

Image and Video Processing

Fourier Transform and Linear Filtering Part 1: 2D Fourier Transform

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Outline of this lecture

- Part 1: 2D Fourier Transforms
- Part 2: 2D Convolution
- Part 3: Basic image processing operations:
 - Noise removal, image sharpening, and edge detection using linear filtering

2D Fourier Transform

- General concept of signals and transforms
 - Representation using basis functions
- Continuous Space Fourier Transform (CSFT)
 - 1D CTFT \rightarrow 2D CSFT
 - Concept of spatial frequency
 - Separable transform
 - Transform of separable signals
- Discrete Space Fourier Transform (DSFT) and DFT
 - 1D \rightarrow 2D

Signal in 2D Space

- General 2D continuous space signal: $f(x,y)$
 - Can have infinite support: $x,y = (-\infty, \dots, \infty)$
 - $f(x,y)$ can generally take on complex values
- General 2D discrete space signal: $f(m,n)$
 - Can have infinite support: $m,n = -\infty, \dots, 0, 1, \dots, \infty$
 - $f(m,n)$ can generally take on complex values
- Each color component of a digital image is a 2D real signal with finite support
 - $M \times N$ image: $m=0, 1, \dots, M-1, n=0, 1, \dots, N-1$
 - We will use first index for row, second index for column
 - We will consider a gray scale image or a single color component only
 - Same operations can be applied to each component

Separable Signals

- Separability
 - $f(m,n)$ is separable if $f(m,n) = f_v(m) f_h(n)$
 - $f_v(m)$: changes vertically
 - $f_h(n)$: changes horizontally
- Separable image = Rank 1 matrix
 - Rank 1 matrix = product of 1D column vector and 1D row vector

$$H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T$$

Transform Representation of 1D Signals

- Transforms are **decompositions** of a function $f(x)$ into some **basis functions** $\phi(x, u)$. u is typically the freq. index.

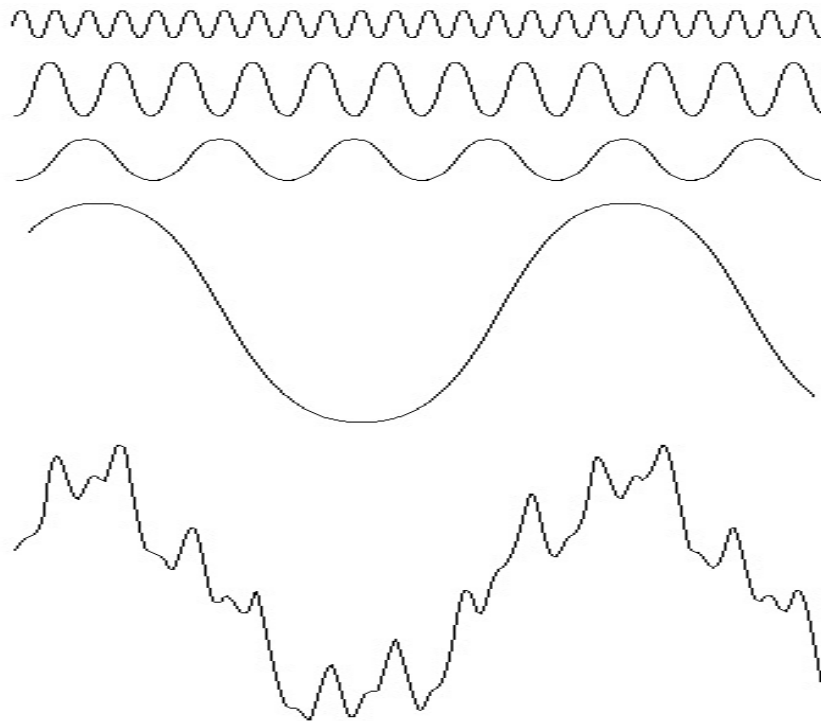
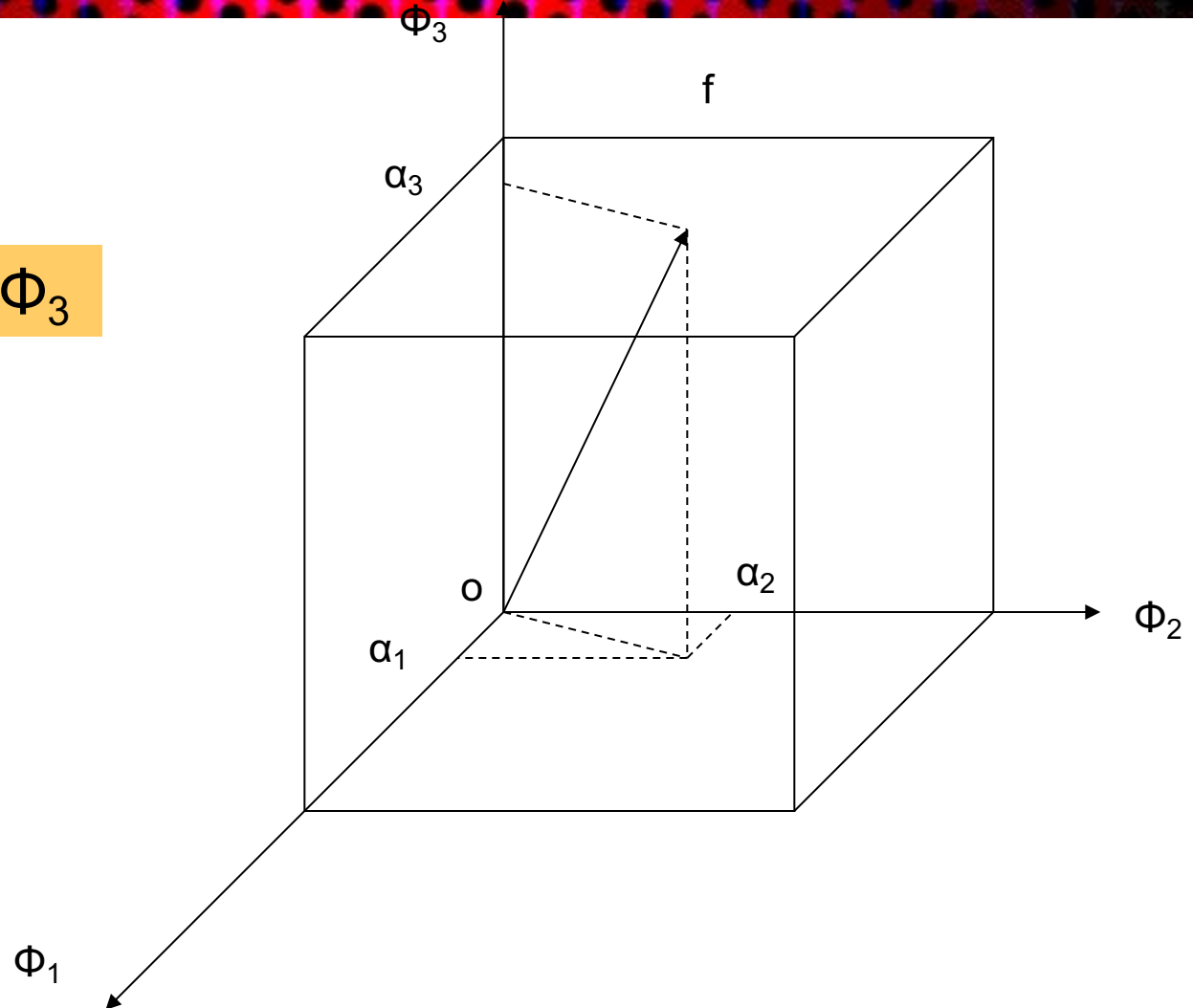


