Multirate Systems

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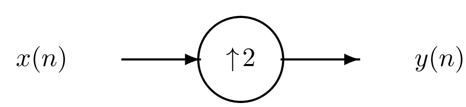
1 Introduction

1.1 Applications

1. Used in A/D and D/A converters.
2. Used to change the rate of a signal. When two devices that operate at different rates are to be interconnected, it is necessary to use a rate changer between them.
3. Interpolation.
4. Some efficient implementations of single rate filters are based on multirate methods.
5. Filter banks and wavelet transforms depend on multirate methods.

2 The Up-sampler

The *up-sampler*, represented by the diagram,



is defined by the relation

$$y(n) = \begin{cases} x(n/2), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd.} \end{cases} \tag{1}$$

The usual notation is

$$y(n) = [\uparrow 2]\, x(n). \tag{2}$$

The up-sampler simply inserts zeros between samples. For example, if $x(n)$ is the sequence

$$x(n) = \{\dots,\ 3,\ \underline{5},\ 2,\ 9,\ 6,\ \dots\}$$

where the underlined number represents $x(0)$, then $y(n)$ is given by

$$y(n) = [\uparrow 2]\, x(n) = \{\dots,\ 0,\ 3,\ 0,\ \underline{5},\ 0,\ 2,\ 0,\ 9,\ 0,\ 6,\ 0,\ \dots\}.$$

Given $X(z)$, what is $Y(z)$? Using the example sequence above we directly write

$$X(z) = \cdots + 3z + 5 + 2z^{-1} + 9z^{-2} + 6z^{-3} + \cdots \tag{3}$$

and

$$Y(z) = \cdots + 3z^2 + 5 + 2z^{-2} + 9z^{-4} + 6z^{-6} + \cdots \tag{4}$$

It is clear that

$$\boxed{y(n) = [\uparrow 2]\, x(n) \quad \Longleftrightarrow \quad Y(z) = X(z^2).} \tag{5}$$

We can also derive this using the definition:

$$Y(z) = \sum_n y(n)\, z^{-n} \tag{6}$$

$$= \sum_{n \text{ even}} x(n/2)\, z^{-n} \tag{7}$$

$$= \sum_n x(n)\, z^{-2n} \tag{8}$$

$$= X(z^2). \tag{9}$$

How does up-sampling affect the Fourier transform of a signal?
The discrete-time Fourier transform of $y(n)$ is given by

$$Y(e^{j\omega}) = X(z^2)\big|_{z=e^{j\omega}} \tag{10}$$

$$= X((e^{j\omega})^2) \tag{11}$$

so we have

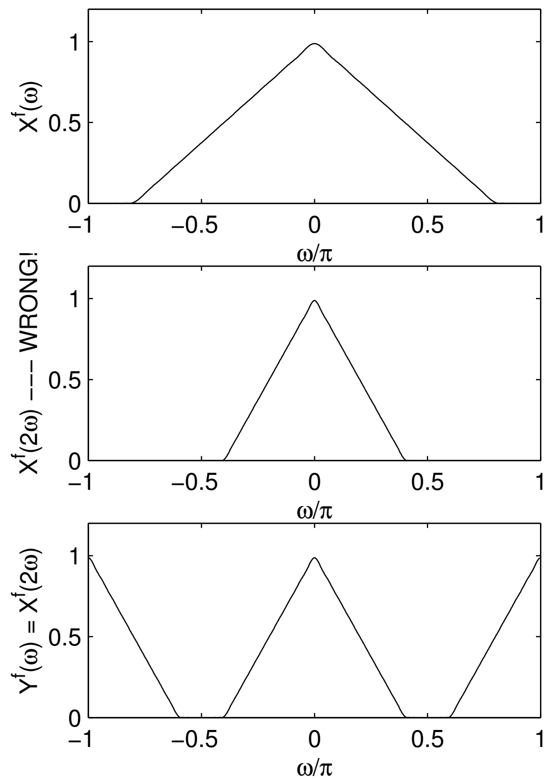
$$\boxed{Y(e^{j\omega}) = X(e^{j2\omega}).} \tag{12}$$

Or using the notation $Y^f(\omega) = Y(e^{j\omega})$, $X^f(\omega) = X(e^{j\omega})$, we have

$$\boxed{y(n) = [\uparrow 2]\, x(n) \quad \Longleftrightarrow \quad Y^f(\omega) = X^f(2\omega).} \tag{13}$$

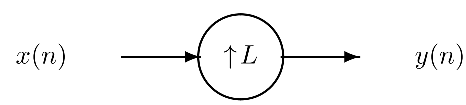
When sketching the Fourier transform of an up-sampled signal, it is easy to make a mistake. When the Fourier transform is as shown in the following figure, it is easy to incorrectly think that the Fourier transform of $y(n)$ is given by the second figure. This is not correct, because the Fourier transform is 2π-periodic. Even though it is usually graphed in the range $-\pi \leqslant \omega \leqslant \pi$ or $0 \leqslant \omega \leqslant \pi$, outside this range it is periodic. *Because $X^f(\omega)$ is a 2π-periodic function of ω, $Y^f(\omega)$ is a π-periodic function of ω.*

The correct graph of $Y^f(\omega)$ is the third subplot in the figure.



Note that the spectrum of $X^f(\omega)$ is repeated — there is an 'extra' copy of the spectrum. This part of the spectrum is called the *spectral image*.

General case: An L-fold up-sampler, represented by the diagram,



is defined as

$$y(n) = [\uparrow L]\, x(n) = \begin{cases} x(n/L), & \text{when } n \text{ is a multiple of } L \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

The L-fold up-sampler simply inserts $L-1$ zeros between samples. For example, if the sequence $x(n)$

$$x(n) = \{\ldots,\ 3,\ \underline{5},\ 2,\ 9,\ 6,\ \ldots\}$$

is up-sampled by a factor $L = 4$, the result is the following sequence

$$\begin{aligned} y(n) &= [\uparrow 4]\, x(n) \\ &= \{\ldots,\ 0,\ 3,\ 0,\ 0,\ 0,\ \underline{5},\ 0,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0,\ 9,\ 0,\ 0,\ 0,\ 6,\ 0,\ \ldots\}. \end{aligned}$$

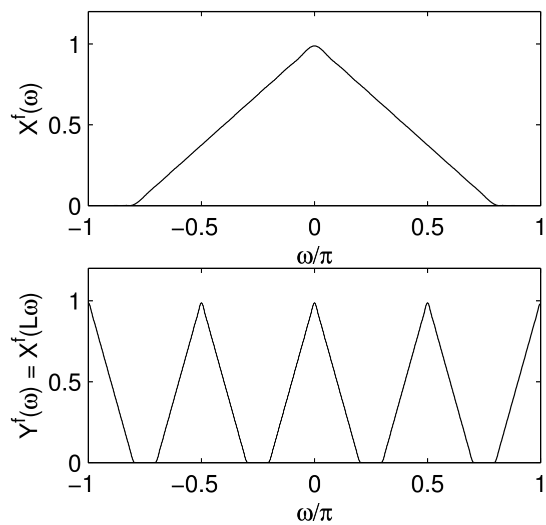
Similarly, we have

$$\boxed{y(n) = [\uparrow L]\, x(n) \quad \Longleftrightarrow \quad Y(z) = X(z^L),} \tag{15}$$

$$\boxed{Y(e^{j\omega}) = X(e^{jL\omega}),} \tag{16}$$

$$\boxed{y(n) = [\uparrow L]\, x(n) \quad \Longleftrightarrow \quad Y^f(\omega) = X^f(L\omega).} \tag{17}$$

The L-fold up-sampler will create $L-1$ spectral images. For example, when a signal is up-sampled by 4, there are 3 spectral images as shown in the following figure.

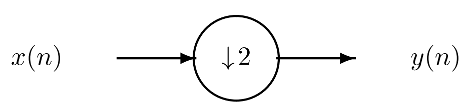


Remarks

1. No information is lost when a signal is up-sampled.
2. The up-sampler is a linear but *not* a time-invariant system.
3. The up-sampler introduces *spectral images*.

