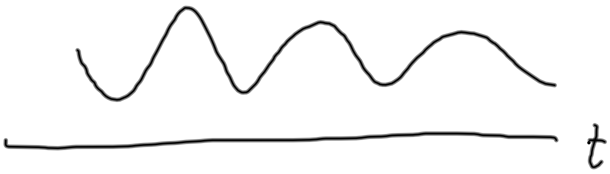
SAMPLING Thm

① Fourier Transform.

$x(t) \quad t \in \mathbb{R}$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad X(\omega) = \mathcal{F}\{x(t)\}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Inverse Fourier Transform.

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

Notation

$$x(t) \longleftrightarrow X(\omega)$$

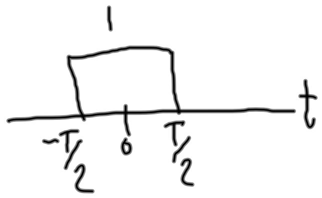
$$x(t-t_0) \longleftrightarrow e^{-jt_0\omega} X(\omega)$$

$$x(t) \cdot g(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * G(\omega)$$

$$x(t) * g(t) \longleftrightarrow X(\omega) \cdot G(\omega).$$

Example

$x(t)$



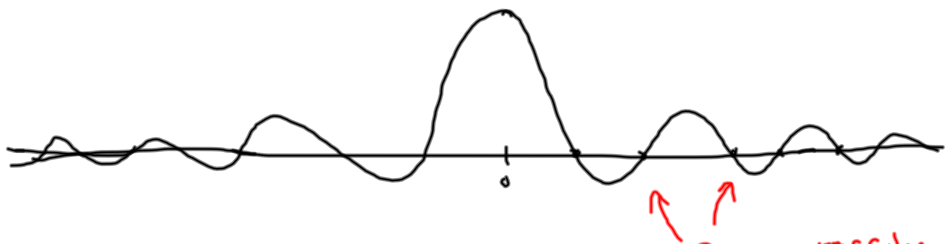
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, d\omega = \int_{-T/2}^{T/2} e^{-j\omega t}\, dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{t=-T/2}^{T/2} = \frac{j}{\omega}\Big[\underbrace{e^{-j\omega\frac{T}{2}} - e^{+j\omega\frac{T}{2}}}_{-2j\sin(\omega\frac{T}{2})}\Big]$$

$$= \frac{2}{\omega}\sin\left(\omega\frac{T}{2}\right) = \frac{\pi}{\omega}\,\mathrm{sinc}\left(\frac{T}{2\pi}\omega\right)\cdot\left(\ \right)$$

$\frac{\pi}{2\pi}\omega$

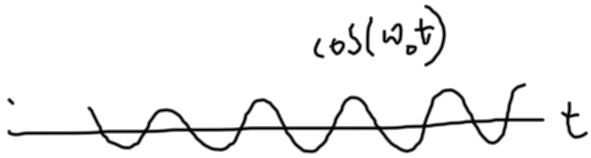
$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta} \qquad \approx= \frac{T}{\pi}\,\mathrm{sinc}\left(\frac{T}{2\pi}\omega\right)$$



zero-crossing are equally-spaced.

Example:

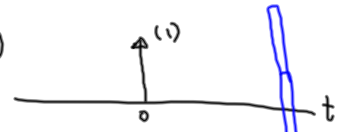
$x(t) = \cos(\omega_0 t)$.



$$X(\omega) = \int_{-\infty}^{\infty} \cos(\omega_0 t)\, e^{-j\omega t}\, dt$$

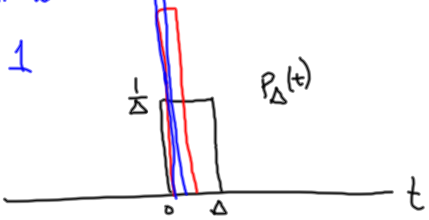
undefined due to ∞ in limit.