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Illustration of Decomposition in Vector Space

$$f = \alpha_1 \Phi_1 + \alpha_2 \Phi_2 + \alpha_3 \Phi_3$$



Representation of 1D Signal Using An Orthonormal Basis

- Orthonormal basis function

$$\int_{-\infty}^{\infty} \phi(x, u_1) \phi^*(x, u_2) dx = \begin{cases} 1, & u_1 = u_2 \\ 0, & u_1 \neq u_2 \end{cases}$$

Inner product of two functions

- Each basis has norm of 1
- Different bases are orthogonal to each other

- Forward transform

$$F(u) = \langle f(x), \phi(x, u) \rangle = \int_{-\infty}^{\infty} f(x) \phi^*(x, u) dx$$

Projection of $f(x)$ onto $\phi(x, u)$
= inner product of $f(x)$ and $\phi(x, u)$

- Inverse transform

$$f(x) = \int_{-\infty}^{\infty} F(u) \phi(x, u) du$$

Representing $f(x)$ as integral (limit of sum) of $\phi(x, u)$ for all u , with weight $F(u)$

1D Continuous Time Fourier Transform

- Basis functions (complex sinusoidal or exponential)

$$\phi(x, u) = e^{j2\pi ux}, \quad u \in (-\infty, +\infty).$$

u =frequency=
cycles per unit of x

$\omega=2\pi u$ (radian freq.)

- Forward Transform

$$F(u) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux} dx$$

- Inverse Transform

$$f(x) = F^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux} du$$

Here we use frequency (rather than radian frequency) to define FT. Inverse transform does not need the factor of $1/2\pi$

Important Transform Pairs

$$f(x) = 1 \Leftrightarrow F(u) = \delta(u)$$

$$f(x) = e^{j2\pi f_0 x} \Leftrightarrow F(u) = \delta(u - f_0)$$

$$f(x) = \cos(2\pi f_0 x) \Leftrightarrow F(u) = \frac{1}{2}(\delta(u - f_0) + \delta(u + f_0))$$

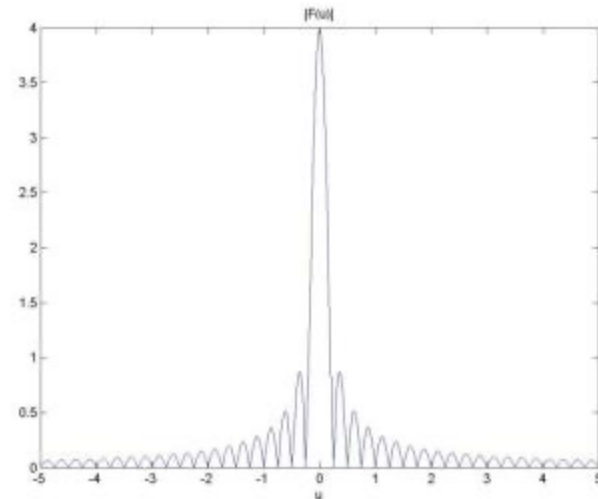
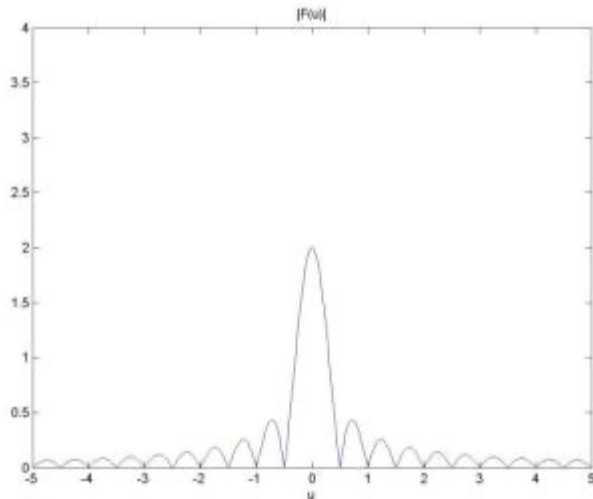
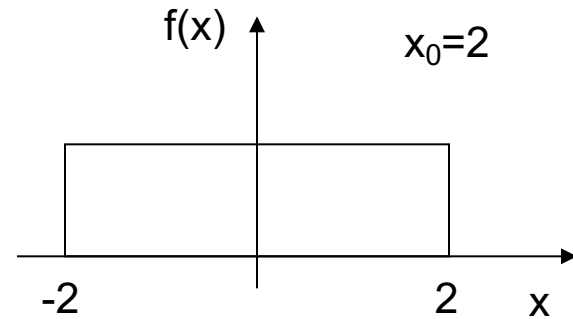
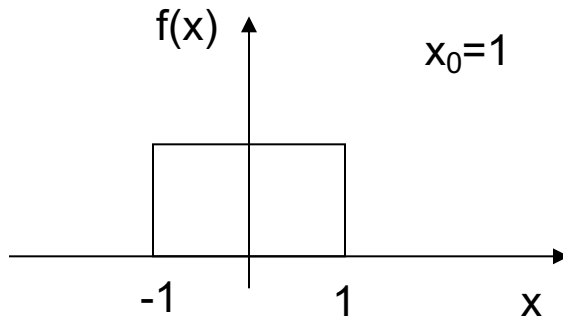
$$f(x) = \sin(2\pi f_0 x) \Leftrightarrow F(u) = \frac{1}{2j}(\delta(u - f_0) - \delta(u + f_0))$$