3 The Down-sampler

The down-sampler, represented by the following diagram,



is defined as

$$y(n) = x(2n). \tag{18}$$

The usual notation is

$$y(n) = [\downarrow 2]\, x(n). \tag{19}$$

The down-sampler simply keeps every second sample, and discards the others. For example, if $x(n)$ is the sequence

$$x(n) = \{\ldots,\ 7,\ 3,\ \underline{5},\ 2,\ 9,\ 6,\ 4,\ \ldots\}$$

where the underlined number represents $x(0)$, then $y(n)$ is given by

$$y(n) = [\downarrow 2]\, x(n) = \{\dots,\ 7,\ \underline{5},\ 9,\ 4,\ \dots\}.$$

Given $X(z)$, what is $Y(z)$? This is not as simple as it is for the up-sampler. Using the example sequence above we directly write

$$X(z) = \cdots + 7z^2 + 3z + 5 + 2z^{-1} + 9z^{-2} + 6z^{-3} + 4z^{-4} + \cdots \tag{20}$$

and

$$Y(z) = \cdots + 7z + 5 + 9z^{-1} + 4z^{-2} + \cdots \tag{21}$$

How can we express $Y(z)$ in terms of $X(z)$? Consider the sum of $X(z)$ and $X(-z)$. Note that $X(-z)$ is given by

$$X(-z) = \cdots + 7z^2 - 3z + 5 - 2z^{-1} + 9z^{-2} - 6z^{-3} + 4z^{-4} + \cdots . \tag{22}$$

The odd terms are negated. Then

$$X(z) + X(-z) = 2 \cdot \left(\cdots + 7z^2 + 5 + 9z^{-2} + 4z^{-4} + \cdots\right) \tag{23}$$

or

$$\boxed{X(z) + X(-z) = 2Y(z^2)} \tag{24}$$

or

$$\boxed{Y(z) = \frac{1}{2}\left[X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right]} \tag{25}$$

How does down-sampling affect the Fourier transform of a signal?
The discrete-time Fourier transform of $y(n)$ is given by

$$Y(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2}\left[X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right]\Big|_{z=\mathrm{e}^{\mathrm{j}\omega}} \tag{26}$$

$$= \frac{1}{2}\left(X(\mathrm{e}^{\mathrm{j}\frac{\omega}{2}}) + X(-\mathrm{e}^{\mathrm{j}\frac{\omega}{2}})\right) \tag{27}$$

$$= \frac{1}{2}\left(X(\mathrm{e}^{\mathrm{j}\frac{\omega}{2}}) + X(\mathrm{e}^{-j\pi}\,\mathrm{e}^{\mathrm{j}\frac{\omega}{2}})\right) \tag{28}$$

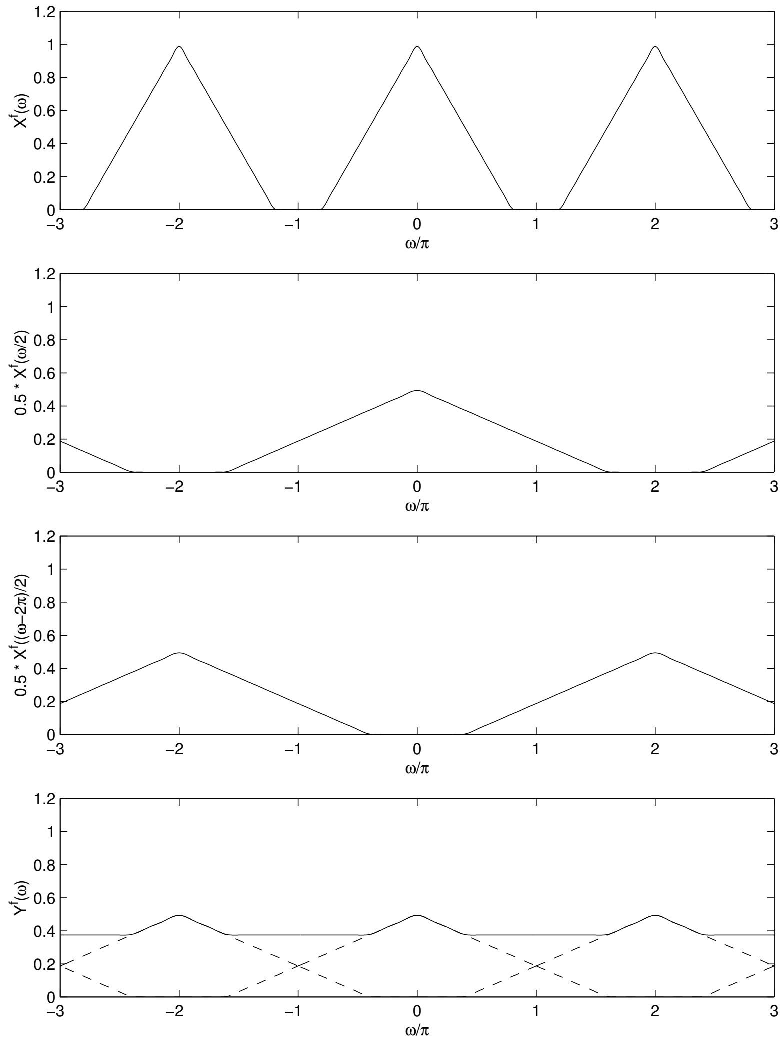
$$= \frac{1}{2}\left(X(\mathrm{e}^{\mathrm{j}\frac{\omega}{2}}) + X(\mathrm{e}^{\mathrm{j}(\frac{\omega}{2}-\pi)})\right) \tag{29}$$

$$= \frac{1}{2}\left(X^f(\frac{\omega}{2}) + X^f(\frac{\omega - 2\pi}{2})\right) \tag{30}$$

$$\boxed{y(n) = [\downarrow 2]\, x(n) \quad \Longleftrightarrow \quad Y^f(\omega) = \frac{1}{2}\left[X^f\left(\frac{\omega}{2}\right) + X^f\left(\frac{\omega - 2\pi}{2}\right)\right]} \tag{31}$$

where we have used the notation $Y^f(\omega) = Y(\mathrm{e}^{\mathrm{j}\omega})$, $X^f(\omega) = X(\mathrm{e}^{\mathrm{j}\omega})$.

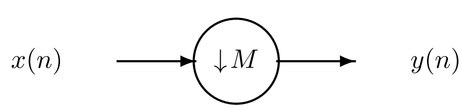
Note that because $X^f(\omega)$ is periodic with a period of 2π, the functions $X^f(\frac{\omega}{2})$ and $X^f(\frac{\omega-2\pi}{2})$ are each periodic with a period of 4π. But as $Y^f(\omega)$ is the Fourier transform of a signal, it must be 2π-periodic. What does $Y^f(\omega)$ look like? It is best illustrated with an example.



Notice that while the two terms $X^f(\frac{\omega}{2})$ and $X^f(\frac{\omega-2\pi}{2})$ are 4π-periodic, because one is shifted by 2π, their sum is 2π-periodic, as a Fourier transform must be.

Notice that when a signal $x(n)$ is down-sampled, the spectrum $X^f(\omega)$ may overlap with adjacent copies, depending on the specific shape of $X^f(\omega)$. This overlapping is called *aliasing*. When aliasing occurs, the signal $x(n)$ can not in general be recovered after it is down-sampled. In this case, information is lost by the down-sampling. If the spectrum $X^f(\omega)$ were zero for $\pi/2 \leqslant |\omega| \leqslant \pi$, then no overlapping would occur, and it would be possible to recover $x(n)$ after it is down-sampled.

General case: An M-fold down-sampler, represented by the diagram,



is defined as

$$y(n) = x(Mn). \tag{32}$$

The M-fold down-sampler keeps only every M^{th} sample. For example, if the sequence $x(n)$

$$x(n) = \{\ldots,\ 8,\ 7,\ 3,\ \underline{5},\ 2,\ 9,\ 6,\ 4,\ 2,\ 1,\ \ldots\}$$

is down-sampled by a factor $M = 3$, the result is the following sequence

$$y(n) = [\downarrow 3]\, x(n) = \{\ldots,\ 8,\ \underline{5},\ 6,\ 1,\ \ldots\}.$$

Similarly, we have

$$\boxed{y(n) = [\downarrow M]\, x(n) \quad \Longleftrightarrow \quad Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(W^k z^{\frac{1}{M}})} \tag{33}$$

where

$$W = \mathrm{e}^{\mathrm{j}\frac{2\pi}{M}}, \tag{34}$$

and

$$\boxed{y(n) = [\downarrow M]\, x(n) \quad \Longleftrightarrow \quad Y^f(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X^f\Big(\frac{\omega - 2\pi\, k}{M}\Big).} \tag{35}$$

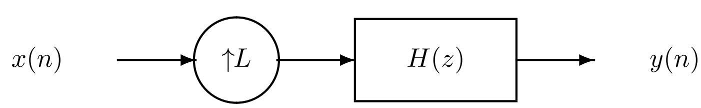
Remarks

1. In general, information is lost when a signal is down-sampled.
2. The down-sampler is a linear but *not* a time-invariant system.
3. In general, the down-sampler causes *aliasing*.

4 Rate-changing

The up-sampler and down-sampler are usually used in combination with filters, not by themselves. For example, to change the rate of a signal, it is necessary to employ low-pass filters in addition to the up-sampler and down-sampler.

The following system is used for interpolation.

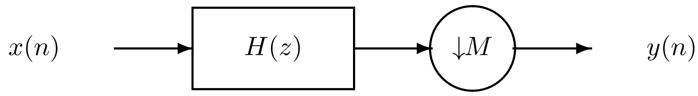


The combined up-sampling and filtering can be written as

$$\boxed{y(n) = ([\uparrow L]\, x * h)(n) = \sum_k x(k)\, h(n - Lk).} \tag{36}$$

The filter fills in the zeros that are introduced by the up-sampler. Equivalently, it is designed to remove the *spectral images*. It should be a low-pass filter with a cut-off frequency $\omega_o = \pi/L$. In this context, the low-pass filter is often called an *interpolation filter*.

The following system is used for decimation.

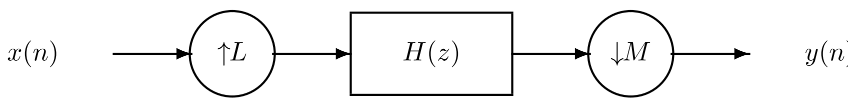


The combined filtering and down-sampling can be written as

$$y(n) = [\downarrow M]\,(x * h)(n) = \sum_k x(k)\,h(Mn - k). \tag{37}$$

The filter is designed to avoid *aliasing*. It should be a low-pass filter with a cut-off frequency $\omega_o = \pi/M$. In this context, the low-pass filter is often called an *anti-aliasing filter*.