$\delta(t)$



① $\delta(t) = 0$ for all $t \neq 0$

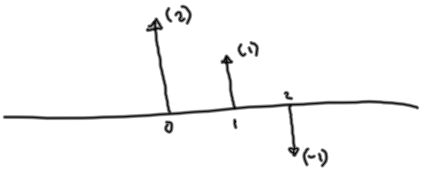
② $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$



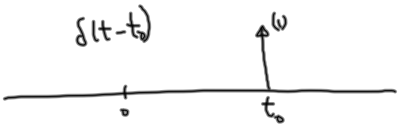
$\int P_\Delta(t)\,dt = 1$ & $P_\Delta(t) = 0$ for all $t \notin [0,\Delta]$

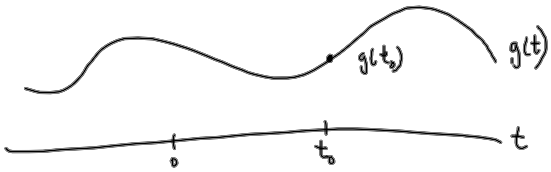
So as $\Delta \to 0$, $P_\Delta(t) \to \delta(t)$

$f(t) = 2\delta(t) + \delta(t-1) - \delta(t-2)$

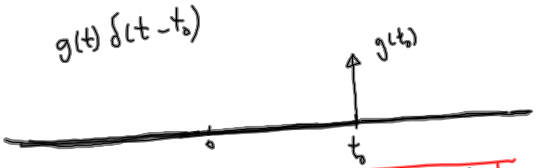


$\delta(t-t_0)$

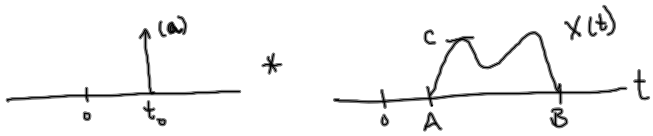




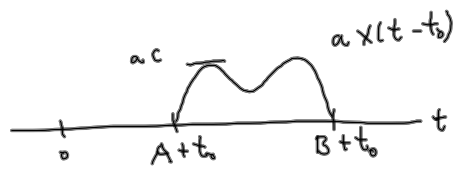
$g(t)\,\delta(t-t_0)$



$$\boxed{g(t)\,\delta(t-t_0) = g(t_0)\cdot\delta(t-t_0)}$$

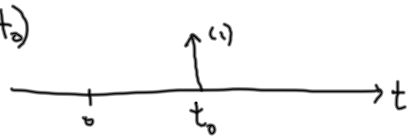


=

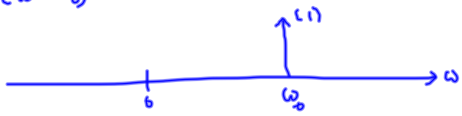


$$\boxed{a\,\delta(t-t_0) * x(t) = a\,x(t-t_0)}$$

$\delta(t-t_0)$



$\delta(\omega-\omega_0)$



---

lets set $X(\omega) = \delta(\omega-\omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\delta(\omega-\omega_0)\, e^{j\omega t}}\, d\omega = \mathcal{F}^{-1}\{X(\omega)\}$$

$\delta(\omega-\omega_0)\cdot f(\omega) =$

$\delta(\omega-\omega_0)\cdot f(\omega_0)$

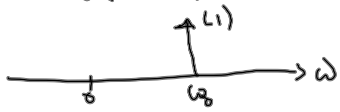
$f(\omega) = e^{j\omega t}$, $f(\omega_0) = e^{j\omega_0 t}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega_0 t}\, d\omega$$

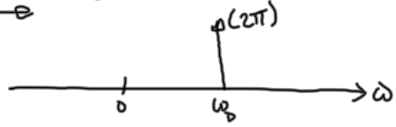
$$= \frac{1}{2\pi} e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, d\omega}_{1}$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