$$f(x) = \begin{cases} 1, & |x| < x_0 \\ 0, & \text{otherwise} \end{cases} \Leftrightarrow F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \operatorname{sinc}(2x_0 u)$$

$$\text{where, } \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

FT of the Rectangle Function

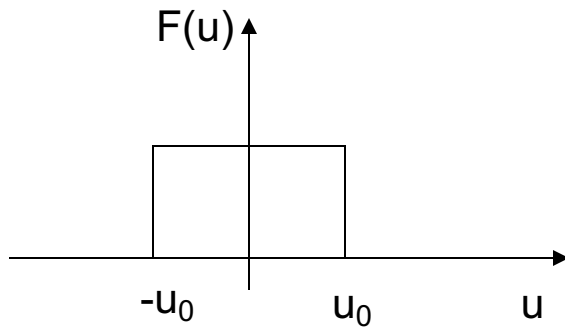
$$F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \operatorname{sinc}(2x_0 u) \quad \text{where, } \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



Note first zero occurs at $u_0=1/(2 x_0)=1/\text{pulse-width}$, other zeros are multiples of this.

IFT of Ideal Low Pass Signal

- What is $f(x)$?



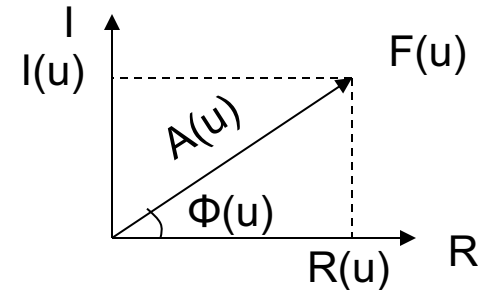
Representation of FT

- Generally, both $f(x)$ and $F(u)$ are complex
- Two representations
 - Real and Imaginary
 - Magnitude and Phase

$$F(u) = R(u) + jI(u)$$

$$F(u) = A(u)e^{j\phi(u)}, \quad \text{where}$$

$$A(u) = \sqrt{R(u)^2 + I(u)^2}, \quad \phi(u) = \tan^{-1} \frac{I(u)}{R(u)}$$



- Relationship

$$R(u) = A(u) \cos \phi(u), \quad I(u) = A(u) \sin \phi(u)$$

- Power spectrum

$$P(u) = A(u)^2 = F(u) \times F(u)^* = |F(u)|^2$$

What if $f(x)$ is real?

- Real world signals $f(x)$ are usually real
- $F(u)$ is still complex, but has special properties

$$F^*(u) = F(-u)$$

$$R(u) = R(-u), A(u) = A(-u), P(u) = P(-u) : \text{even function}$$

$$I(u) = -I(-u), \phi(u) = -\phi(-u) : \text{odd function}$$

Properties of Fourier Transform

- Duality

$$f(t) \Leftrightarrow F(u)$$

$$F(t) \Leftrightarrow f(-u)$$

- Linearity

$$F\{a_1 f_1(x) + a_2 f_2(x)\} = a_1 F\{f_1(x)\} + a_2 F\{f_2(x)\}$$

- Scaling

$$F\{af(x)\} = aF\{f(x)\}$$

- Translation

$$f(x - x_0) \Leftrightarrow F(u)e^{-j2\pi x_0 u}, \quad f(x)e^{j2\pi u_0 x} \Leftrightarrow F(u - u_0)$$

- Convolution

$$f(x) \otimes g(x) = \int f(x - \alpha)g(\alpha)d\alpha$$

$$f(x) \otimes g(x) \Leftrightarrow F(u)G(u)$$

We will review convolution later!

Two Dimension Continuous Space Fourier Transform (CSFT)

- Basis functions (separable functions)

$$\phi(x, y; u, v) = e^{j(2\pi ux + 2\pi vy)} = e^{j2\pi ux} e^{j2\pi vy}, \quad u, v \in (-\infty, +\infty).$$

- Forward Transform

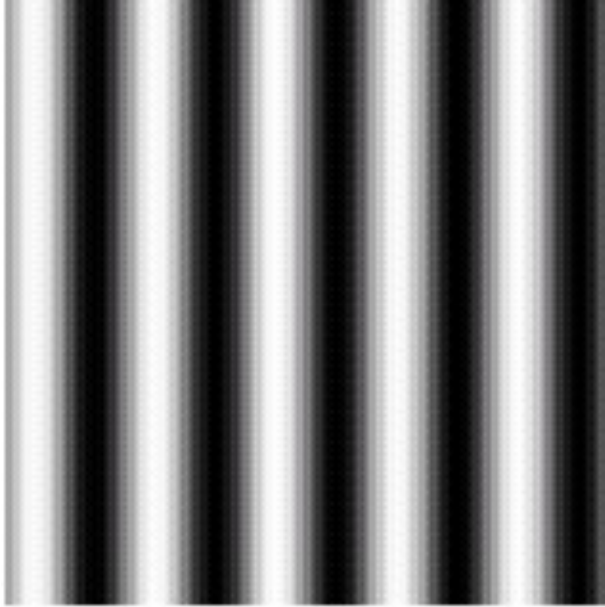
$$F(u, v) = F\{f(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx dy$$

- Inverse Transform

$$f(x, y) = F^{-1}\{F(u, v)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} du dv$$