A rate changer for a fractional change (like 2/3) can be obtained by cascading an interpolation system with a decimation system. Then, instead of implementing two separate filters in cascade, one can implement a single filter. Structure for rational rate changer:



The filter is designed to both eliminate spectral images and to avoid aliasing. The cascade of two ideal low-pass filters is again a low-pass filter with a cut-off frequency that is the minimum of the two cut-off frequencies. So, in this case, the cut-off frequency should be

$$\omega_o = \min\left\{\frac{\pi}{L}, \frac{\pi}{M}\right\}. \tag{38}$$

4.1 Interpolation Example 1

In this example (Fig. 1), we interpolate a signal $x(n)$ by a factor of 4, using the interpolation system described above. We use a linear-phase Type I FIR lowpass filter of length 21 to follow the 4-fold up-sampler. Note that because the filter is causal, a delay in introduced by the interpolation system. $y(n)$ could be aligned with $x(n)$ by shifting it.

4.2 Interpolation Example 2

In this example (Fig. 2) we use a filter of length 7,

$$h(n) = \frac{1}{4} \cdot (1,\ 2,\ 3,\ 4,\ 3,\ 2,\ 1). \tag{39}$$

Note that this filter has the effect of implementing *linear* interpolation between the existing samples $x(n)$. The result is rather poor — the signal $y(n)$ is not very smooth. Similarly, *quadratic* interpolation can be implemented by using an appropriate filter $h(n)$.

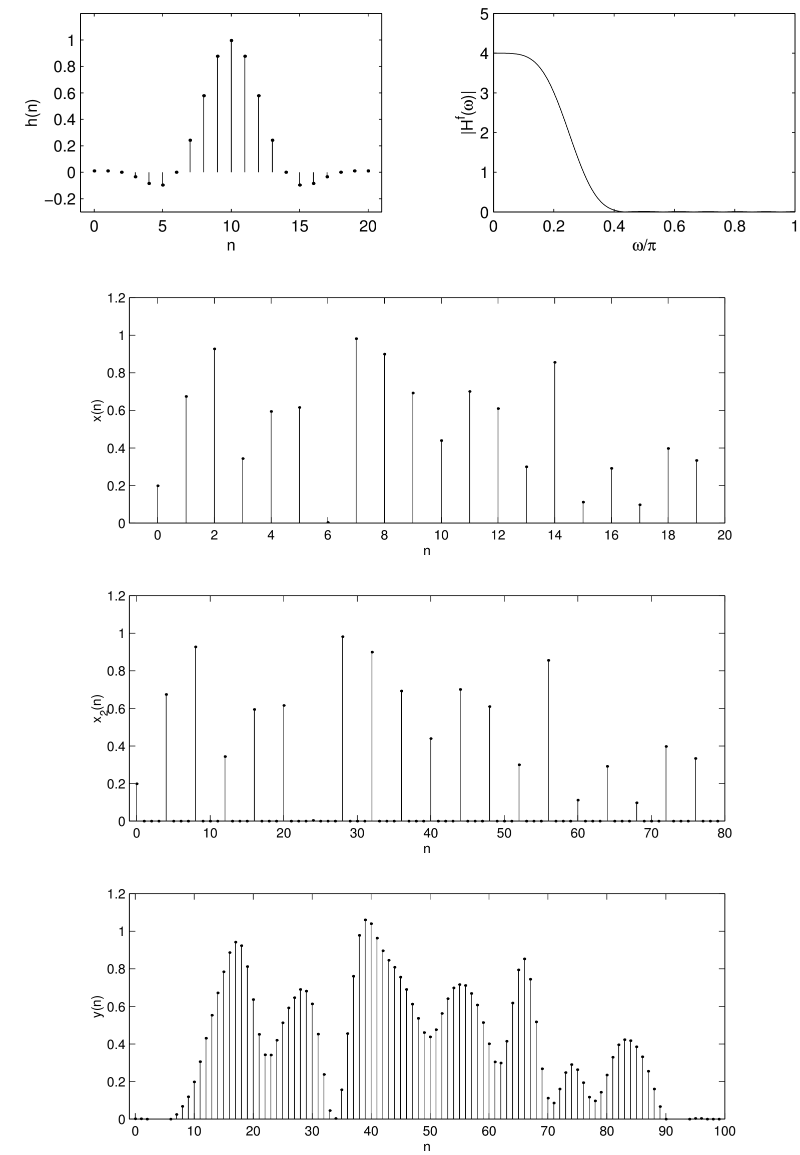


Figure 1: Interpolation example 1

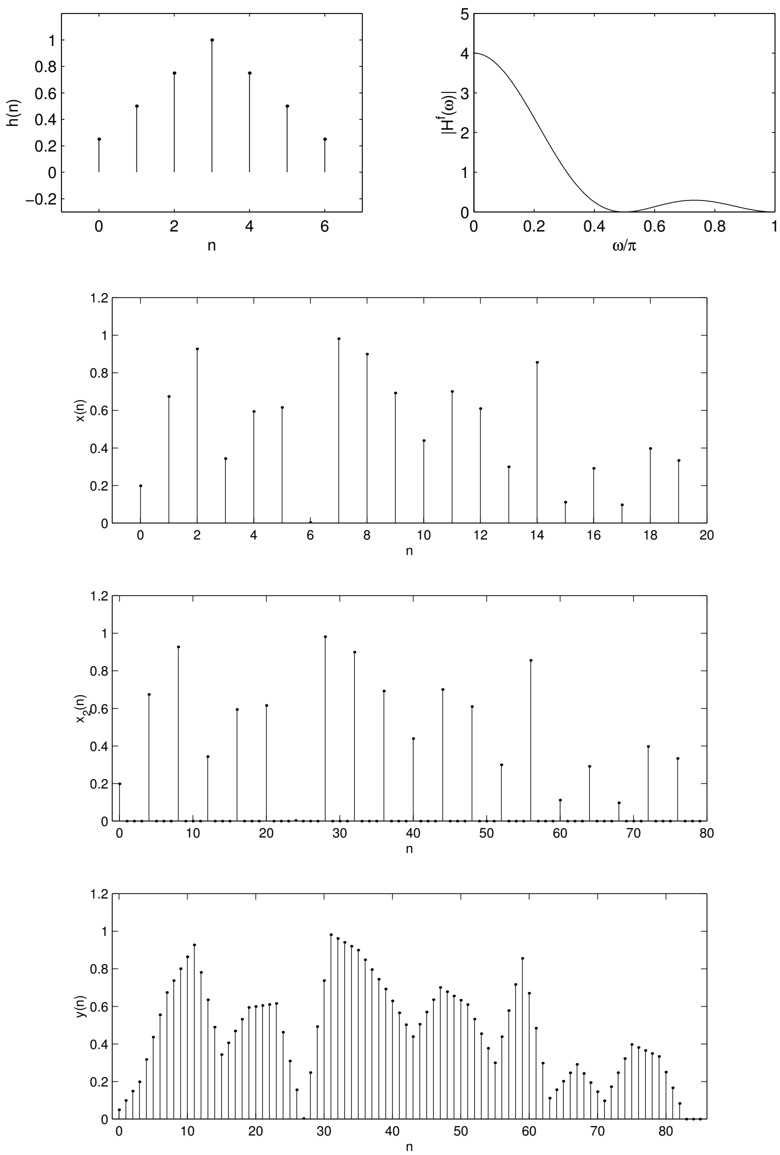


Figure 2: Interpolation example 2

5 Half-band Filters

When interpolating a signal $x(n)$, the interpolation filter $h(n)$ will in general change the samples of $x(n)$ in addition to filling in the zeros. It is natural to ask if the interpolation filter can be designed so as to preserve the original samples $x(n)$.

To be precise, if

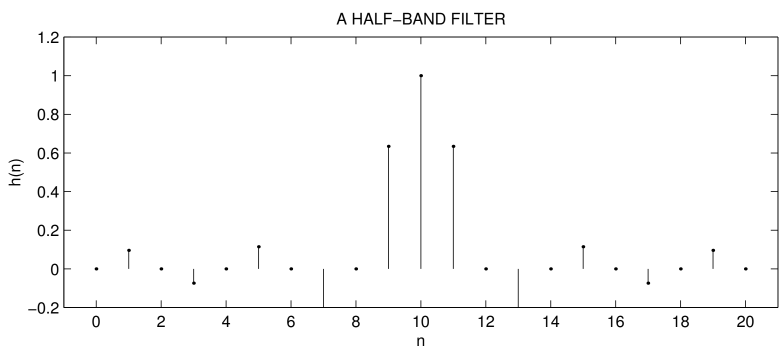
$$y(n) = (h * [\uparrow 2]\, x)(n)$$

then can we design $h(n)$ so that $y(2n) = x(n)$, or more generally, so that $\boxed{y(2n + n_o) = x(n)}$?

