterize events regarding whether a switch of D_{ij_r} blocks a chain of non-robust link formation decisions given the network L^* that was pairwise stable given the restrictions on l, D_i, D_j. For any $l \notin \{i, j_r\}$, let $\mathbf{A}_{kl}$ denote an indicator for the event that there exists a collection of proposals $D^* := (D_{ij})_{i,j\leq n}$ satisfying the pairwise stability conditions given the link proposals D_i, D_j such that for the resulting network $L^* := \left(D^*_{ij} D^*_{ji}\right)_{i,j}$,

$$\mathbb{1}\{\mathbf{U}_{kl}(L^* + (ij_r)) \geq \mathbf{MC}_k, \mathbf{U}_{lk}(L^* + (ij_r)) - \mathbf{MC}_l\} \neq \mathbb{1}\{\mathbf{U}_{kl}(L^* - (ij_r)) \geq \mathbf{MC}_k, \mathbf{U}_{lk}(L^* - (ij_r)) - \mathbf{MC}_l\}$$

In words, $\mathbf{A}_{kl}$ is an indicator for whether a change to the link L_{ij_r} changes whether the link kl is pairwise stable.

For $h \in \{1, \ldots, n\}$ we then let $\mathbf{B}_h(q; F)$ be an indicator for the event that there exists a collection of proposals $D^* := (D_{ij})_{i,j\leq n}$ satisfying the pairwise stability conditions given the link proposals D_i, D_j such that $D^*_{j_r i} = 1$, $\sum_l \mathbf{A}_{hl} = q$ and the conditional empirical distribution of $\mathbf{x}_l, \mathbf{s}^*_l$ given $\mathbf{A}_{hl} = 1$ is equal to F. By Assumptions 2.1 (iii) and 3.1 (ii), q is stochastically bounded. We then define $\mathcal{B}_{h,ij_r}$ as the sigma-field generated by $\{B_h(q; F) : q = 0, 1, \ldots, n-1, \quad F \text{ is a c.d.f.}\}$.

Now conditional on the two nodes $\tau(k) = h$ and k satisfying $D^*_{\tau'(i)h} = D_{ik}, D^*_{\tau'(j)h} = D_{jk}$ for some permutation τ', possibly different from τ, we have by the inductive hypothesis that for each q and F, $\mathbf{B}_h(q; F)$ and $\mathbf{B}_k(q; F)$ have the same probability. We then distinguish all possible cases whether or not a change to D_{ij_r} triggers a chain of adjustments through h or k, or neither, corresponding to the events $\mathcal{B}_{h,ij_r}(q_h, F_h)$ and $\mathcal{B}_{k,ij_r}(q_k, F_k)$ for all combinations of (q_h, F_h) and (q_k, F_k). If we can show that conditional on a partition in terms of these events, the distribution of network statistics is the same for nodes h and k, then conditional invariance given $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$ and $D^*_{\tau'(i)k} = D_{ik}, D^*_{\tau'(j)k} = D_{jk}$ follows from the law of total probability.

Specifically we first consider the (potentially overlapping) events

$$\mathbf{C}(q_1, q_2, F_1, F_2) := \{\mathbf{B}_h(q_1, F_1) = \mathbf{B}_k(q_2, F_2) = 1\}$$

as the arguments q_1, q_2 and F_1, F_2 vary freely. We first consider the case $q_2 = q_1$ and $F_2 = F_1$: by definition of the event $\mathbf{C}(q_1, q_1, F_1, F_1)$, a switch in D_{ij_r} triggers q_1 chains of adjustments starting at node h. Any such chain may reach node k after a change to l, and depending only on the signs of $U^*(x_l, x_k; s_l, s_k) + \sigma\varepsilon_{lk} - \mathbf{MC}_l$ and $U^*(x_k, x_l; s_k, s_l) + \sigma\varepsilon_{kl} - \mathbf{MC}_k$. Similarly, the chain reaches h after that same change depending only on the signs of $U^*(x_l, x_h; s_l, s_h) + \sigma\varepsilon_{lh} - \mathbf{MC}_l$ and $U^*(x_h, x_l; s_h, s_l) + \sigma\varepsilon_{hl} - \mathbf{MC}_h$. By inspection, conditional on $\mathcal{B}_{\mathbf{E_1},h}$ and $\mathcal{B}_{\mathbf{E_1}^\tau,k}$ the distributions of $(\boldsymbol{\varepsilon}_{lk}, \boldsymbol{\varepsilon}_{kl}, \boldsymbol{MC}_k, \boldsymbol{x}_k)$ and $(\boldsymbol{\varepsilon}_{lh}, \boldsymbol{\varepsilon}_{hl}, \mathbf{MC}_h, \mathbf{x}_h)$ are the same, so that for $s_h = s_k$, the probability that the chain proceeds to h is equal to that of the chain proceeding to k. Finally, conditional on $\mathbf{D}^*_{\tau'(i)h} = D_{ik}, \mathbf{D}^*_{\tau'(j)h} = D_{jk}$, the resulting changes on whether $\mathbf{S}_{hij}(l, D_i, D_j) = s_k$ and $\mathbf{S}_{kij}(l, D_i, D_j) = s_k$ are supported are the same.

Hence, given $\mathbf{D}^*_{\tau'(i)h} = D_{ik}, \mathbf{D}^*_{\tau'(j)h} = D_{jk}$, the conditional probability that $\mathbf{S}_{hij}(l, D_i, D_j) = s_k$ is supported given $\mathbf{C}(q_1, q_1; F_1, F_1)$ is equal to that of $\mathbf{S}_{kij}(l, D_i, D_j) = s_k$ being supported given $\mathbf{C}(q_1, q_1; F_1, F_1)$. For the case $(q_1, F_1) \neq (q_2, F_2)$, we can follow an analogous line of argument to conclude that conditional on $\mathbf{D}^*_{\tau'(i)h} = D_{ik}, \mathbf{D}^*_{\tau'(j)h} = D_{jk}$, the conditional probability that $\mathbf{S}_{hij}(l, D_i, D_j) = s_k$ is supported given $\mathbf{C}(q_1, q_2; F_1, F_2)$ is equal to that of $\mathbf{S}_{kij}(l, D_i, D_j) = s_k$ being supported given $\mathbf{C}(q_2, q_1; F_2, F_1)$. Since the events $\mathbf{C}(q_2, q_1; F_2, F_1)$ and $\mathbf{C}(q_2, q_1; F_2, F_1)$ have equal probability, it follows from the law of total probability that the conditional distributions are the same given the union $\mathbf{C}(q_1, q_2; F_1, F_2) \cup \mathbf{C}(q_2, q_1; F_2, F_1)$. A similar line of reasoning gives us the analogous conclusion conditional on any intersections of "symmetrized" events $C(q_1, q_2; F_1, F_2) \cup \mathbf{C}(q_2, q_1; F_2, F_1), \dots, \mathbf{C}(q_R, q_{R+1}; F_R, F_{R+1}) \cup \mathbf{C}(q_{R+1}, q_R; F_{R+1}, F_R)$, allowing us to construct a partition of the event $\mathbf{D}^*_{\tau'(i)h} = D_{ik}, \mathbf{D}^*_{\tau'(j)h} = D_{jk}$ such that invariance with respect to τ holds conditional on each element in that partition. By the law of total probability, this completes the inductive step from $r-1$ to r.