- Representing a 2D signal as sum of 2D complex exponential signals

Illustration of Spatial Frequency

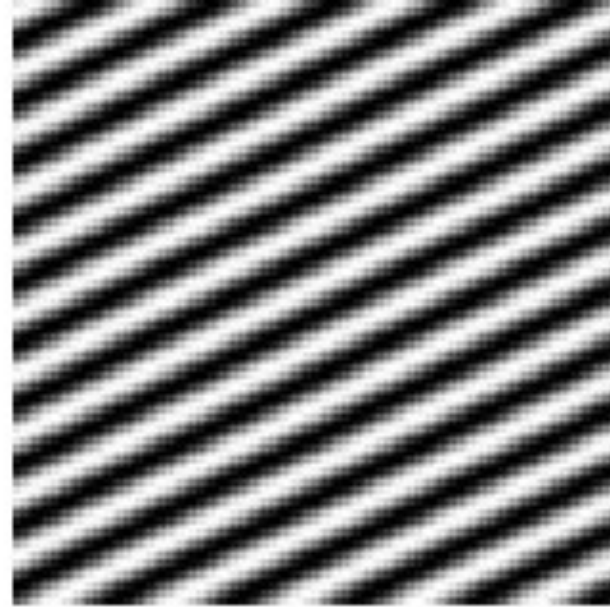


$$f(x, y) = \sin(10\pi x)$$

$$f_x = 5, f_y = 0, f_m = 5, \phi = 0$$

f_x : unit= 1 cycle/image-width

f_y : unit=1 cycle/image-height



$$f(x, y) = \sin(10\pi x - 20\pi y)$$

$$f_x = 5, f_y = -10, f_m = \sqrt{125}, \phi = \text{atan}(-2)$$

f_x and f_y are used to represent the spatial freq. instead of u, v .

Need two components to describe spatial frequency: either frequency in two orthogonal directions (does not have to be horizontal and vertical) or amplitude and direction

Example 1

$$f(x, y) = \sin 4\pi x + \cos 6\pi y$$

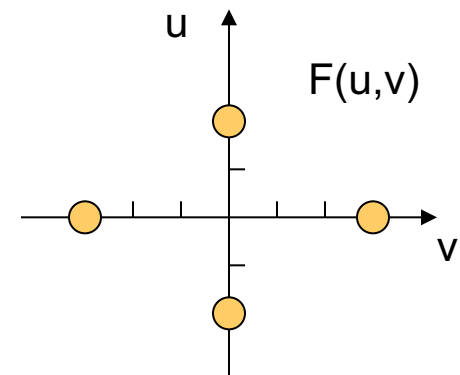
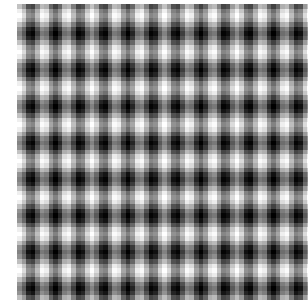
$$\begin{aligned} F\{\sin 4\pi x\} &= \iint \sin 4\pi x e^{-j2\pi(ux+vy)} dx dy \\ &= \int \sin 4\pi x e^{-j2\pi ux} dx \int e^{-j2\pi vy} dy \\ &= \int \sin 4\pi x e^{-j2\pi ux} dx \delta(v) \\ &= \frac{1}{2j} (\delta(u-2) - \delta(u+2)) \delta(v) \\ &= \frac{1}{2j} (\delta(u-2, v) - \delta(u+2, v)) \end{aligned}$$

$$\text{where } \delta(x, y) = \delta(x)\delta(y) = \begin{cases} \infty, & x = y = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Likewise, } F\{\cos 6\pi y\} = \frac{1}{2} (\delta(u, v-3) + \delta(u, v+3))$$

$$\text{Recall } F\{e^{j2\pi u_0 t}\} = \delta(u - u_0)$$

$f(x,y)$



Example 2

$$f(x, y) = \sin(2\pi x + 3\pi y) = \frac{1}{2j} \left(e^{j(2\pi x + 3\pi y)} - e^{-j(2\pi x + 3\pi y)} \right)$$

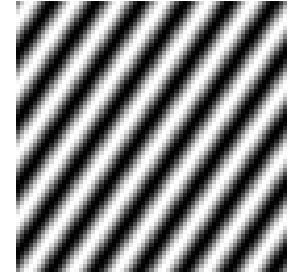
$$F\{e^{j(2\pi x + 3\pi y)}\} = \iint e^{j(2\pi x + 3\pi y)} e^{-j2\pi(xu + yv)} dx dy$$
$$= \int e^{j2\pi x} e^{-j2\pi ux} dx \int e^{j3\pi y} e^{-j2\pi yv} dy$$

$$= \delta(u - 1) \delta(v - \frac{3}{2}) = \delta(u - 1, v - \frac{3}{2})$$

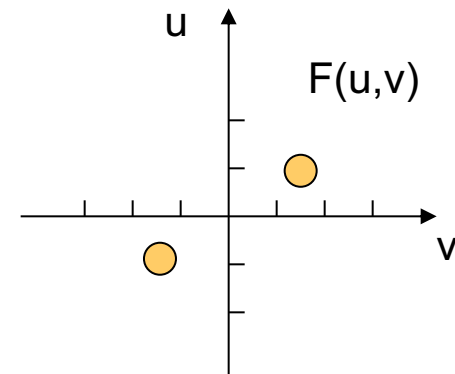
Likewise, $F\{e^{-j(2\pi x + 3\pi y)}\} = \delta(u + 1, v + \frac{3}{2})$

Therefore,

$$F\{\sin(2\pi x + 3\pi y)\} = \frac{1}{2j} \left(\delta(u - 1, v - \frac{3}{2}) - \delta(u + 1, v + \frac{3}{2}) \right)$$



```
[X,Y]=meshgrid(-2:1/16:2,-2:1/16:2);  
f=sin(2*pi*X+3*pi*Y);  
imagesc(f); colormap(gray)  
Truesize, axis off;
```



In this and following example, x / u is vertical, y / v is horizontal

A Little Teaser ☺

$$f(x, y) = \sin(10\pi x)$$

$$f_x = 5, f_y = 0, f_m = 5, \phi = 0$$

f_x : unit= 1 cycle/image-width

f_y : unit=1 cycle/image-height

What is its 2D FT?