It turns out that this is possible. When interpolating by a factor of 2, if $h(n)$ is a *half-band*, then it will not change the samples $x(n)$. A n_o-centered half-band filter $h(n)$ is a filter that satisfies

$$h(n) = \begin{cases} 1, & \text{for } n = n_o \\ 0, & \text{for } n = n_o \pm 2, 4, 6, \ldots \end{cases} \tag{40}$$

That means, every second value of $h(n)$ is zero, except for one such value, as shown in the figure.



In the figure, the center point is $n_o = 10$. The definition of a half-band filter can be written more compactly using the Kronecker delta function $\delta(n)$.
A *half-band* filter is one where the impulse response h satisfies

$$\boxed{h(2n + n_o) = \delta(n).} \tag{41}$$

When $n_o = 0$, we get simply

$$\boxed{h(2n) = \delta(n).} \tag{42}$$

Note that the transfer function of a half-band filter (centered at $n_o = 0$) can be written as

$$H(z) = 1 + z^{-1} H_1(z^2). \tag{43}$$

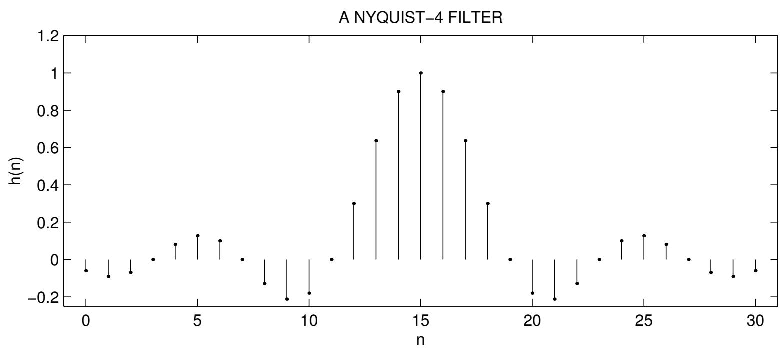
Here $H_1(z)$ contains the odd samples of $h(n)$.

6 Nyquist Filters

When interpolating a signal $x(n)$ by a factor L, the original samples of $x(n)$ are preserved if the interpolation filter $h(n)$ is a *Nyquist-L* filter. A Nyquist-L filter simply generalizes the notion of the half-band filter to $L > 2$. A (0-centered) Nyquist-L filter $h(n)$ is one for which

$$\boxed{h(Ln) = \delta(n).} \tag{44}$$

A Nyquist-4 filter is shown in the following figure.



7 The Noble Identity for the Up-sampler

The two following equivalences are sometimes called the *noble* identities.

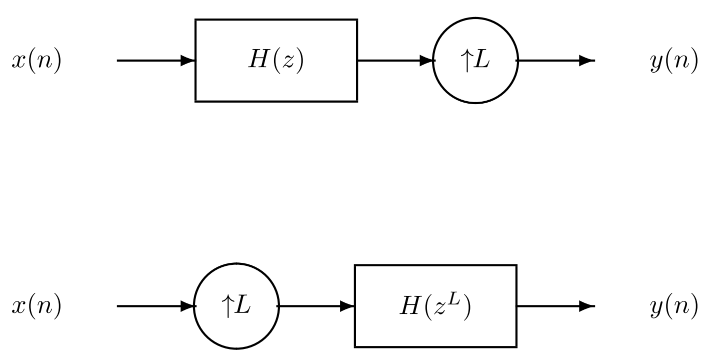
Can you reverse the order of an up-sampler and a filter?

Yes and no — it depends. There are two cases.

1. If the up-sampler comes *after* the filter, then you can reverse the order of the filter and the up-sampler, but the filter needs to be modified as shown in the figure.

2. If the up-sampler comes *before* the filter, then you can *not* reverse their order unless the filter is of the special form $H(z^L)$.

This can be summarized by the following figure.

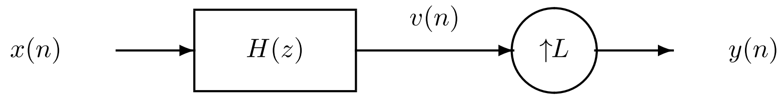


Equivalently:

$$\boxed{[\uparrow L]\ (h * x)\,(n) = ([\uparrow L]\, h * [\uparrow L]\, x)(n)} \tag{45}$$

7.1 Proof

This identity is most easily derived using the Z-transform and equation (15). In the following figure the intermediate signal $v(n)$ is shown.



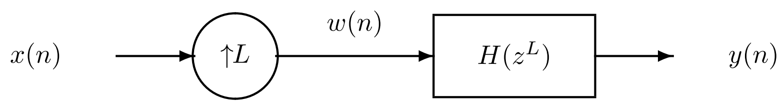
Then, using the Z-transform, we have

$$V(z) = H(z)X(z) \qquad \text{and} \qquad Y(z) = V(z^L)$$

and therefore,

$$Y(z) = H(z^L)X(z^L).$$

Now consider the system that we claim to be equivalent. In the following figure the intermediate signal $w(n)$ is shown.



Then, using the Z-transform, we have

$$W(z) = X(z^L) \qquad \text{and} \qquad Y(z) = H(z^L)W(z)$$

and therefore,

$$Y(z) = H(z^L)X(z^L).$$

This shows that the systems are equivalent.

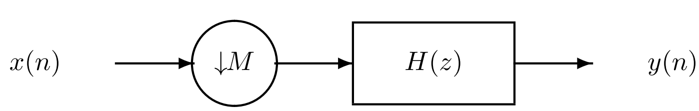
8 The Noble Identity for the Down-sampler

Can you reverse the order of an down-sampler and a filter?
Yes and no — it depends. There are two cases.

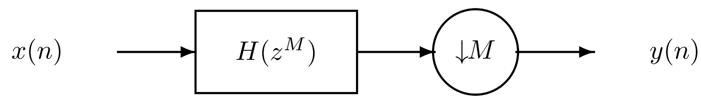
1. If the down-sampler comes *before* the filter, then you can reverse the order of the filter and the down-sampler, but the filter needs to be modified as shown in the figure.

2. If the down-sampler comes *after* the filter, then you can *not* reverse their order unless the filter is of the special form $H(z^M)$.

This can be summarized by the following figure.



$\Updownarrow$

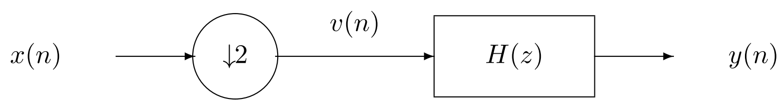


Equivalently:

$$(h * [\downarrow M]\, x)(n) = [\downarrow M]\ ([\uparrow M]\, h * x)\,(n) \tag{46}$$

8.1 Proof

For convenience, we prove it just for $M = 2$. This identity is most easily derived using the Z-transform and equation (25). In the following figure the intermediate signal $v(n)$ is shown.



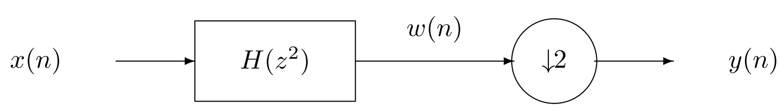
Then, using the Z-transform, we have

$$V(z) = \frac{1}{2}X(z^{\frac{1}{2}}) + \frac{1}{2}X(-z^{\frac{1}{2}}) \qquad \text{and} \qquad Y(z) = H(z)V(z)$$

and therefore,

$$Y(z) = \frac{1}{2}H(z)X(z^{\frac{1}{2}}) + \frac{1}{2}H(z)X(-z^{\frac{1}{2}})$$

Now consider the system that we claim to be equivalent. In the following figure the intermediate signal $w(n)$ is shown.



Then, using the Z-transform, we have

$$W(z) = H(z^2)X(z) \qquad \text{and} \qquad Y(z) = \frac{1}{2}W(z^{\frac{1}{2}}) + \frac{1}{2}W(-z^{\frac{1}{2}})$$

and therefore,

$$Y(z) = \frac{1}{2}H(z)X(z^{\frac{1}{2}}) + \frac{1}{2}H(z)X(-z^{\frac{1}{2}}).$$

This shows that the systems are equivalent.

9 Polyphase Decomposition

The polyphase decomposition of a signal is simply the even and odd samples,

$$x_0(n) = x(2n) \tag{47}$$

$$x_1(n) = x(2n+1). \tag{48}$$

Then the Z-transform $X(z)$ is given by

$$\boxed{X(z) = X_0(z^2) + z^{-1} X_1(z^2)} \tag{49}$$

where $X_0(z)$ and $X_1(z)$ are the Z-transforms of $x_0(n)$ and $x_1(n)$.