This establishes the claim of the Lemma for a binary permutation τ such that $\tau(k) = h$. Since an arbitrary permutation can be generated by a sequence of pairwise swaps of indices, this is sufficient to establish the main conclusion of the Lemma. The generalization of this argument to two (or any finite number of) potential values is immediate $\square$

Proof of Lemma 3.3. This result is a generalization of Lemma B.1 in Menzel (2015). We therefore refer to the proof of that result for some of the intermediate technical steps below.

By independence of $\boldsymbol{\varepsilon}_{i1}, \dots, \boldsymbol{\varepsilon}_{iN}$,

$$\begin{aligned}
J^r \Phi_n(i, j_1, \dots, j_r) &= J^r \int \left(\prod_{q=1}^r \mathbb{P}(\mathbf{U}_{ij_q} \geq \sigma w) \right) \left(\prod_{q \geq r+1} \mathbb{P}(\mathbf{U}_{ij_q} < \sigma w)^{D_{j_q i}} \right) J G(w)^{J-1} g(w) dw \\
&= J^r \int \left(\prod_{q=1}^r (1 - G(w - \sigma^{-1} \tilde{U}_{ij_q})) \right) \left(\prod_{q \geq r+1} G(w - \sigma^{-1} \tilde{U}_{ij_q})^{D_{j_q i}} \right) J G(w)^{J-1} g(w) dw \\
&= \int \left(\prod_{q=1}^r J(1 - G(w - \sigma^{-1} \tilde{U}_{ij_q})) \right) J \frac{g(w)}{G(w)} \\
&\quad \times \exp \left\{ J \log G(w) + \frac{1}{J} \sum_{q \geq r+1} J D_{j_q i} \log G(w - \sigma^{-1} \tilde{U}_{ij_q}) \right\} dw
\end{aligned}$$

Now let $b_J := G^{-1}\left(1 - \frac{1}{J}\right)$ and $a_J = a(b_J)$, where $a(\cdot)$ is the auxiliary function in Assumption 2.2 (ii). By Assumption 3.1 (iii), $\sigma = \frac{1}{a(b_J)}$, so that a change of variables $w = a_J t + b_J$ yields

$$\begin{aligned}
J^r \Phi_n(i, j_1, \dots, j_r) &= \int \left(\prod_{q=1}^r J(1 - G(b_J + a_J(t - \tilde{U}_{ij_q}))) \right) J \frac{a_J g(b_J + a_J t)}{G(b_J + a_J t)} \\
&\quad \times \exp \left\{ J \log G(b_J + a_J t) + \frac{1}{J} \sum_{q \geq r+1} J D_{j_q i} \log G(b_J + a_J(t - \tilde{U}_{ij_q})) \right\} dt
\end{aligned}$$

By Assumption 2.2 (ii), $J(1 - G(b_J + a_J t)) \to e^{-t}$ and

$$J a_J g(b_J + a_J t) = J a(b_J) g(b_J + a(b_J) t) = a(b_J) \frac{1 - G(b_J + a_J t)}{a(b_J + a_J t)(1 - G(b_J))} \to e^{-t}$$

where the last step uses Lemma 1.3 in Resnick (1987). Also, following steps analogous to the proof of Lemma B.1 in Menzel (2015), we can take limits and obtain

$$\begin{aligned}\prod_{q=1}^{r} J(1-G(b_J+a_J(t-\tilde{U}_{ij_q}))) &\to \exp\left\{-rt+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}\\ J\log G(b_J+a_J(t-\tilde{U}_{ij_q})) &\to -e^{-t}\exp\{\tilde{U}_{ij_q}\}\end{aligned}$$

Combining the different components, we can take the limit of the integrand in (B.2),

$$\begin{aligned}R_J(t) &:= \left(\prod_{q=1}^{r} J(1-G(b_J+a_J(t-\tilde{U}_{ij_q})))\right) J\frac{a_J g(b_J+a_J t)}{G(b_J+a_J t)}\\ &\quad\times\exp\left\{J\log G(b_J+a_J t)+\frac{1}{J}\sum_{q\geq r+1} JD_{j_q i}\log G(b_J+a_J(t-\tilde{U}_{ij_q}))\right\}\\ &= \exp\left\{-e^{-t}\left(1+\frac{1}{J}\sum_{q\geq r+1} D_{j_q i}\exp\{\tilde{U}_{ij_q}\}\right)-(r+1)t+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}+o(1) \qquad \text{(B.2)}\end{aligned}$$

for all $t\in\mathbb{R}$. Using the same argument as in the proof of Lemma B.1 in Menzel (2015), pointwise convergence and boundedness of the integrand imply convergence of the integral by dominated convergence, so that we obtain

$$\begin{aligned}J^r\Phi_n(i,j_1,\dots,j_r) &\to \int_{-\infty}^{\infty}\exp\left\{-e^{-t}\left(1+\frac{1}{J}\sum_{q\geq r+1} D_{j_q i}\exp\{\tilde{U}_{ij_q}\}\right)-(r+1)t+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}dt\\ &= \int_{-\infty}^{0}\exp\left\{s\left(1+\frac{1}{J}\sum_{q\geq r+1} D_{j_q i}\exp\{\tilde{U}_{ij_q}\}\right)+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}s^r ds\\ &= \frac{r!\exp\{\sum_{q=1}^{r}\tilde{U}_{ik_q}\}}{\left(1+\frac{1}{J}\sum_{q\geq r+1} D_{j_q i}\exp\left\{\tilde{U}_{ik_q}\right\}\right)^{r+1}}\end{aligned}$$

where the first step uses a change of variables $s=-e^{-t}$, and the last step can be obtained recursively via integration by parts. Furthermore, if $\frac{r}{J}\to 0$, boundedness of the systematic parts from Assumption 2.1 implies that