Properties of 2D FT (1)

- Linearity

$$F\{a_1 f_1(x, y) + a_2 f_2(x, y)\} = a_1 F\{f_1(x, y)\} + a_2 F\{f_2(x, y)\}$$

- Translation / modulation

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(x_0 u + y_0 v)},$$

$$f(x, y) e^{j2\pi(u_0 x + v_0 y)} \Leftrightarrow F(u - u_0, v - v_0)$$

- Conjugation

$$f^*(x, y) \Leftrightarrow F^*(-u, -v)$$

Properties of 2D FT (2)

- Symmetry

$$f(x, y) \text{ is real} \Leftrightarrow |F(u, v)| = |F(-u, -v)|$$

- Convolution

- Definition of convolution

$$f(x, y) \otimes g(x, y) = \iint f(x - \alpha, y - \beta) g(\alpha, \beta) d\alpha d\beta$$

- Convolution theory

$$f(x, y) \otimes g(x, y) \Leftrightarrow F(u, v)G(u, v)$$

We will describe 2D convolution later!

Separability of 2D FT and Separable Signal

- Separability of 2D FT

$$F_2\{f(x, y)\} = F_y\{F_x\{f(x, y)\}\} = F_x\{F_y\{f(x, y)\}\}$$

- where F_x, F_y are 1D FT along x and y .
- one can do 1DFT for each row of original image, then 1D FT along each column of resulting image

- Separable Signal

- $f(x, y) = f_x(x)f_y(y)$
- $F(u, v) = F_x(u)F_y(v)$,
 - where $F_x(u) = F_x\{f_x(x)\}$, $F_y(v) = F_y\{f_y(y)\}$
- For separable signals, one can simply compute two 1D transforms and take their product!

Example 1

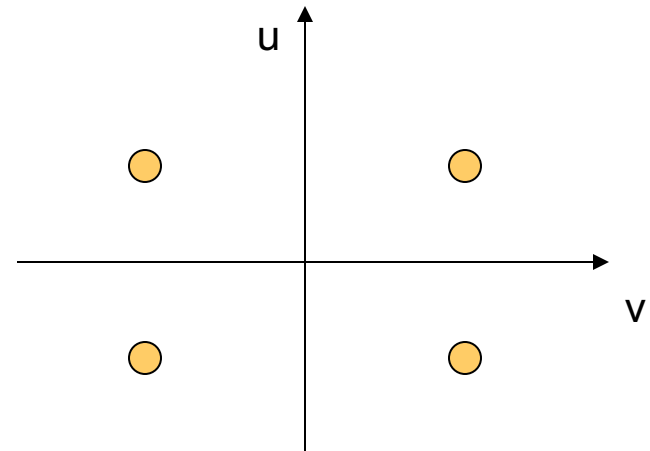
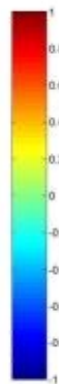
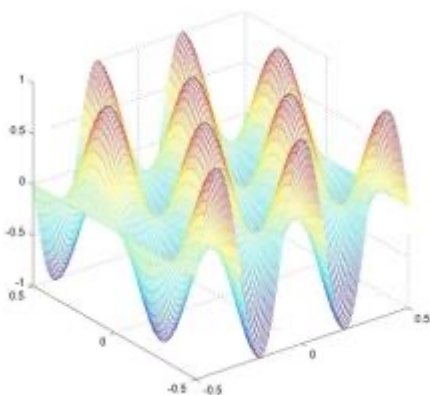
$$f(x, y) = \sin(3\pi x) \cos(5\pi y)$$

$$f_x(x) = \sin(3\pi x) \Leftrightarrow F_x(u) = \frac{1}{2j} (\delta(u - 3/2) - \delta(u + 3/2))$$

$$f_y(y) = \cos(5\pi y) \Leftrightarrow F_y(v) = \frac{1}{2} (\delta(v - 5/2) + \delta(v + 5/2))$$

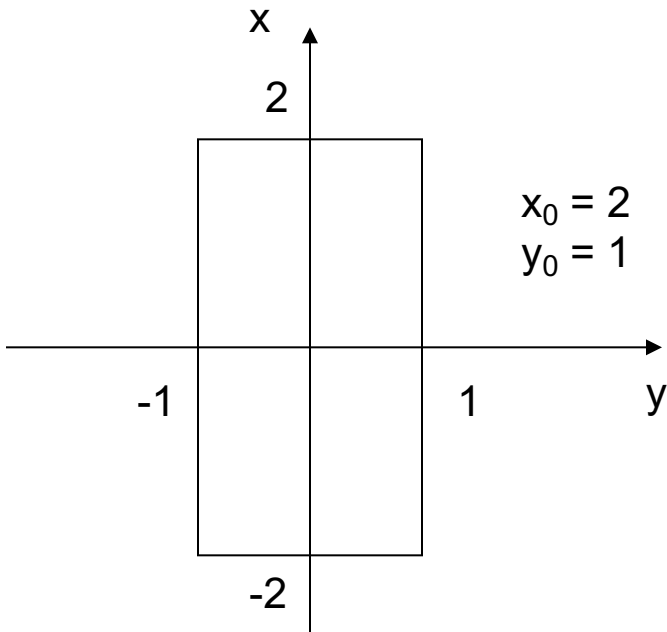
$$F(u, v) = F_x(u)F_y(v)$$

$$= \frac{1}{4j} \left(\delta\left(u - \frac{3}{2}, v - \frac{5}{2}\right) - \delta\left(u + \frac{3}{2}, v - \frac{5}{2}\right) + \delta\left(u - \frac{3}{2}, v + \frac{5}{2}\right) - \delta\left(u + \frac{3}{2}, v + \frac{5}{2}\right) \right)$$