For example, if $x(n)$ is:

$$x(n) = \{\underline{3},\ 1,\ 5,\ 6,\ 2,\ 4,\ -3,\ 7\}$$

then the polyphase components are

$$x_0(n) = \{\underline{3},\ 5,\ 2,\ -3\} \tag{50}$$

$$x_1(n) = \{\underline{1},\ 6,\ 4,\ 7\}. \tag{51}$$

The Z-transforms for this example are given by

$$X(z) = 3 + z^{-1} + 5z^{-2} + 6z^{-3} + 2z^{-4} + 4z^{-5} - 3z^{-6} + 7z^{-7}$$

$$X_0(z) = 3 + 5z^{-1} + 2z^{-2} - 3z^{-3}$$

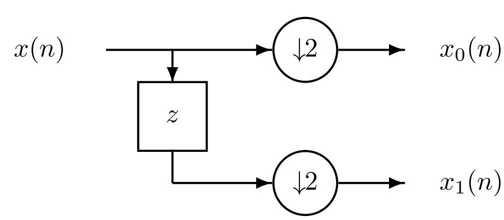
$$X_1(z) = 1 + 6z^{-1} + 4z^{-2} + 7z^{-3}.$$

In general, $X_0(z)$ and $X_1(z)$ can be obtained from $X(z)$ as,

$$X_0(z^2) = \frac{1}{2}\left(X(z) + X(-z)\right) \tag{52}$$

$$X_1(z^2) = \frac{z}{2}\left(X(z) - X(-z)\right). \tag{53}$$

The polyphase components $x_0(n)$, $x_1(n)$ can be obtained with the following structure.



General case: An M-component polyphase decomposition of $x(n)$ is given by

$$x_0(n) = x(Mn) \tag{54}$$

$$x_1(n) = x(Mn+1) \tag{55}$$

$$\vdots \tag{56}$$

$$x_{M-1}(n) = x(Mn + M - 1). \tag{57}$$

The Z-transform $X(z)$ is then given by

$$\boxed{X(z) = X_0(z^M) + z^{-1}X_1(z^M) + \cdots + z^{-(M-1)}X_{M-1}(z^M)} \tag{58}$$

where $X_i(z)$ is the Z-transform of $x_i(n)$. The polyphase component $X_i(z)$ can be found from $X(z)$ with

$$X_i(z^M) = \frac{z^i}{M} \sum_{k=0}^{M-1} W^{ik}\, X(W^k z) \tag{59}$$

where

$$W = \mathrm{e}^{\mathrm{j}\frac{2\pi}{M}}. \tag{60}$$

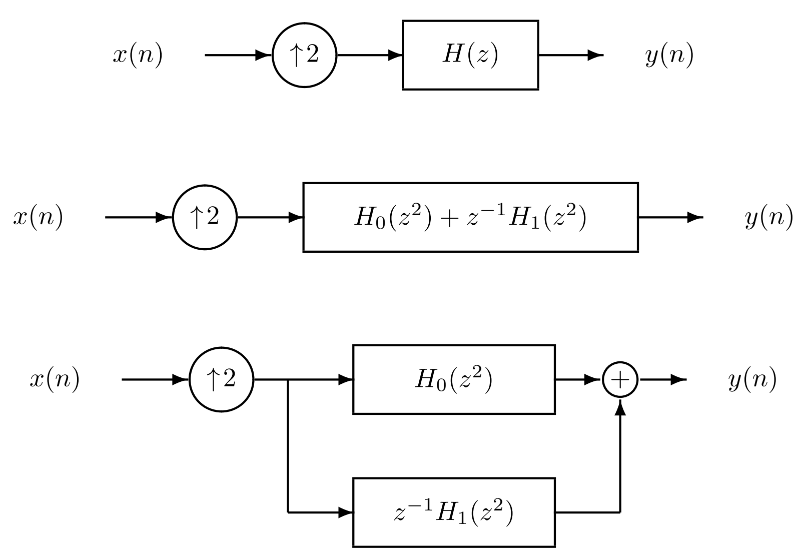
10 Efficient Implementation

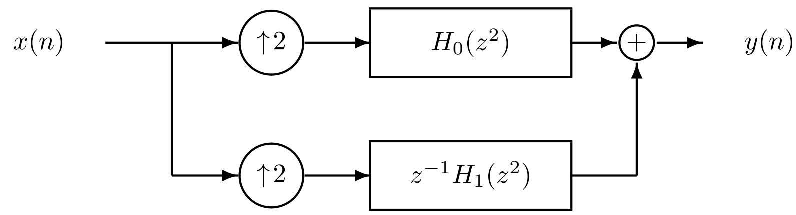
The noble identities and the polyphase decomposition can be used together to obtain efficient structures. Consider again the system for interpolation: an up-sampler is followed by a filter. In this system, the up-sampler inserts zeros between the samples $x(n)$. There are two disadvantages.

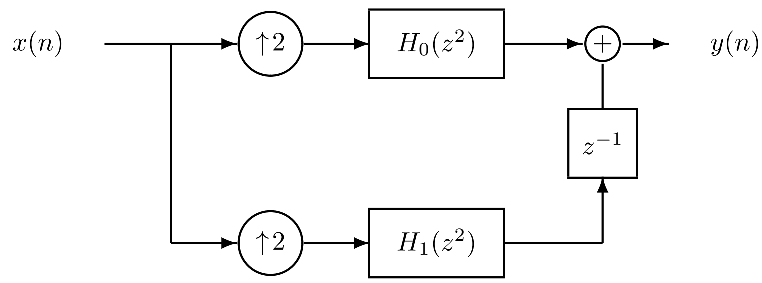
1. Half the samples of the input to the filter are zero. That means the filter is doing unnecessary computations (multiplications by zero, adding zeros).

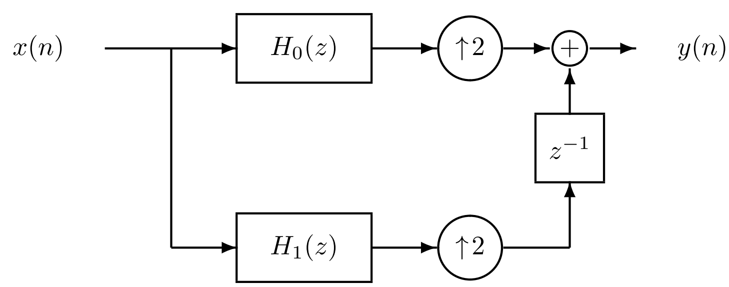
2. The filter operates at the higher rate.

A more efficient implementation can be obtained by writing the filter in polyphase form, and then using the noble identities. This is done through the following transformation of the block diagram.





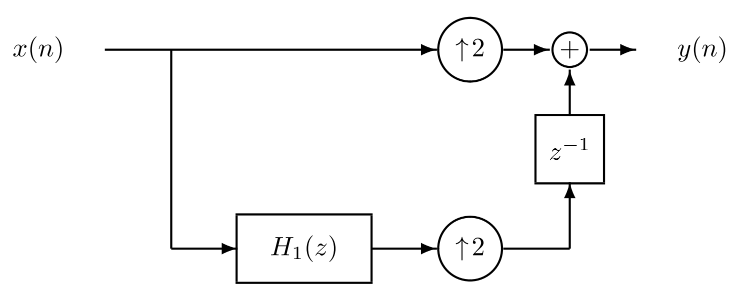




Note that in the last block diagram, the filters operate at the slower rate, and the filter inputs are not zero. Also note that the filters $h_0(n)$, $h_1(n)$ are each half the length of the original filter $h(n)$. The adding node in the last diagram does not incur any actual additions — it implements an interleaving of the two branches.

10.1 Half-band case

If $h(n)$ is a half-band filter, then the polyphase component $H_0(z)$ is 1 (assuming the half-band filter is centered at $n_o = 0$). In this case, the block diagram becomes more simple as shown.



11 Polynomial Signals

A (discrete-time) polynomial signal $x(n)$ is a signal of the form

$$x(n) = c_0 + c_1 n + c_2 n^2 + \cdots + c_d\, n^d.$$

The degree is d. The set of polynomial signals of degree d or less is denoted by $\mathcal{P}_d$.

Consider a system described by the rule

$$y(n) = x(n) - x(n-1).$$

This system gives the *first difference* of the signal $x(n)$. It has the impulse response

$$h(n) = \delta(n) - \delta(n-1),$$

and the transfer function

$$H(z) = 1 - z^{-1}$$

and so we can write

$$y(n) = (h * x)(n)$$

or

$$Y(z) = \left(1 - z^{-1}\right) X(z).$$

Clearly if $x(n)$ is a constant signal ($x(n) = c$, so we can write $x(n) \in \mathcal{P}_0$), then the first difference of $x(n)$ is identically zero,

$$Y(z) = \left(1 - z^{-1}\right) X(z) = 0 \quad \text{for} \quad x(n) \in \mathcal{P}_0.$$

Moreover, the first difference $Y(z)$ is identically zero *only* if $x(n)$ is a constant signal.

Similarly, if $x(n)$ is a ramp signal ($x(n) = c_0 + c_1 n$, so we can write $x(n) \in \mathcal{P}_1$), then the first difference is a constant signal. Therefore the *second difference*, (defined as the first difference of the first difference), must be identically zero. Writing this using the Z-transform gives

$$\left(1 - z^{-1}\right)^2 X(z) = 0 \quad \text{for} \quad x(n) \in \mathcal{P}_1.$$

Moreover, the second difference of $x(n)$ is identically zero *only* if $x(n)$ is of the form $c_0 + c_1 n$. Therefore, the set of first degree polynomial signals $\mathcal{P}_1$ is exactly the set of signals that is annihilated by $(1 - z^{-1})^2$.