$$\left|\frac{1}{J}\sum_{j=1}^{J}\exp\left\{\tilde{U}_{ij}\right\}-\frac{1}{J}\sum_{q=r+1}^{J}\exp\left\{\tilde{U}_{ik_q}\right\}\right|\to 0$$

so that

$$J^r\Phi_n(i,j_1,\dots,j_r|\mathbf{z}_i^*)\to\frac{r!\prod_{q=0}^{r}\exp\{\tilde{U}_{ik_q}\}}{\left(1+\frac{1}{J}\sum_{j=1}^{J}\exp\left\{\tilde{U}_{ij}\right\}\right)^{r+1}}$$

which completes the proof □

Proof of Lemma 3.4. For completeness, we give the following proof explicitly for the general case of a (potentially) non-unique edge response. Without loss of generality, we develop the formal argument only for the case in which the payoff-relevant network characteristic is binary, $\mathcal{S}=\{\underline{s},\bar{s}\}$, where $U^*(x,x';\underline{s},s')\leq U^*(x,x';\bar{s},s')$ and $U^*(x,x';s,\underline{s})\leq U^*(x,x';s,\bar{s})$ for all values of x,x',s'.

Let $S_i^*\subset\mathcal{S}$ denotes the set of values for s_i supported by the edge response for i, and let

$$B_0:=\{j:S_j^*=\mathcal{S}\}$$

denote the set of nodes for whom both values for s_j are supported by j's edge response. For each node i we also define

$$A_i := \{j : U^*(x_j, x_i; \underline{s}, s_i) \geq MC_j - \sigma\varepsilon_{ji}\} \cap B_0$$

and

$$B_i := \{j : U^*(x_j, x_i; \underline{s}, s_i) < MC_j - \sigma\varepsilon_{ji} \leq U^*(x_j, x_i; \bar{s}, s_i)\} \cap B_0$$

be the set of nodes with a non-unique edge response that are available to i for any value of s_j. As a notational convention, $i \notin A_i \cup B_i$. Note that by Assumption 3.1 and Lemma 3.3, $\mathbb{P}(j \in \mathbf{A}_i), \mathbb{P}(j \in \mathbf{B}_i) = O(n^{-1/2})$.

Define $a_{ij} := \mathbb{1}\{j \in A_i\} \exp\{U^*(x_i, x_j; s_i, \underline{s})\}$, $b_{ij} := \mathbb{1}\{j \in B_i\} \exp\{U^*(x_i, x_j; s_i, \bar{s})\}$, and $c_{ij} := \mathbb{1}\{j \in A_i\}(\exp\{U^*(x_i, x_j; s_i, \bar{s})\} - \exp\{U^*(x_i, x_j; s_i, \underline{s})\})$, and $g_{ij} := b_{ij} + c_{ij}$. Note that given $\mathbf{x}_i, \mathbf{x}_j$, (b_{ij}, c_{ij}) are conditionally independent across i, j. We also let $\Delta_i a_{ij} := a_{ij} - \mathbb{E}[\mathbf{a}_{ij} | \mathbf{x}_i = x, s \in \mathbf{S}_i^*]$ and $\Delta_i g_{ij} := g_{ij} - \mathbb{E}[\mathbf{g}_{ij} | \mathbf{x}_i = x, s \in \mathbf{S}_i^*]$.

We now introduce the allocation parameter $\alpha_j \in [0, 1]$ corresponding to the probability with which node j is assigned to choose the edge response $s_j = \bar{s}$, so that $s_j = \underline{s}$ will be chosen with probability $1 - \alpha_j$. In particular, for a given choice of $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n)'$, the inclusive value for agent i is given by

$$I_i[\boldsymbol{\alpha}] = n^{-1/2} \sum_{j=1}^n (a_{ij} + \alpha_j g_{ij}),$$

and the inclusive value function

$$\hat{H}_n^*(x, s; \boldsymbol{\alpha}) := n^{-1/2} \sum_{j=1}^n (\mathbb{E}[\mathbf{a}_{ij} | \mathbf{x}_i = x, s \in \mathbf{S}_i^*] + \alpha_j \mathbb{E}[\mathbf{g}_{ij} | \mathbf{x}_i = x, s \in \mathbf{S}_i])$$

Hence, we can write

$$\begin{aligned} I_i[\boldsymbol{\alpha}] - \hat{H}_n^*(x, s; \boldsymbol{\alpha}) &= n^{-1/2} \sum_{j=1}^n (a_{ij} - \mathbb{E}[\mathbf{a}_{ij} | \mathbf{x}_i = x, s \in \mathbf{S}_i^*] + \alpha_j (\mathbf{g}_{ij} - \mathbb{E}[\mathbf{g}_{ij} | \mathbf{x}_i = x, s \in \mathbf{S}_i])) \\ &= n^{-1/2} \sum_{j=1}^n (\Delta_i a_{ij} + \alpha_j \Delta_i g_{ij}) \end{aligned}$$

We can now quantify the average dispersion of I_i about its conditional mean by

$$\hat{V}_n[\boldsymbol{\alpha}] := \frac{1}{n} \sum_{i=1}^n (I_i[\alpha] - \hat{H}_n^*(x_i, s_i; \alpha))^2$$

for a given value of $\boldsymbol{\alpha}$. To find an upper bound for a given realization of payoffs, we can solve the problem

$$\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}] \text{ subject to } \alpha_1, \ldots, \alpha_n \in [0, 1]. \tag{B.3}$$

This upper bound is generally not sharp since for some nodes j only either value of s_j may be supported by the edge response. Multiplying out the square, we obtain

$$\begin{aligned} \hat{V}_n[\boldsymbol{\alpha}] &= \frac{1}{n} \sum_{i=1}^n \left(n^{-1/2} \sum_{j=1}^n (\Delta_i a_{ij} + \alpha_j \Delta_i g_{ij}) \right)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left(n^{-1/2} \sum_{j=1}^n \Delta_i a_{ij} \right)^2 + 2 \left(n^{-1/2} \sum_{j=1}^n \Delta_i a_{ij} \right) \left(n^{-1/2} \sum_{j=1}^n \alpha_j \Delta_i g_{ij} \right) + \left(n^{-1/2} \sum_{j=1}^n \alpha_j \Delta_i g_{ij} \right)^2 \end{aligned}$$

where by a LLN, $n^{-1/2}\sum_{j=1}^n \Delta_i \mathbf{a}_{ij} \stackrel{p}{\to} 0$ (see also Lemma B.5 in Menzel (2015) for a detailed proof), so that

$$\begin{aligned}\max_{\boldsymbol{\alpha}} \hat{V}_n[\alpha] &= \frac{1}{n}\max_{\boldsymbol{\alpha}}\sum_{i=1}^n\left(n^{-1/2}\sum_{j=1}^n \alpha_j \Delta_i \mathbf{g}_{ij}\right)^2 + o_p(1)\\ &= \frac{1}{n^2}\max_{\boldsymbol{\alpha}}\sum_{j=1}^n\sum_{k=1}^n \alpha_j\alpha_k\sum_{i=1}^n \Delta_i\mathbf{g}_{ij}\Delta_i\mathbf{g}_{ik} + o_p(1)\end{aligned}$$

where in the last step we multiplied out the square and changed the order of summation.