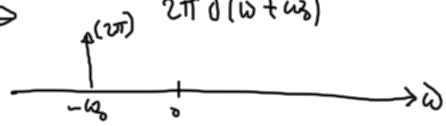
$2\pi \left[\frac{1}{2\pi} e^{j\omega_0 t} \longleftrightarrow \delta(\omega-\omega_0) \right]$



$$\boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)}$$

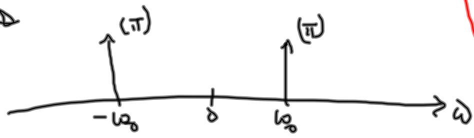


$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega+\omega_0)$$



$$\frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \longleftrightarrow$$

$= \cos(\omega_0 t)$



Fourier xform: is Linear.

$$\mathcal{F}\{a\,x_1(t) + b\,x_2(t)\} = a\,\mathcal{F}\{x_1(t)\} + b\,\mathcal{F}\{x_2(t)\}$$

$$a\,x_1(t) + b\,x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} a\,X_1(\omega) + b\,X_2(\omega)$$

Fourier Series

if $x(t+T) = x(t)$, i.e. "$x(t)$ is T-periodic"

then $$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

$\omega_0 = \frac{2\pi}{T}$

$f_0 = \frac{1}{T}$

$\omega_0 = 2\pi f_0$

frequencies $k\omega_0$.

integer multiples of the fond. freq.

"harmonics"

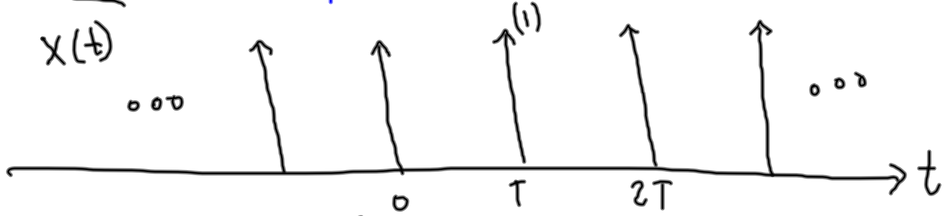
c_k : "F.S. coeffs."

$$\boxed{c_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt}$$

integrate of any interval of length T.

Example

"Impulse Train"



$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{\delta(t)\, e^{-jk\omega_0 t}} dt$$

$\delta(t) f(t) = \delta(t) f(0)$

$f(t) = e^{-jk\omega_0 t}$

$f(0) = e^0 = 1$

$$= \frac{1}{T}\underbrace{\int_{-T/2}^{T/2} \delta(t)\, dt}_{1} = \frac{1}{T}$$

$$\boxed{c_k = \frac{1}{T}}$$

Suppose $x(t)$ is T-periodic, then what is its F.T.?

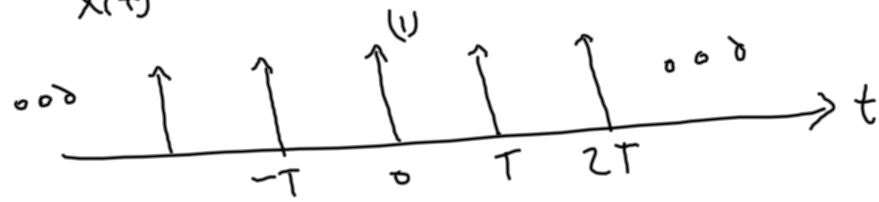
First, $$x(t) = \sum_k c_k e^{jk\omega_0 t}$$

$$X(\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_k c_k e^{jk\omega_0 t}\right\}$$

$$= \sum_k c_k \mathcal{F}\{e^{jk\omega_0 t}\}$$

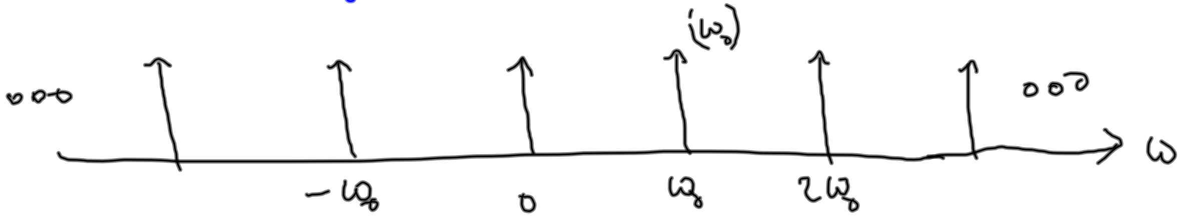
$$= \sum_k c_k \, 2\pi \delta(\omega - k\omega_0) \quad \text{Ⓐ}$$