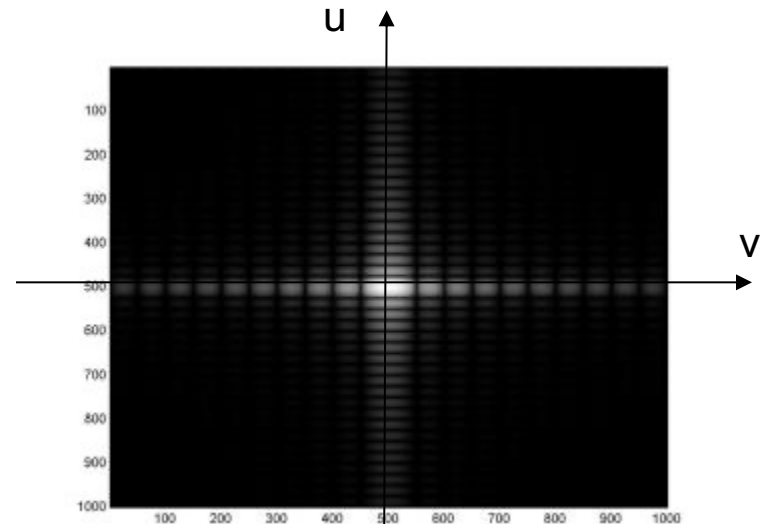
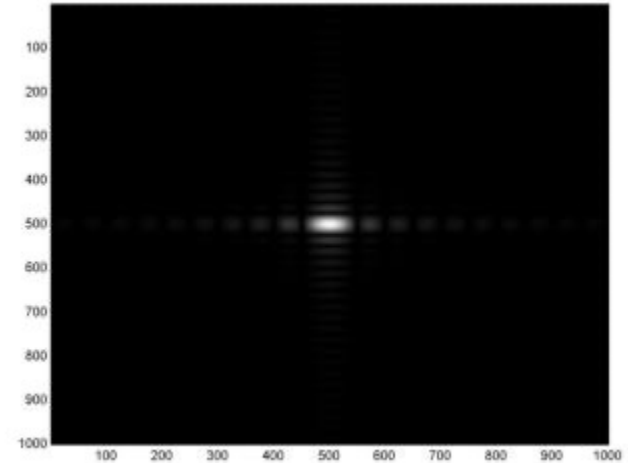


Example 2

$$f(x, y) = \begin{cases} 1, & |x| \leq x_0, |y| \leq y_0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow$$
$$F(u, v) = 4x_0y_0 \operatorname{sinc}(2x_0u) \operatorname{sinc}(2y_0v)$$



Is this signal separable?



w/ contrast enhancement through log mapping

Important Transform Pairs

- All following signals are separable and the relations can be proved by applying the separability of the CSFT

$$f(x, y) = 1 \Leftrightarrow F(u, v) = \delta(u, v), \text{ where } \delta(u, v) = \delta(u)\delta(v)$$

$$f(x) = e^{j2\pi(f_1x + f_2y)} \Leftrightarrow F(u) = \delta(u - f_1, v - f_2)$$

$$f(x, y) = \cos(2\pi(f_1x + f_2y)) \Leftrightarrow F(u) = \frac{1}{2}(\delta(u - f_1, v - f_2) + \delta(u + f_1, v + f_2))$$

$$f(x, y) = \sin(2\pi(f_1x + f_2y)) \Leftrightarrow F(u) = \frac{1}{2j}(\delta(u - f_1, v - f_2) - \delta(u + f_1, v + f_2))$$

$$f(x, y) = \begin{cases} 1, & |x| < x_0, |y| < y_0 \\ 0, & \text{otherwise} \end{cases} \Leftrightarrow$$

$$F(u, v) = \frac{\sin(2\pi x_0 u)}{\pi u} \frac{\sin(2\pi y_0 v)}{\pi v} = 4x_0 y_0 \operatorname{sinc}(2x_0 u) \operatorname{sinc}(2y_0 v)$$

$$\text{where } \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$$\frac{1}{2\pi\sigma^2} \exp\left\{-\frac{x^2 + y^2}{2\sigma^2}\right\} \Leftrightarrow \exp\left\{-\frac{u^2 + v^2}{2\beta^2}\right\}, \beta = \frac{1}{2\pi\sigma}$$

- Constant \Leftrightarrow impulse at (0,0) freq.
- Complex exponential \Leftrightarrow impulse at a particular 2D freq.
- 2D box \Leftrightarrow 2D sinc function
- Gaussian \Leftrightarrow Gaussian

Rotation

- Let $x = r \cos \theta$, $y = r \sin \theta$, $u = \rho \cos \omega$, $v = \rho \sin \omega$.
- 2D FT in polar coordinate (r, θ) and (ρ, ϕ)