Now, for $j \neq k$,

$$\mathrm{Var}(\Delta_i\mathbf{g}_{ij}\Delta_i\mathbf{g}_{ik}) = \mathbb{E}[(\Delta_i\mathbf{g}_{ij})^2(\Delta_i\mathbf{g}_{ik})^2] - (\mathbb{E}[\Delta_i\mathbf{g}_{ij}\Delta_i\mathbf{g}_{ik}])^2 = O(n^{-1}) - O(n^{-2})$$

and

$$\mathrm{Var}(\Delta_i\mathbf{g}_{ij}^2) = \mathbb{E}[(\Delta_i\mathbf{g}_{ij})^4] - \left(\mathbb{E}[\Delta_i\mathbf{g}_{ij}^2]\right)^2 = O(n^{-1/2}) - O(n^{-1})$$

Since $\Delta_i\mathbf{g}_{ik}$ is bounded for all i, k, we can use a CLT to conclude that for any $j \neq k$

$$\mathbf{Z}_{jk,n} := \sum_{i=1}^n \Delta_i\mathbf{g}_{ij}\Delta_i\mathbf{g}_{ik} = O_p(1), \text{ and } \mathbf{Z}_{jj,n} := n^{-1/4}\sum_{i=1}^n \Delta_i\mathbf{g}_{ij}^2 = O_p(1)$$

where Assumption 2.1 implies that the asymptotic variances of $\mathbf{Z}_{jk,n}$ and $\mathbf{Z}_{jj,n}$ are bounded. Furthermore, $\mathbb{E}[\mathbf{Z}_{jk}] = 0$ for $j \neq k$, and $\mathbf{Z}_{jk,n}$ are independent across $1 \le j \le k \le n$.

Next, we can bound the sum corresponding to the "diagonal" elements $\mathbf{Z}_{jj,n}$ by

$$\frac{1}{n^2}\sum_{j=1}^n \alpha_j^2 \sum_{i=1}^n \Delta_i\mathbf{g}_{ij}^2 \le \frac{1}{n^2}\max_{\boldsymbol{\alpha}}\sum_{j=1}^n \alpha_j^2 n^{1/4}\mathbf{Z}_{jj,n} = n^{-7/4}\sum_{j=1}^n \mathbf{Z}_{jj,n} = O_p(n^{-3/4})$$

noting that $\mathbf{Z}_{jj,n} \ge 0$ a.s., so that the maximum in the second expression is attained at $\alpha_1 = \cdots = \alpha_n = 1$. In the following, we let $\mathbf{Z}_n$ be the symmetric matrix whose (j,k)th element is $\mathbf{Z}_{jk,n}$ for $j \neq k$, and where we set $\mathbf{Z}_{jj}$ equal to zero.

Given these definitions, we can express the maximum in matrix notation and bound

$$\max_{\boldsymbol{\alpha}} \hat{V}_n[\alpha] = \frac{1}{n}\max_{\boldsymbol{\alpha}}\frac{1}{n}\boldsymbol{\alpha}'\mathbf{Z}_n\boldsymbol{\alpha} + o_p(1) \le \frac{1}{n}\max_{\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}'\mathbf{Z}_n\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\alpha}} + o_p(1) \equiv n^{-1/2}\lambda_{max}(n^{-1/2}\mathbf{Z}_n) + o_p(1)$$

where $\lambda_{max}(\mathbf{A}$ denotes the largest eigenvalue of a symmetric matrix $\mathbf{A}$. For the second step, notice that $|\alpha_j|^2 \le 1$ for each j, so that the scalar product $\boldsymbol{\alpha}'\boldsymbol{\alpha} \le n$ for each permissible $\boldsymbol{\alpha}$.

Also, $\mathbf{Z}_n$ is a symmetric matrix with entries which, conditional on $x_1, \dots, x_n$, are independent although in general not identically distributed, bounded, mean zero random variables. Furthermore, if we pre- and post-multiply the matrix $\mathbf{Z}_n$ with the diagonal matrix $H := \mathrm{diag}(1/\sigma_i)$, where $\sigma_i^2 := \frac{1}{n}\sum_{j\neq i}\mathrm{Var}(\Delta_i\mathbf{g}_{ij}^2|x_i)$, then the entries also have constant variance. It therefore follows from Theorem 2 of Füredi and Komlós (1981) that the maximal eigenvalue of $n^{-1/2}H\mathbf{Z}_nH$ is bounded from above by a finite constant with probability approaching 1, so that

$$\mathbb{E}\left[\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}]\right] = O(n^{-1/2}) \tag{B.4}$$

which converges to zero.

Now let $\tilde{j}$ be drawn uniformly at random from the set $\{1, \dots, n\}$. For any $\eta > 0$ we can use Chebyshev's Inequality to bound

$$\begin{aligned} P\left((\mathbf{I}_{\tilde{j}} - \hat{H}_n^*(x_{\tilde{j}}, s_{\tilde{j}}))^2 > \eta^2\right) &= \frac{1}{n}\sum_{i=1}^{n} P\left((\mathbf{I}_i - \hat{H}_n^*(x_i, s_i))^2 > \eta^2\right) \\ &\leq \frac{1}{\eta^2}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(\mathbf{I}_i - \hat{H}_n^*(x_i, s_i))^2] \\ &\leq \frac{1}{\eta^2}\mathbb{E}\left[\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}]\right] = o(1) \end{aligned}$$

for an arbitrary selection from the edge responses, where the right-hand side bound is uniform across all possible selections from pairwise stable networks and converges to zero by (B.4). This establishes convergence that is pointwise in x, s but uniform in all selections from the best response.