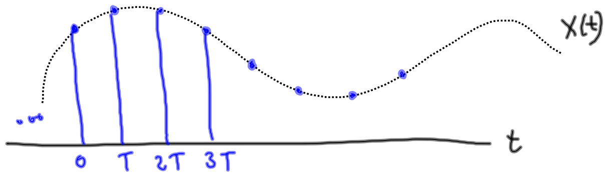
Ex. $x(t)$

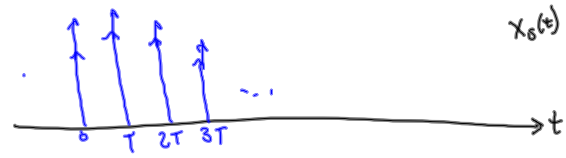


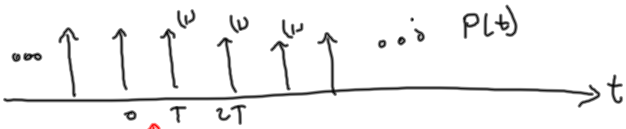
use Ⓐ with $c_k = \frac{1}{T}$

$$X(\omega) = \underbrace{\frac{2\pi}{T}}_{\omega_0} \sum_k \delta(\omega - k\omega_0)$$

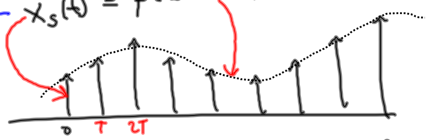








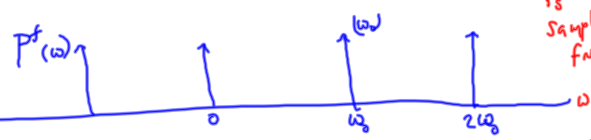
$x_s(t) = p(t) \cdot x(t)$

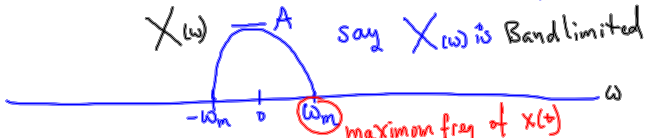


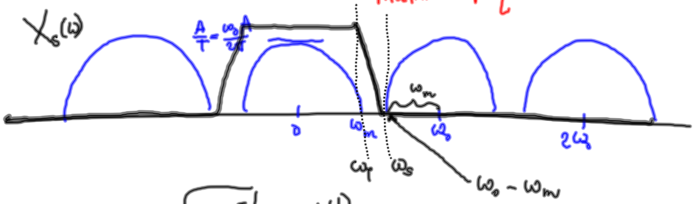
① Can we recover x(t) from its samples?
② How?

$$X_s(\omega) = \frac{1}{2\pi} P^f(\omega) * X(\omega) \qquad \omega_0 = \frac{2\pi}{T} = 2\pi F_s$$

F_s is sampling freq.







$x_s(t) \rightarrow \boxed{LPF} \rightarrow x(t)$

$$\boxed{\omega_m < \omega_p < \omega_s < \omega_0 - \omega_m}$$

$$\therefore \quad \omega_m < \omega_0 - \omega_m$$

$$2\omega_m < \omega_0$$

$$2 f_m < F_s$$

The sampling freq, F_s, must be greater than two-time the maximum frequency present in the analog signal x(t).

otherwise the adjacent copies of the spectrum will overlap, making reconstruction of x(t) impossible "ALIASING"