Similarly, if $x(n)$ is a polynomial signal of degree d, then

$$Y(z) = \left(1 - z^{-1}\right)^{d+1} X(z) = 0 \quad \text{for} \quad x(n) \in \mathcal{P}_d.$$

or equivalently,

$$y(n) = \underbrace{h(n) * h(n) * \cdots * h(n)}_{d+1 \text{ terms}} * x(n) = 0 \quad \text{for} \quad x(n) \in \mathcal{P}_d.$$

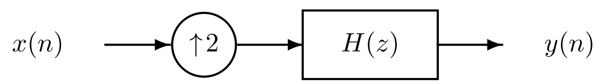
Moreover, $y(n) = 0$ *only* if $x(n)$ has the form $x(n) = c_0 + c_1 n + c_2 n^2 + \cdots + c_d\, n^d$.

Therefore we have the following result.

$$\boxed{x(n) \in \mathcal{P}_d \qquad \Longleftrightarrow \qquad \left(1 - z^{-1}\right)^{d+1} X(z) = 0} \tag{61}$$

11.1 Interpolation of polynomial signals

We saw before that the interpolation of discrete-time signals can be carried out by using an upsampler together with a filter. For interpolation by a factor of two (2X interpolation) we have the following diagram.



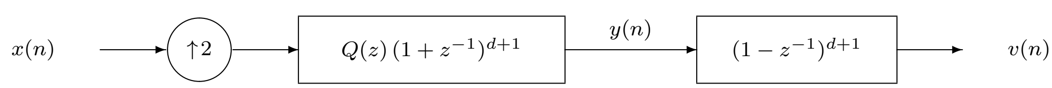
Suppose $x(n)$ is a polynomial signal of degree d. Then it is natural to ask that $y(n)$ also be a polynomial signal of degree d. But for just any filter $h(n)$ that will not be the case. What condition must $h(n)$ satisfy, to ensure that $y(n)$ is also a polynomial signal of degree d?

It turns out that if $(1+z^{-1})^{d+1}$ is a factor of $H(z)$,

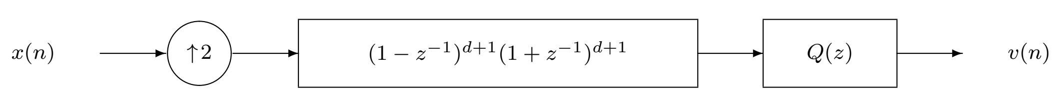
$$H(z) = Q(z)\,(1+z^{-1})^{d+1}$$

then $y(n) \in \mathcal{P}_d$ whenever $x(n) \in \mathcal{P}_d$. This can be verified using the boxed result on the previous page together with the noble identity, as we will now show.

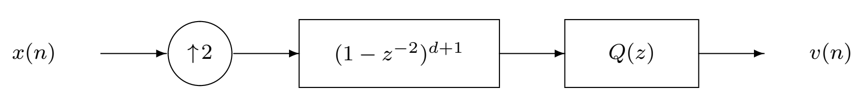
We may determine if $y(n) \in \mathcal{P}_d$ by filtering $y(n)$ with the transfer function $(1-z^{-1})^{d+1}$ and checking that the result is zero. If the signal $v(n)$ in the following figure is zero, then we know that $y(n) \in \mathcal{P}_d$ as explained earlier.



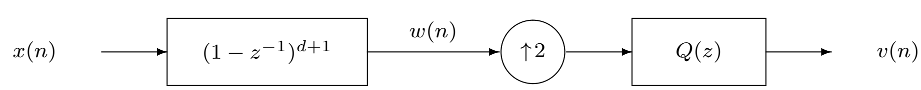
We can rearrange the order of the filters to get the following diagram which is an equivalent structure (end-to-end).



Recognizing that $(1-z^{-1})(1+z^{-1}) = 1-z^{-2}$ we get the following diagram.



Observing that the transfer function $(1-z^{-2})^{d+1}$ is a function of z^2, and using the noble identity for upsampling, we get the final diagram.



As explained above, if $x(n) \in \mathcal{P}_d$, then $w(n) = 0$ and therefore $v(n) = 0$. Because $v(n) = 0$, we know that $y(n) \in \mathcal{P}_d$.

In other words, if $H(z) = Q(z)\,(1+z^{-1})^{d+1}$ then when it is used for the 2X interpolation, it preserves $\mathcal{P}_d$, the set of polynomial signals of degree d.

$$\boxed{H(z) = Q(z)\,(1+z^{-1})^{d+1} \quad \Longleftrightarrow \quad (h * [\uparrow 2]\, x)(n) \text{ preserves } \mathcal{P}_d} \tag{62}$$

It should be said that in the interpolation structure above, even if $H(z)$ is chosen so that $y(n)$ is ensured to be a polynomial signal of degree d like $x(n)$ is, it does not mean that $y(2n) = x(n)$. That is only true when the filter $H(z)$ is in addition a half-band filter, as discussed above.

11.2 Polynomial interpolation by L

How should the condition above be modified if we are interpolating by a factor L rather than just by a factor of 2? If we guess that $H(z)$ should be of the form $H(z) = Q(z)R(z)$ and follow the same procedure used above, we will see that we will want the product $R(z)\,(1-z^{-1})^{d+1}$ to be equal to $(1-z^{-L})^{d+1}$. For in that case, we could again exchange the order of the (L-fold) upsampler and this term. This gives

$$R(z)\,(1-z^{-1})^{d+1} = (1-z^{-L})^{d+1}$$

or

$$R(z) = \frac{(1-z^{-L})^{d+1}}{(1-z^{-1})^{d+1}} = \left[1 + z^{-1} + z^{-2} + \cdots + z^{-(L-1)}\right]^{d+1}$$

where we have used

$$(1-z)(1+z+z^2+\cdots+z^{L-1}) = 1 - z^L.$$

$$H(z) = Q(z)\left[1 + z^{-1} + z^{-2} + \cdots + z^{-(L-1)}\right]^{d+1} \tag{63}$$

$$\Longleftrightarrow \qquad (h * [\uparrow L]\, x)(n) \text{ preserves } \mathcal{P}_d \tag{64}$$

It should be said that in the LX interpolation structure, even if $H(z)$ is chosen so that $y(n)$ is ensured to be a polynomial signal of degree d like $x(n)$ is, it does not mean that $y(Ln) = x(n)$. That is only true when the filter $H(z)$ is in addition a Nyquist-L filter.

12 Exercises

1. The signal $x(n)$

   $$x(n) = \{\ldots, 0,\ 0,\ \underline{1},\ 2,\ 3,\ 2,\ 1,\ 0,\ 0,\ \ldots\},$$

   where $\underline{1}$ represents $x(0)$, is applied as the input to the following system.

   $$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n)$$

   If the impulse response $h(n)$ is given by

   $$h(n) = \{\underline{1},\ 2\}$$

   then what is the output signal $y(n)$?

2. The signal $x(n)$

$$x(n) = \{\ldots, 0,\ 0,\ \underline{1},\ 2,\ -1,\ 0,\ 1,\ 0,\ 0,\ \ldots\}$$

where $\underline{1}$ represents $x(0)$ is applied as the input to the following system.

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H(z^2)} \longrightarrow y(n)$$

If the impulse response $h(n)$ is given by

$$h(n) = \{\underline{1},\ 1\}$$

then what is the output signal $y(n)$?

3. A signal $x(n)$ is down-sampled by M, and the result is up-sampled by M to yield a signal $y(n)$. Express $Y(z)$ in terms of $X(z)$. Write the expression also in the special case when $M = 2$.

4. If the IIR filter $h(n)$ has the transfer function

$$H(z) = \frac{1}{1 - c\, z^{-1}}$$

find the polyphase components $H_0(z)$ and $H_1(z)$ so that

$$H(z) = H_0(z^2) + z^{-1}\, H_1(z^2).$$

How many poles does each polyphase component have?

5. If $h(n)$ is the impulse response of a linear-phase FIR filter, are the two polyphase components of $h(n)$ linear-phase as well? Consider separately the case when the length N is even and odd.

6. (From Mitra 10.20) The *running-sum* filter, also called the *boxcar filter*,

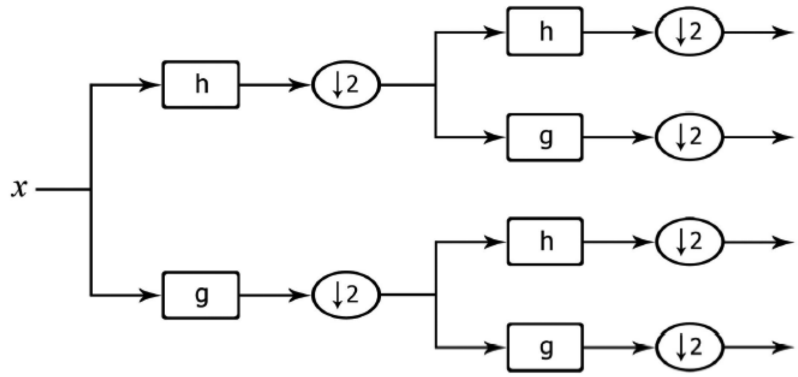
$$H(z) = \sum_{n=0}^{N-1} z^{-n}$$

can be expressed in the form

$$H(z) = (1 + z^{-1})\,(1 + z^{-2})\,(1 + z^{-4})\,\cdots(1 + z^{-2^{K-1}})$$

where $N = 2^K$. Verify this for $N = 16$. What is the impulse response $h(n)$? Using a length 16 boxcar filter, develop a realization of a factor-16 decimator using a *cascade* of simple filters with downsampling by two between each filter. (Use the noble identities.)