This establishes claim (a) of the Lemma for the case in which $\mathcal{S}$ has only two elements. An generalization to the case in which $\mathcal{S}$ has $K < \infty$ elements follows the exact same steps but requires additional case distinctions and an allocation parameter α in the K-dimensional probability simplex. Finally, for the general case in which $\mathcal{S}$ may have infinitely many elements, note that by Assumption 2.1, the systematic part of payoffs only varies over a bounded interval $[-\bar{U}, \bar{U}]$ as s_1, s_2 vary. Furthermore, $U(x_1, x_2; s_1, s_2)$ is Lipschitz in s_2, so that by compactness of $\mathcal{S}$, we can cover the set of functions $\{\exp\{U(x_1, x_2; s_1, s)\} : s \in \mathcal{S}\}$ with a finite number K of L_2-norm brackets of width $\eta/2$, using standard arguments (see e.g. Example 19.7 in van der Vaart (1998)). Identifying the kth bracket with an element $\exp\{U(x_1, x_2; s_1, s^{(k)})\}$, for any $s \in \mathcal{S}$ we can therefore find $s^{(k)} \in \{s^{(1)}, \dots, s^{(K)}\} \subset \mathcal{S}$ such that

$$\int |\exp\{U(x_1, x_2; s_1, s)\} - \exp\{U(x_1, x_2; s_1, s^{(k)})\}| w(x_1)w(x_2)dx_1dx_2 < \eta$$

for each $s_1 \in \mathcal{S}$. A simple calculation then shows that the difference between the analogs of the worst-case bounds in B.3 for the discrete set $s^{(1)}, \dots, s^{(K)} \subset \mathcal{S}$ and the full set $\mathcal{S}$ is less than ε, which can be made arbitrarily small.

For claim (b), note however that the argument for point-wise convergence in part 1 still goes through after multiplying the contribution of node i with bounded weights $\omega(x_i; s_i)$. Uniformity with respect to $\omega(\cdot)$ then follows from the GC condition and using arguments that are analogous as for part (b) of Lemma B.5 in Menzel (2015). For the case of $2 < |\mathcal{S}| < \infty$, the argument is identical except that allocation parameter α_j is now $(|\mathcal{S}| - 1)$-dimensional which increases the bounding constant by a finite multiple □

Size of Opportunity Sets. The next auxiliary result concerns the rate at which the number of available potential spouses increases for each individual in the market. For a given PSN L^*, we let

$$\mathbf{J}_i^* := \mathbf{J}_i[L^*] := \sum_{j=1}^{n} \mathbb{1}\{\mathbf{U}_{ji}(L^*) \geq \mathbf{MC}_j\}$$

denote the size of the link opportunity set available to agent i. Similarly, we let

$$\mathbf{K}_i^* := \mathbf{K}_i[L^*] := \sum_{j=1}^{n} \mathbb{1}\{\mathbf{U}_{ij}(L^*) \geq \mathbf{MC}_i\}$$

so that $\mathbf{K}_i^*$ is the number of nodes to whom i is available.

Lemma B.1 below establishes that in our setup, the number of available potential matches grows at a root-n rate as the size of the market grows.

Lemma B.1. *Suppose Assumptions 2.1-2.2 and 3.1 hold. Then for any pairwise stable network and each* $i = 1, \dots, n$,

$$\begin{aligned} \exp\{-\bar{U}\} \leq \quad & n^{-1/2}\mathbf{J}_i^* \quad \leq \exp\{\bar{U}\} \\ \exp\{-\bar{U}\} \leq \quad & n^{-1/2}\mathbf{K}_i^* \quad \leq \exp\{\bar{U}\} \end{aligned}$$

with probability approaching 1 as $n \to \infty$.

PROOF OF LEMMA B.1: Notice that in the absence of interaction effects across links, D_{ji} does not depend on the number of "proposals" that can be reciprocated, but only the magnitude of $\mathbf{MC}_i$. Furthermore, by Assumption 2.1, the systematic parts of payoffs are uniformly bounded for all values of s_i, s_j. Hence the proof closely parallels the argument for the matching case, and the conclusion of this lemma follows the same sequence of steps as in the proof of Lemma B.2 in Menzel (2015) □

Proof of Lemma 3.5. Aggregating over $j \neq i$, we define

$$\begin{aligned} \hat{H}_n^*(x_i; s_i) \quad &:= \quad n^{-1/2} \sum_{j \neq i} \exp\{U^*(x_i, x_j; s_i, s_j)\} \mathbb{P}(\mathbf{D}_{ji} = 1 | \mathbf{W}_j(\mathbf{L}^*)) \\ &= \quad n^{-1/2} \sum_{i \in W_j} \exp\{U^*(x_i, x_j; s_i, s_j)\} \mathbb{P}(\mathbf{D}_{ji} = 1 | \mathbf{W}_j(\mathbf{L}^*) = W_j) \\ & \quad\quad + n^{-1/2} \sum_{i \notin W_j} \exp\{U^*(x_i, x_j; s_i, s_j)\} \mathbb{P}(\mathbf{D}_{ji} = 1 | W_j(L^*) = W_j) \end{aligned}$$

The asymptotic approximation to the edge response in Lemma 3.3 implies

$$\begin{aligned} n^{1/2}\mathbb{P}(\mathbf{D}_{ji} = 1 | \mathbf{W}_j(\mathbf{L}^*) = W_j) \quad &= \quad n^{1/2}\mathbb{E}[\Phi(j, i_1, \dots, i_r | \mathbf{z}_j^*) | \mathbf{W}_j(L^*) = W_j] \mathbb{1}\{i \in \{i_1, \dots, i_r\}\} \\ &= \quad n^{1/2} \sum_{r \geq 0} \sum_{i_1, \dots, i_r} \Phi(j, i_1, \dots, i_r | \mathbf{z}_j^*) \mathbb{1}\{i \in \{i_1, \dots, i_r\}\} \end{aligned}$$

$$= \sum_{r \geq 0} \frac{(r+1)!}{r!} \frac{\exp\{U^*(x_j, x_i; (r, s_{2j}')', s_i) + U^*(x_i, x_j; s_i, (r, s_{2j}')')\}(I_j^*)^r}{(1 + I_j^*)^{r+2}} + o_p(1) < \infty$$

Since the last expression is uniformly bounded in s_i and $I_j^* \geq 0$, it follows that

$$n^{-1/2} \sum_{i \in W_j(L^*)} \exp\{U^*(x_i, x_j; s_i, s_j)\} \mathbb{P}(D_{ji} = 1 | W_j(L^*)) = o_p(1)$$

noting that by Lemma B.1, $|\{j : i \in W_j(L^*)\}|/n \to 0$ almost surely. Hence the contribution of nodes j such that $i \in W_j(L^*)$ to the inclusive value is negligible to first order.