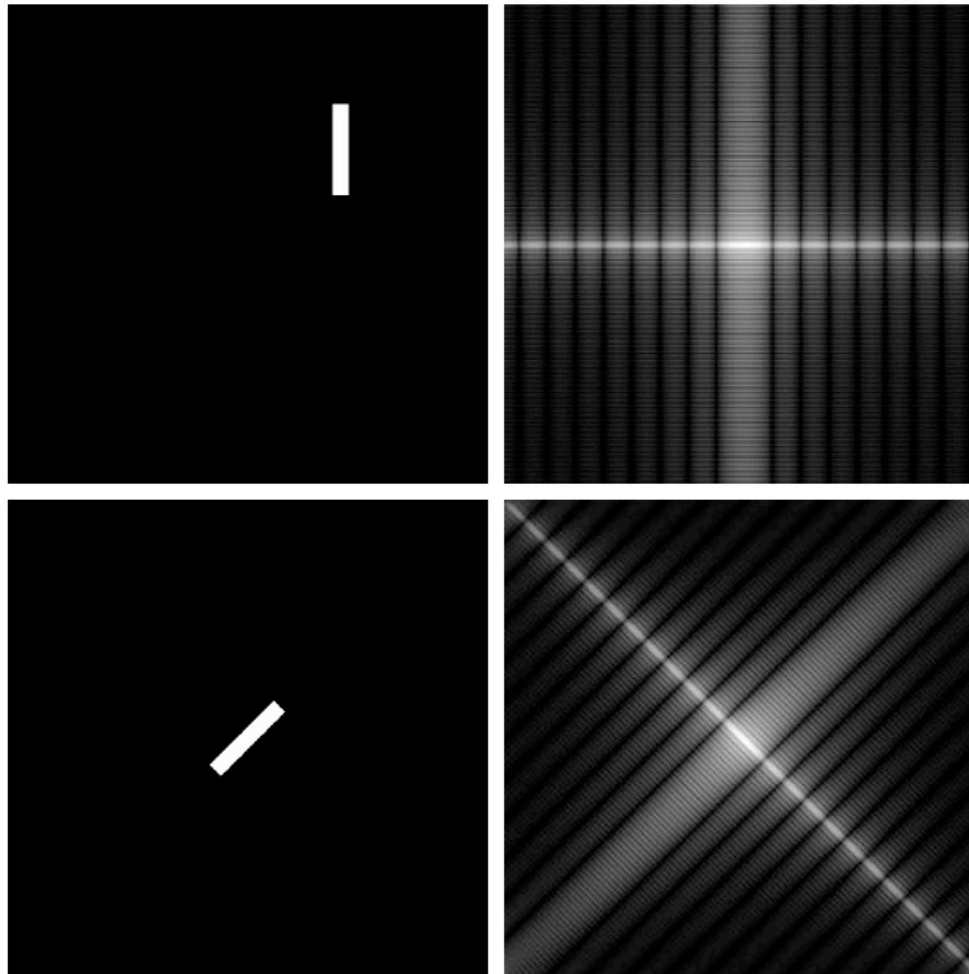
$$\begin{aligned} F(\rho, \phi) &= \int_0^{\infty} \int_0^{2\pi} f(r, \theta) e^{-j2\pi(r \cos \theta \rho \cos \phi + r \sin \theta \rho \sin \phi)} r dr d\theta \\ &= \iint f(r, \theta) e^{-j2\pi r \rho \cos(\theta - \phi)} r dr d\theta \end{aligned}$$

- Property

$$f(r, \theta + \theta_0) \Leftrightarrow F(\rho, \phi + \theta_0)$$

- Proof: Homework!
- Note: In this notation: x is horizontal index, y is vertical

Example of Rotation



| | |
|---|---|
| a | b |
| c | d |

FIGURE 4.25

(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

From Gonzalez

1D Fourier Transform For Discrete Time Sequence (DTFT) (Review)

- $f(n)$ is a 1D discrete time sequence
- Forward Transform

$$F(u) = \sum_{n=-\infty}^{\infty} f(n)e^{-j2\pi un}$$

- Inverse Transform

$$f(n) = \int_{-1/2}^{1/2} F(u)e^{j2\pi un} du$$

- Representing $f(n)$ as a weighted sum of many complex sinusoidal signals with frequency u , $F(u)$ is the weight
- $F(u)$ indicate the “amount” of sinusoidal component with freq. u , $e^{j2\pi un}$, in signal $f(n)$
- $|u|$ = digital frequency = the number of cycles per integer sample.
Period = $1/|u|$ (must be equal or greater than 2 samples $\rightarrow |u| \leq 1/2$)

Properties unique for DTFT

- Periodicity
 - $F(u) = F(u+1) = F(u+k)$, $k = \text{integers}$
 - The FT of a discrete time sequence is only considered for $u \in (-\frac{1}{2}, \frac{1}{2})$, and $u = \pm\frac{1}{2}$ is the highest discrete frequency
- Symmetry for real sequences

$$f(n) = f^*(n) \Leftrightarrow F(u) = F^*(-u)$$

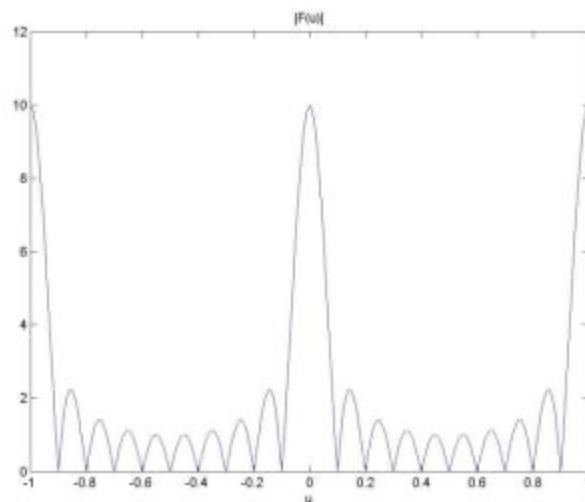
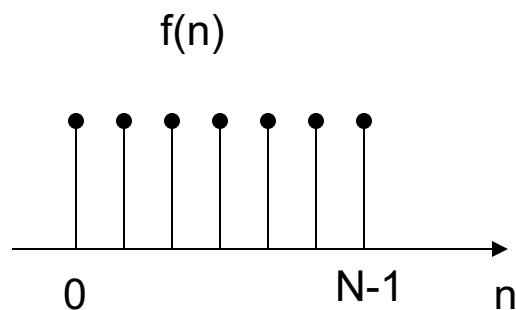
$$\Rightarrow |F(u)| = |F(-u)|$$

$$\Rightarrow |F(u)| \text{ is symmetric}$$

Example

$$f(n) = \begin{cases} 1, & n = 0, 1, \dots, N-1; \\ 0, & \text{others} \end{cases}$$

$$F(u) = \sum_{n=0}^{N-1} e^{-j2\pi nu} = \frac{1 - e^{-j2\pi uN}}{1 - e^{-j2\pi u}} = e^{-j\pi(N-1)u} \frac{\sin 2\pi u(N/2)}{\sin 2\pi u(1/2)}$$



N=10

Digital sinc: There are $N/2-1$ zeros in $(0, 1/2]$, $1/N$ apart

1D Discrete Fourier Transform (DFT)

- DTFT of N-pt sequence

$$F(u) = \sum_{n=0}^{N-1} f(n)e^{-j2\pi un}$$

- N-pt DFT of N-pt sequence

$$F_N(k) = \sum_{n=0}^{N-1} f(n)e^{-j2\pi \frac{kn}{N}} = F\left(u = \frac{k}{N}\right)$$

- DFT is the sampled version of DTFT with samples at $0, 1/N, \dots, (N-1)/N$, if $f(n)$ has length $\leq N$
- FFT: Fast algorithm for computing DFT
 - Direct computation takes N^2 operations
 - FFT takes $\sim N \log(N)$ operations!