7. In the following discrete-time multirate system the filter h is a lowpass filter, and the filter g is a highpass filter. The system produces four *subband* signals, we can call them $s_1(n), \ldots, s_4(n)$.



The path from the input signal to each of the four output signals can be rewritten using the Noble identities as

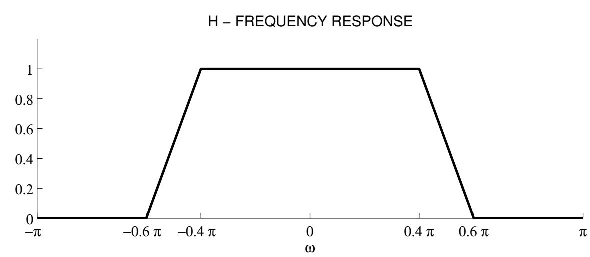
$$x(n) \longrightarrow \boxed{F_k(z)} \longrightarrow \boxed{\downarrow 4} \longrightarrow s_k(n)$$

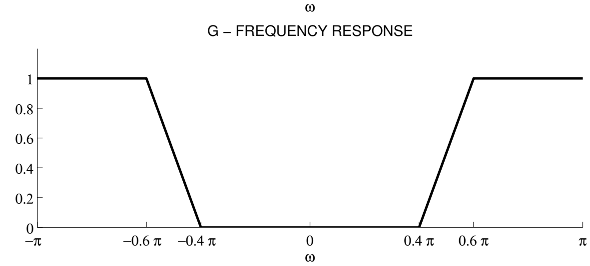
for $1 \leqslant k \leqslant 4$.

(a) What are the four transfer functions $F_k(z)$?

(b) Given the frequency response of h and g shown below, sketch the frequency responses of the four transfer functions $F_k(z)$.

(c) Classify each of the four transfer functions $F_k(z)$ as lowpass, bandpass, bandstop, or highpass.





8. Can you change the order of an up-sampler and a down-sampler with out change the total system? In other words, are the following two systems equivalent?

System A:

$$\longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow L} \longrightarrow$$

System B:

$$\longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\downarrow M} \longrightarrow$$

Consider the following cases:

(a) $M = 2$, $L = 2$.

(b) $M = 2$, $L = 3$.

(c) $M = 2$, $L = 4$.

Try an example in each case with a simple test input signal.

9. Simplify each of the following systems.

(a)

$$\longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\downarrow 3} \longrightarrow$$

(b)

$$\longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{\uparrow 2} \longrightarrow$$

(c)

$$\longrightarrow \boxed{\uparrow 5} \longrightarrow \boxed{\downarrow 10} \longrightarrow \boxed{\uparrow 2} \longrightarrow$$

10. Rewrite the following multirate system

$$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 5} \longrightarrow \boxed{G(z)} \longrightarrow \boxed{\uparrow 3} \longrightarrow y(n)$$

in the following form:

$$x(n) \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{\text{LTI}} \longrightarrow \boxed{\downarrow N} \longrightarrow y(n)$$

What is M, N, and what is the transfer function of the LTI system?

11. **Rate changing.** A discrete-time signal $x(n)$ having a rate of 12 samples per second must be converted into a new discrete-time signal $y(n)$ having a rate of 9 samples per second.

Sketch a multirate system that performs the necessary sampling rate conversion. Sketch the frequency responses of the required filter(s).

12. Suppose the DTFT of $x(n)$ is $X^f(\omega)$:

$$X^f(\omega) = 1 - \frac{|\omega|}{\pi}, \qquad \text{for } |\omega| \leqslant \pi$$

Suppose we generate the sequences $y(n)$ and $s(n)$ from $x(n)$ with the following system

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 2} \xrightarrow{s(n)} \boxed{\uparrow 4} \longrightarrow y(n)$$

where

$$H^f(\omega) = \begin{cases} 1, & |\omega| < \pi/2 \\ 0, & \pi/2 \leqslant |\omega| < \pi \end{cases}$$

Sketch $X^f(\omega)$, $H^f(\omega)$, $S^f(\omega)$ and $Y^f(\omega)$.

13. **Order of systems.**

The sampling rate of a discrete-time signal $x(n)$ is to be reduced to 3/4 its rate. One way to do this is to concatenate two systems: one that increases the rate by 3, another that reduces the rate by 4. To increase the rate by 3, we generally use the system

$$\longrightarrow \boxed{\uparrow 3} \longrightarrow \boxed{H(z)} \longrightarrow$$

where H is ideally a low-pass filter with cut-off frequency $\pi/3$. To reduce the rate by 4, we generally use the system

$$\longrightarrow \boxed{G(z)} \longrightarrow \boxed{\downarrow 4} \longrightarrow$$

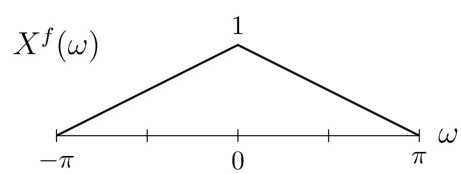
where G is ideally a low-pass filter with cut-off frequency $\pi/4$. These two system could be concatenated in either order:
A:

$$\longrightarrow \boxed{\uparrow 3} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{G(z)} \longrightarrow \boxed{\downarrow 4} \longrightarrow$$

B:

$$\longrightarrow \boxed{G(z)} \longrightarrow \boxed{\downarrow 4} \longrightarrow \boxed{\uparrow 3} \longrightarrow \boxed{H(z)} \longrightarrow$$

Suppose the Fourier transform of the input signal $x(n)$ is



(a) Suppose configuration A is used. Find and sketch the spectrum of the final output signal. In your work, you should also show the spectrum of the intermediate signals. Assume the filters H and G are ideal.

(b) Repeat, supposing configuration B is used.

(c) Which system (A or B) is better for changing the rate of signal? Explain.

14. **Half-band filters.** The following process doubles the rate of a signal x.

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Suppose h is a *half-band* filter. Then show that the output y contains as a subset the input signal values x. Specifically, show that if h is half-band, then

$$[\downarrow 2]\, y(n) = x(n). \tag{65}$$

Hint: use multirate identities.

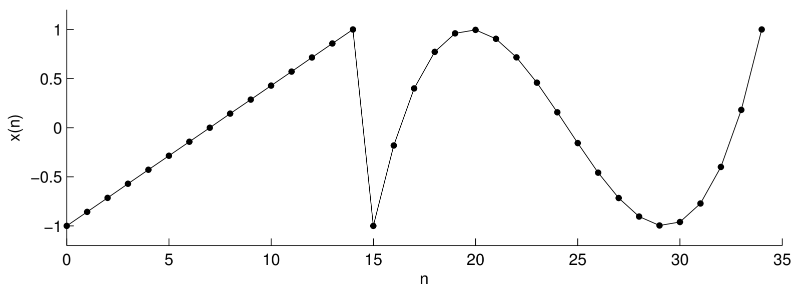
15. Let $h(n)$ be a low-pass half-band filter with real coefficients. Then show that

$$H^f(\omega) + H^f(\pi - \omega) = 2.$$

(That means the ripple in the pass-band of $H^f(\omega)$ is the same as the ripple in the stop-band of $H^f(\omega)$). You can assume the Nyquist filter is centered at $n_o = 0$.

Here, the DC gain will be about 2, rather than 1 as is usual for a low-pass filter. What must the cut-off frequency ω_o be? ($H^f(\omega_o) = 1$) Does this explain the terminology *half-band*?

16. **Polynomial signal interpolation.** The signal consists of two polynomial segments.



Interpolate this signal by 2 (upsample by 2 and filter) using each of the following filters $h(n)$ and plot the new signal. (You should obtain signals which are about twice as long as the test signal shown above.)

(a) $h(n) = [1,\ 1]/2$

(b) $h(n) = [1,\ 2,\ 1]/4$

(c) $h(n) = [1,\ 3,\ 3,\ 1]/8$

For each filter, give $H(z)$ in factored form.

Explain your observations using the results in the lecture notes concerning the interpolation of polynomial signals. That Matlab code to generate the test signal shown above is available on the course webpage.

17. **Half-band filter design.** Consider the interpolation of a signal $x(n)$ to increase its rate by two. This can be done by first up-sampling the signal and then filtering with a filter $H(z)$. For interpolation by a factor of two, it common to ask that the be half-band,

$$h(2n) = \delta(n)$$

where $h(n)$ is centered at $n = 0$. In addition, if the interpolation process is to conserve a polynomial signal of degree d, then $H(z)$ must have $(1+z^{-1})^{d+1}$ as a factor. Therefore, we consider the design of a minimal-length symmetric odd-length (Type I) half-band filter $H(z)$ of the form $H(z) = Q(z)(1 + z^{-1})^{d+1}$.

Given an odd integer d, we seek the minimal-length $q(n)$ such that $H(z)$ is a Type I half-band filter, where $H(z)$ is given by

$$H(z) = Q(z)\,(1 + z^{-1})^{d+1}$$

or equivalently,

$$h = q * \underbrace{[1,\ 1] * \cdots * [1,\ 1]}_{d+1 \text{ terms}}.$$

By symmetry, $Q(z)$ should also be a Type I filter.