Next consider the nodes j such that $i \notin W_j(L^*)$. Note that in that case, a link proposal to i does not result in a new link, and therefore D_{ji} does not affect the network structure. Hence, for given values of s_i, s_j and payoff shocks, the link proposal indicator D_{ji} is uniquely determined. Hence, using Lemma 3.3 again,

$$\begin{aligned} \hat{H}_n^*(x_i; s_i) \quad &= \quad n^{-1/2} \sum_{i \notin W_j} \exp\{U^*(x_i, x_j; s_i, s_j)\} \mathbb{P}(\mathbf{D}_{ji} = 1 | \mathbf{W}_j(L^*) = W_j) + o_p(1) \\ &= \quad \frac{1}{n} \sum_{j=1}^{n} \frac{s_{1j,+i}^* \exp\{U^*(x_i, x_j; s_i, s_j) + U^*(x_j, x_i; s_j, s_i)\}}{1 + I_j^*} + o_p(1) \end{aligned}$$

where the last expression depends on the empirical distribution of endogenous network characteristics given exogenous traits. We can therefore write

$$\hat{H}_n^*(x_i; s_i) \quad = \quad \frac{1}{n} \sum_{j=1}^{n} \frac{s_{1j,+i}^* \exp\{U^*(x_i, x_j; s_i, s_j) + U^*(x_j, x_i; s_j, s_i)\}}{1 + \hat{H}_n^*(x_j; s_j)} + o_p(1) \quad \text{(B.5)}$$

Substituting in the definition of $\hat{\Psi}_n$ in (3.10), we obtain pointwise convergence in x, s. Uniformity follows from the Donsker property of the fixed point mapping (see Example 2.10.25 in van der Vaart and Wellner (1996)), noting that I_j^* and $\hat{H}_n^*$ are guaranteed to be nonnegative $\square$

Proof of Corollary 3.1: Given part (i) of Proposition 3.1, it is sufficient to show that $\mathbb{E}[s_{i1}|x_i = x]$ is uniformly bounded for $x \in \mathcal{X}$. To this end, notice that for payoffs of the form $U^*(x_1, x_2; s_1, s_2) = U^*(x_1, x_2)$, the inclusive value function only depends on x, i.e. $H^*(x; s) = H^*(x)$. Furthermore, the edge response is unique so that the conditional degree distribution given $x_i = x$ has p.d.f. $\mathbb{P}(s_{i1} = s|x_i = x) = \frac{H^*(x)^s}{(1+H^*(x))^{s+1}}$. Hence, the conditional expectation of s_{i1} is given by

$$\begin{aligned}
\mathbb{E}[\mathbf{s}_{i1}|\mathbf{x}_i = x] &= \sum_{s=0}^{\infty} s \frac{H^*(x)^s}{(1+H^*(x))^{s+1}} = \frac{1}{1+H^*(x)} \sum_{s=0}^{\infty} s \left(\frac{H^*(x)}{1+H^*(x)}\right)^s \\
&=: \frac{1}{1+H^*(x)} \sum_{s=0}^{\infty} s\delta^s = \frac{1}{1+H^*(x)} \frac{\delta}{(1-\delta)^2} = H^*(x)
\end{aligned}$$

where $\delta := \frac{H^*(x)}{1+H^*(x)}$. Finally, it remains to be shown that $H^*(x)$ is uniformly bounded: from the fixed-point condition (3.7),

$$\begin{aligned}
\Psi[H, M](x) &= \int \frac{s_{j1} \exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1 + H(x_j)} M(s_j|x_j, x) w(x_j) ds_j dx_j \\
&= \int \frac{H^*(x_j) \exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1 + H(x_j)} M(s_j|x_j, x) w(x_j) ds_j dx_j \\
&\leq \exp\{2\bar{U}\}
\end{aligned}$$

where $\bar{U} < \infty$ is the bound in Assumption 2.1. Hence the range of Ψ_0 is uniformly bounded, so that the fixed point H^* also has to satisfy this bound $\square$

Proof of Lemma 3.6. For the first claim of the Lemma, notice that the fixed point condition (3.11) is a direct consequence of Lemmas 3.4 and 3.5. Furthermore, (3.13) holds by construction of the mapping $\hat{\Omega}_n$, where the exact form of the fixed-point mapping has to be derived separately for the problem at hand. For the proof of the second claim, we first state the following Lemma:

Lemma B.2. *Suppose the conditions for Proposition 3.1 hold. Then the mapping*

$$\hat{\Psi}_n[H, M](x; s) \xrightarrow{p} \Psi_0[H, M](x; s)$$

uniformly in $H \in \mathcal{H}$, $M \in \mathcal{M}$, and $(x', s)' \in \mathcal{X} \times \mathcal{S}$ as $n \to \infty$.

This result is a straightforward extension of Lemma B.6 in Menzel (2015), a separate proof will therefore be omitted.

Now let

$$\mathcal{Z}^* := \{(H^*, M^*) : H^* \in \Psi_0[H^*, M^*], M^* = \Omega_0[H^*, M^*]\}$$

be the set of fixed points of (3.7) and (3.8). Since the respective ranges of Ψ_0 and Ω_0 are contained in $\mathcal{H}$ and $\mathcal{M}$, respectively, any fixed points must be in $\mathcal{H} \times \mathcal{M}$, so that it is sufficient to consider the fixed-point mapping restricted to that compact space.

Now fix $\delta > 0$ and define

$$\eta := \inf\left\{\sup_{x_1, x_2, s} |M(s_1|x_1, x_2) - \Omega_0[H, M](s_1|x_1, x_2)| + \sup_{x,s} |\Psi_0[H, M](x; s) - H(x; s)| : d((H, M), \mathcal{Z}^*) \geq \delta\right\}. \tag{B.6}$$

By definition of $\mathcal{Z}^*$, we must have that either

$$\sup_{x_1,x_2,s} |M(s_1|x_1,x_2) - \Omega_0[H,M](s_1|x_1,x_2)| > 0$$

or

$$\sup_{x,s} |\Psi_0[H,M](x;s) - H(x;s)| > 0$$

for any $(H,M) \notin \mathcal{Z}^*$. Furthermore the δ-enlargement $(\mathcal{Z}^*)^\delta := \{(H,M) \in \mathcal{H} \times \mathcal{M} : d((H,M),\mathcal{Z}^*) < \delta\}$ is open, so that its complement is closed. Since any closed subset of a compact space is compact, the set $\{(H,M) \in \mathcal{H} \times \mathcal{M} : d(H,M) \geq \delta\}$ is compact. Since furthermore the quantities $\sup_{x_1,x_2,s} |M(s_1|x_1,x_2) - \Omega_0[H,M](s_1|x_1,x_2)|$ and $\sup_{x,s} |\Psi_0[H,M](x;s) - H(x;s)|$ are continuous in H, M, the infimum in the definition of η in (B.6) is attained, which implies that $\eta > 0$.