Exercise: For $d = 1, 3, 5$, find $Q(z)$ such $H(z)$ is a Type I half-band filter. For $d = 1$ the problem is trivial. For $d = 3$ you can solve the linear equations problem by hand. For $d = 5$ you may use a computer to solve the system of linear equations. In each case, verify that $h(n)$ is a half-band filter, make a plot of $h(n)$, the zero diagram, and frequency response $|H^f(\omega)|$.

Hint: Given d, the filter $q(n)$ can be obtained by writing $h(n)$ in terms of $q(n)$ and solving a set of linear equations. If $q(n)$ is chosen with a suitable length, then there will be the same number of equations and unknowns variables.

18. Design a Type I linear-phase FIR digital half-band filter of minimal length having a transfer function $H(z)$ of the form

$$H(z) = Q(z)\,(1 + z^{-1})^2\,(1 + z^{-1} + z^{-2}).$$

   (a) Find the impulse response $h(n)$ of the filter.
   (b) Sketch the zeros of $H(z)$ in the complex z-plane.
   (c) Roughly sketch the frequency response $|H(e^{j\omega})|$ based on the filter's zero-diagram and half-band property.
   (d) What particular properties does the filter have, when it is used for interpolation?

19. Design a Type I linear-phase FIR digital half-band filter of minimal length having a transfer function $H(z)$ of the form

$$H(z) = Q(z)\,(1 + 2z^{-1} + z^{-2})\,(1 + 3z^{-1} + z^{-2}).$$

   (a) Find the impulse response $h(n)$ of the filter.
   (b) Sketch the zeros of $H(z)$ in the complex z-plane.
   (c) Roughly sketch the frequency response $|H(e^{j\omega})|$ based on the filter's zero-diagram and half-band property.
   (d) What particular properties does the filter have, when it is used for interpolation?

20. **Nyquist filters.** For a linear-phase half-band filter with impulse response $h(n)$ centered at $n = 0$, we have

$$H(\omega) + H(\omega - \pi) = 2.$$

   What is the frequency-domain equivalent condition for a Nyquist-3 filter?

21. **Interpolation.** Consider the interpolation of a discrete-time signal $x(n)$ by a factor of three. The new higher rate signal is denoted $y(n)$. It is required that the values $x(n)$ be preserved in $y(n)$, i.e. $y(3n) = x(n)$ for all $n \in \mathbb{Z}$.

(a) What condition should the impulse response of the interpolation filter satisfy?

(b) Verify that your condition satisfies the requirement.

22. **Nyquist filter design.** Design a Type I linear-phase FIR Nyquist-3 filter of minimal length having a transfer function $H(z)$ of the form

$$H(z) = Q(z)\,(1 + z^{-1} + z^{-2})^3.$$

(a) Find the impulse response $h(n)$ of the filter.

(b) Sketch the zeros of $H(z)$ in the complex z-plane.

(c) What particular properties does the filter have, when it is used for interpolation?

23. **Rate changing.** To increase the sampling rate of a discrete-time signal $x(n)$, to four thirds its original rate, the following system is used.

$$x(n) \longrightarrow \boxed{\uparrow 4} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n)$$

where the filter $H(z)$ is a low-pass filter.

If the filter $H(z)$ is furthermore a Nyquist-4 filter, are any samples of the input signal $x(n)$ preserved in the output signal $y(n)$? In other words, does the output signal satisfy:

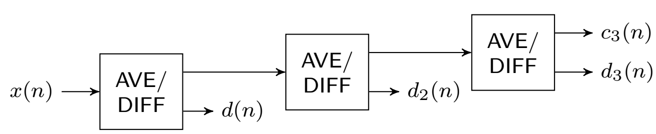
$$y(Kn) = x(Ln)$$

for some integers K and L? If so, show a derivation and identify K and L.

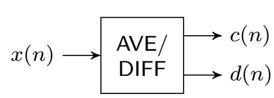
You can assume the Nyquist filter is centered at $n = 0$, i.e. $h(4n) = \delta(n)$.

Demonstrate your answer by a MATLAB example.

24. **The wavelet transform.** The Haar wavelet transform is implemented using the system



where the sub-system



is described by the two equations:

$$c(n) = 0.5\,x(2n) + 0.5\,x(2n+1)$$
$$d(n) = 0.5\,x(2n) - 0.5\,x(2n+1).$$

(a) Draw a block diagram for obtaining $c(n)$ from $x(n)$. The block diagram may combine an up-sampler, a down-sampler, and an LTI filter; but not more than one of each. What is the impulse response of the LTI filter? Sketch the zeros of the filter.

Similarly, draw a block diagram for obtaining $d(n)$ from $x(n)$.

(b) Draw a block diagram expressing the system between $x(n)$ and $d_2(n)$ in the 3-level system above. The block diagram may combine up-samplers, down-samplers, and LTI filters; but it should not have more than one of each. (Hint: use part a and the Noble Identities.) Sketch the impulse response and the zeros of the LTI system. Based on the zero diagram, roughly sketch the frequency response of the LTI system. Show your work.

(c) Repeat part (b), but for $d_3(n)$ instead of $d_2(n)$.

25. An Interpolated FIR (IFIR) filter is an FIR filter implemented as a cascade of two filters in the following structure.

$$x(n) \longrightarrow \boxed{F(z)} \longrightarrow \boxed{G(z^2)} \longrightarrow y(n)$$

To satisfy some filter specifications, this kind of multistage filter can be more efficient than a single FIR filter. Suppose $G(z)$ is a lowpass filter with the transition band from 0.2π to 0.3π. (Although it is not realistic, suppose $G^f(\omega)$ is exactly 1 in the passband and exactly 0 in the stopband, and linear in between. Make the same assumption about $F(z)$ in part (c).)

(a) Sketch the frequency response of the transfer function $G(z)$.

(b) Sketch the frequency response of the transfer function $G(z^2)$.

(c) $F(z)$ is to be a lowpass filter so that the total system is a lowpass filter with transition band from 0.1π to 0.15π. What should be the transition band of the lowpass filter $F(z)$? Sketch the frequency response of the total system with your $F(z)$.

(d) Given an expression for the impulse response $h(n)$ of the total system in terms of $f(n)$ and $g(n)$. Explain why this system is called an *Interpolated* FIR filter.

26. An IFIR (Interpolated FIR) filter is an FIR filter of the form:

$$H(z) = H_1(z^M)\, H_2(z).$$

Sometimes an IFIR filter can meet given specifications with a lower implementation complexity than a generic FIR filter.

Consider the design of a lowpass filter $H(z)$ having cut-off frequency $0.125\,\pi = \pi/8$. Suppose

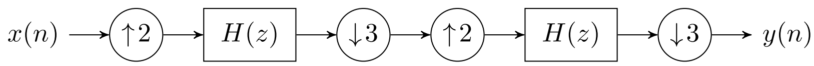
$$H(z) = H_1(z^4)\, H_2(z)$$

and that $H_1(z)$ is the ideal lowpass filter with cut-off frequency $0.5\,\pi$. That means, $H_1(z)$ is given, and $H_2(z)$ is to be designed so that the total filter $H(z)$ is a lowpass filter with cut-off frequency $0.125\,\pi = \pi/8$.

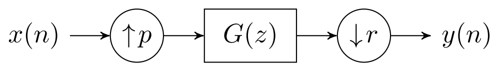
(a) Sketch the frequency response of $H_1(z)$ and of $H_1(z^4)$.

(b) What should be the passband edge and stopband edge of the filter $H_2(z)$? What is the widest transition band $H_2(z)$ can have? What is the advantage of choosing a wide transition band for the design of $H_2(z)$?

27. The following system consists of a rate changer in sequence with itself.



(a) The total system can be rewritten as a simpler one:



What is p, r, and the transfer function $G(z)$?

(b) Let $H(z)$ be a LPF with cut-off frequency $\omega_c = \pi/3$ with a transition band of width $\Delta\omega$. Let the frequency response of $H(z)$ be exactly unity and zero in the pass-band and stop-band respectively. Let the transition-band be linear over the frequency interval $[\omega_c - \Delta\omega/2,\ \omega_c + \Delta\omega/2]$. Then sketch the frequency response of $G(z)$.

28. **Linear-phase FIR Nyquist-L filter design.** Using Matlab, design a Type I linear-phase *Nyquist*-3 FIR filter of length 29. A Nyquist-3 filter $h(n)$ is one for which every third coefficient is zero, except for one such value. (See the notes on multirate systems.) The cut-off frequency ω_o should be $\pi/3$. Use the following design methods.

(a) Spline method.
Why does the spline method produce a Nyquist filter?

(b) (Constrained) Least squares.
Use $K_p = K_s = 1$ and $\omega_p = \omega_o - \frac{\pi}{20}$, $\omega_s = \omega_o + \frac{\pi}{20}$.

(c) Constrained Chebyshev with linear programming.
Use $K_p = K_s = 1$ and $\omega_p = \omega_o - \frac{\pi}{20}$, $\omega_s = \omega_o + \frac{\pi}{20}$.

For (b) and (c), the Nyquist property can be included in the design by adding constraints on $h(n)$.

Demonstrate the use of one of the filters to do interpolation of a signal $x(n)$ by a factor of 3. First upsample $x(n)$ by 3, and then filter it to get $y(n)$. Verify that the values $x(n)$ are not changed by the process. A data set will be available on the course webpage.