Finally, by Lemma B.2 and Assumption 3.2 (iii), the fixed-point mappings $\hat{\Omega}_n$ and $\hat{\Psi}_n$ converge uniformly to the respective limits, Ω_0 and Ψ_0. In particular, for any $\zeta > 0$, we can find $n_\zeta < \infty$ such that for all $n \geq n_\zeta$, $\sup_{M,H} \|\hat{\Omega}_n[H,M] - \Omega_0[H,M]\| < \zeta/2$ and $\sup_{M,H} \|\hat{\Psi}_n[H,M] - \Psi_0[H,M]\| < \zeta/2$ with probability greater than $1-\zeta$. It follows that as n increases, any point $(\hat{H}^*, \hat{M}^*)$ satisfying the fixed point conditions (3.11) and (3.13) is contained in $(\mathcal{Z}^*)^\delta$ w.p.a.1, establishing the second claim □

Proof of Lemma 3.7. We establish the conclusion by considering the conditional probability of link proposals D_i^*, D_j^* given potential outcomes $\mathbf{S}_{kij}^*(l, D_i^*, D_j^*)$. D_i^*, D_j^* are determined by potential values $\mathbf{S}_{ki}(l, d_i, d_j)$ where $d_i, d_j \in \{0,1\}^n$. Whether those potential values are supported by a pairwise stable network is independent from payoff shocks $\{\boldsymbol{\varepsilon}_{ik}, \boldsymbol{\varepsilon}_{jk}, \mathbf{MC}_i, \mathbf{MC}_j : k \notin \{i,j\}\}$ by Lemma 3.3. Furthermore, $\mathbf{S}_{ij}^*(l) := \mathbf{S}_{ij}(l, D_i^*, D_j^*)$, where $\mathbf{S}_{ij}(l, d_i, d_j)$ and $\mathbf{S}_{ji}(l, d_j, d_i)$ are independent of $\{\boldsymbol{\varepsilon}_{ik}, \boldsymbol{\varepsilon}_{jk}, \mathbf{MC}_i, \mathbf{MC}_j : k \notin \{i,j\}\}$ for each d_i, d_j by Lemma 3.1.

Now, let $D_i^* \in \{0,1\}^n$ denote the vector of indicators $(D_{ik}^*)_{k=1}^n$ and consider any values $d_i, d_j \in \{0,1\}^n$ that support the potential values $\mathbf{S}_{ij}^*(l, d_i, d_j) = s_i, \mathbf{S}_{ji}^*(l, d_j, d_i) = s_j$. Without loss of generality, we set $d_{ik} = d_{jk} = 0$ for $k = i, j$. Define $\Phi_n(d_i, d_j)$ as the conditional probability of the event $D_{ik}^* D_{ki} = d_{ik}, D_{jk}^* D_{kj} = d_{jk}$ for all $k \notin \{i,j\}$ given potential values $\mathbf{Z}_{kij}(l, D_i^*, D_j^*) = (s_k, D_{ki}, D_{kj})$.

Then by Lemmas 3.3 and 3.4,

$$\lim_n r_n \Phi_n(d_i, d_j) = \Phi(d_i, d_j) := \|d_i\|!\|d_j\|! \frac{\exp\left\{\sum_{k=1}^n (d_{ik} U_{ik}^* + d_{jk} U_{jk}^*)\right\}}{(1 + H^*(x_i; s_i))(1 + H^*(x_j; s_j))}$$

where $r_n := n^{(\|d_i\| + \|d_j\|)/2}$ and $\|d\| := \left(\sum_{k=1}^n d_k^2\right)^{1/2}$ denotes the Euclidean norm.

By the same argument, nr_n times the conditional probability of that same elementary outcome regarding $D_{i,-j}^*, D_{j,-i}^*$ together with availability D_{ij}^*, D_{ji}^* is

$$\Phi(d_i + (ij), d_j + (ji)) := (\|d_i\| + 1)(\|d_j\| + 1) \frac{\exp\left\{U_{ij}^* + U_{ji}^*\right\}}{(1 + H^*(x_i; s_i))(1 + H^*(x_j; s_j))} \Phi(d_i, d_j)$$

Taking ratios, n times the conditional probability that $D_{ij}^* = D_{ji}^* = 1$ given $D_{ik}^* D_{ki} = d_{ik}, D_{jk}^* D_{kj} = d_{jk}$ for all and potential outcomes for s_k for $k \notin \{i,j\}$ converges to

$$\frac{\Phi(d_i + (ij), d_j + (ji))}{\Phi(d_i, d_j)} = (\|d_i\| + 1)(\|d_j\| + 1) \frac{\exp\left\{U_{ij}^* + U_{ji}^*\right\}}{(1 + H^*(x_i; s_i))(1 + H^*(x_j; s_j))} \tag{B.7}$$

Since network statistics include degree centrality s_{i1}, s_{j1} by assumption, $\|d_i\| + l = s_{i1}$ and $\|d_j\| + l = s_{j1}$. For any of these events we have $\|d_i\| + l = s_{i1}$ and $\|d_j\| + l = s_{j1}$, respectively, so that the claim follows immediately from (B.7) after summing over elementary outcomes supporting s_i, s_j and integrating out potential values $\mathbf{S}_{kij}(l, D_i^*, D_j^*)$, noting that the right-hand side expression is constant across the conditioning event with $\|d_{i,-j}\| + l = s_{i1}$ and $\|d_j\| + l = s_{j1}$ $\square$

By inspection of the previous proof, we can generalize the conclusion of Lemma 3.7 to pairs of links and state the following corollary:

Corollary B.1. *Suppose that Assumptions 2.1-3.1 hold. Then for i, j, k, l and $v := |\{(ij), (kl)\}|$ denoting the number of distinct edges, the conditional probability of proposals $\mathbf{D}_{ij}^* = \mathbf{D}_{ji}^* = \mathbf{D}_{kl}^* = \mathbf{D}_{lk}^* = 1$ is approximated by*

$$\begin{aligned}
&\lim_n \quad n^v \bar{\mathbb{P}}\left(\mathbf{D}_{ij}^* = \mathbf{D}_{ji}^* = \mathbf{D}_{kl}^* = \mathbf{D}_{lk}^* = 1 \middle| \mathbf{S}_{ij}^*(l) = s_i, \mathbf{S}_{ji}^*(l) = s_j, \mathbf{S}_{kl}^*(l) = s_k, \mathbf{S}_{lk}^*(l) = s_l, \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_l\right) \\
&= \quad \prod_{q \in \{i,j,k,l\}} \left(\frac{s_{q1} + 1}{1 + H^*(x_q; s_q)}\right) \prod_{(qr) \in \{(ij),(kl)\}} \exp\left\{U_{qr}^* + U_{rq}^*\right\}
\end{aligned}$$

where $U_{ij}^*, U_{ji}^*, U_{kl}^*, U_{lk}^*$ are defined as in Lemma 3.7.

This Corollary can be established using steps analogous to the proof of Lemma 3.7 with minor changes requiring additional notation. Note also that this corollary allows for the edges $(ij), (kl)$ to have one or both nodes in common, in which case the products on the right-hand side skip any duplicate factors.

B.6. **Proof of Proposition 4.1.** For notational simplicity, we let $\mathbf{U}_{i0} := \mathbf{MC}_i$ which we can approximate by $\mathbf{U}_{i0} = \log J + \sigma \boldsymbol{\varepsilon}_{i0}^*$ as J grows large, where $\boldsymbol{\varepsilon}_{i0}^*$ is a random draw from the same distribution as $\boldsymbol{\varepsilon}_{i1}, \dots, \boldsymbol{\varepsilon}_{iJ}$. We therefore let $\tilde{U}_{i0} := \log J$. Consider the case in which degree is equal to s, so that $\mathbf{MC}_i$ is the $(s+1)$ highest order statistic. Following Assumption 3.1 (ii), let $J = n^{1/2}$ and denote the number of elements in $\mathbf{W}_i(\mathbf{L}^*)$ with $\mathbf{J}_W$. Note also that by Lemma B.1, $\mathbf{J}_W = O_P(n^{1/2})$, and by Lemma 3.1, taste shifters $\boldsymbol{\varepsilon}_{ij}$ are asymptotically independent of $\mathbf{W}_i(\mathbf{L}^*)$.

In the following we let $\mathbf{A}_i^+(r; s)$ denote the event that payoffs support $s_i = s$ and network degree $s_{i1} \geq r$. From the law of iterated expectations, the partial mean of the rth order statistic $\mathbf{U}_{i;r}$ given $\mathbf{A}_i^+(r; s)$ is

$$\begin{aligned}
&\mathbb{E} \quad \left[\left(\mathbf{U}_{i;r} - \frac{1}{2}\log n\right) \mathbb{1}\{\mathbf{A}_i^+(r; s)\}\right] = \sum_{j=1}^{J_W} \mathbb{E}[(\tilde{U}_{ij} + \sigma\varepsilon_{ij} - \log J)\mathbb{1}\{\mathbf{A}_i^+(r; s), \mathbf{U}_{i;r} = \tilde{U}_{ij} + \sigma\boldsymbol{\varepsilon}_{ij}\}] \\
&= \quad \frac{1}{J^{r-1}} \sum_{j=1}^{J_W} \sum_{j_1 \neq \dots \neq j_{r-1} \neq j} \int_{-\infty}^{\infty} w \left(\prod_{q=1}^{r} J(1 - G(w - \sigma^{-1}\tilde{U}_{ij_q}))\right) \left(\prod_{q=r+1}^{J_W} G(w - \sigma^{-1}\tilde{U}_{ij_q})\right) \\
&\quad \times g(\sigma^{-1}(w - \tilde{U}_{ij} + \log J)dw + o(1) \\
&= \quad \frac{1}{(r-1)!} \int_{-\infty}^{\infty} w \left(\frac{1}{J}\sum_{j=1}^{J_W} \exp\{\tilde{U}_{ij}\}\right)^r \exp\left\{-rw - e^{-w}\left(1 + \frac{1}{J}\sum_{k=1}^{J_W} \exp\{\tilde{U}_{ik}\}\right)\right\} dw + o(1)
\end{aligned}$$

where the last step follows from the approximation in equation (B.2).

Now note that for any $\lambda \geq 0$, $\frac{d}{d\lambda} v^{\lambda}\big|_{\lambda=r} = \log(v) v^r$. Hence, if we define $a := \frac{1}{J}\sum_{j=1}^{J_W} \exp\{\tilde{U}_{ij}\}$, and after a change of variables $v = e^{-w}$, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbf{U}_{i;r} - \frac{1}{2}\log n\right)\mathbb{1}\{\mathbf{A}_i(r;s)\}\right] &= \frac{1}{(r-1)!}\int_{-\infty}^{\infty} a^r w \exp\{-rw - (1+a)e^{-w}\}dw + o(1)\\
&= -\frac{1}{(r-1)!}\left(\frac{a}{1+a}\right)^r \int_0^{\infty} \left[\log(v) - \log(1+a)\right] v^{r-1} e^{-v} dv + o(1)\\
&= \left(\frac{a}{1+a}\right)^r \left(\frac{1}{(r-1)!}(\log(1+a)\Gamma(r) - \Gamma'(r)) + o(1)\right)\\
&= \left(\frac{a}{1+a}\right)^r \left(\log(1+a) + \gamma - \sum_{q=1}^{r-1}\frac{1}{q} + o(1)\right)
\end{aligned}$$

where $\Gamma(r+1) := \int_0^{\infty} v^r e^{-v} dv$ denotes the Gamma function. Since $\mathbb{P}(\mathbf{A}_i^+(r;s)) = \left(\frac{a}{1+a}\right)^r$, it follows that

$$\lim_n \mathbb{E}\left[\left(\mathbf{U}_{i;r} - \frac{1}{2}\log n\right)\middle| \mathbf{A}_i^+(r;s)\right] = \log(1+a) + \gamma - \sum_{q=1}^{r-1}\frac{1}{q}$$

Finally note that by Lemmas 3.4 and 3.6, $\frac{1}{J}\sum_{j=1}^{J_W}\exp\{\tilde{U}_{ij}\} \xrightarrow{p} H^*(x_i;s_i)$. Since the draws $\mathbf{U}_{i:1}, \dots, \mathbf{U}_{i:J_W}$ are independent, this also establishes the first claim of the Lemma.

Similarly, the partial mean of $\mathbf{MC}_i$ given that $\mathbf{MC}_i$ is the $(s+1)$th order statistic is given by

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbf{MC}_i - \frac{1}{2}\log n\right)\mathbb{1}\{\mathbf{A}_i(t;s)\}\right] &= \frac{1}{s!}\int_{-\infty}^{\infty} a^s w \exp\{-(s+1)w - (1+a)e^{-w}\}dw + o(1)\\
&= \frac{a^s}{(1+a)^{s+1}}\left(\log(1+a) + \gamma - \sum_{q=1}^{s}\frac{1}{q} + o(1)\right)
\end{aligned}$$

where $\mathbb{P}(\mathbf{A}_i(r;s)) = \frac{a^s}{(1+a)^{s+1}}$, which establishes the second claim □

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