Discrete Space Fourier Transform (DSFT) for Two Dimensional Signals

- Let $f(m,n)$ represent a 2D sequence
- Forward Transform

$$F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m, n) e^{-j2\pi(mu+nv)}$$

- Inverse Transform

$$f(m, n) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u, v) e^{j2\pi(mu+nv)} du dv$$

- Representing an image as weighted sum of many 2D complex sinusoidal images $e^{j2\pi(mu+nv)}$ for different u, v
- u : number of cycles per vertical sample (vertical freq.)
- v : number of cycles per horizontal sample (horizontal freq)

Spatial Frequency for Digital Images

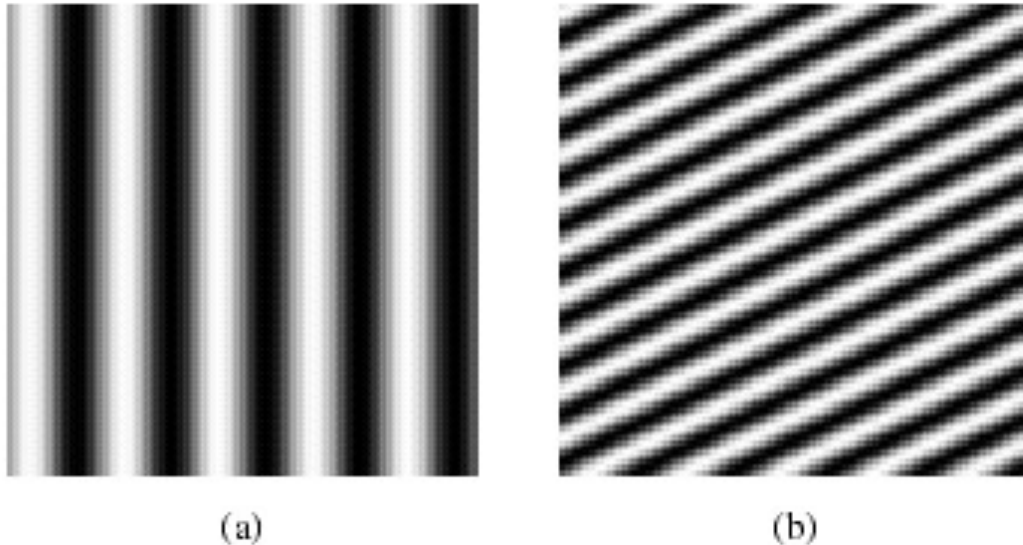


Figure 2.1 Two-dimensional sinusoidal signals: (a) $(f_x, f_y) = (5, 0)$; (b) $(f_x, f_y) = (5, 10)$. The horizontal and vertical units are the width and height of the image, respectively. Therefore, $f_x = 5$ means that there are five cycles along each row.

$$f_s = \sqrt{125}, \quad \varphi = \arctan(2)$$

If the image has 256x256 pixels, $f_x=5$ cycles per width (analog frequency) $\rightarrow u = 5$ cycles/256 pixels = 5/256 cycles/sample (digital frequency)

When both horizontal and vertical frequency are non-zero, we see directional patterns. f_s is the frequency along the direction with the maximum change (orthogonal to the lines)

Note that $1/u$ may not correspond to integer.

If $u=a/b$, digital period = b . Ex $u=3/8$, Digital Period = 8

Periodicity

$$F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m, n) e^{-j2\pi(mu+nv)}$$

- $F(u, v)$ is periodic in both u, v with period 1, i.e., for all integers k, l :
 - $F(u+k, v+l) = F(u, v)$
 - Only need to consider the rectangle area defined by $(-1/2, 1/2)$
- To see this consider

$$\begin{aligned} & e^{-j2\pi(m(u+k)+n(v+l))} \\ &= e^{-j2\pi(mu+nv)} e^{-j2\pi(mk+nl)} \\ &= e^{-j2\pi(mu+nv)} \end{aligned}$$

Example: Delta Function

- Fourier transform of a delta function

$$F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(m, n) e^{-j2\pi(mu+nv)} = 1$$

$$\delta(m, n) \Leftrightarrow 1$$

- Delta function contains all frequency components with equal weights!

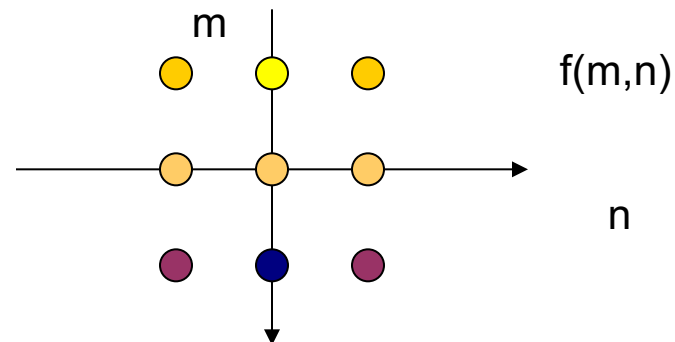
- Inverse Fourier transform of a delta function

$$f(m, n) = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} \delta(u, v) e^{j2\pi(mu+nv)} du dv = 1$$

$$1 \Leftrightarrow \delta(u, v)$$

Example

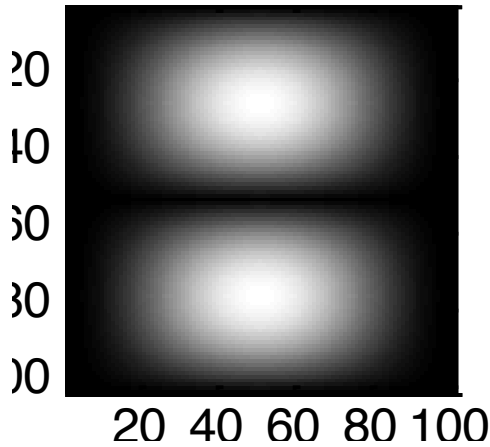
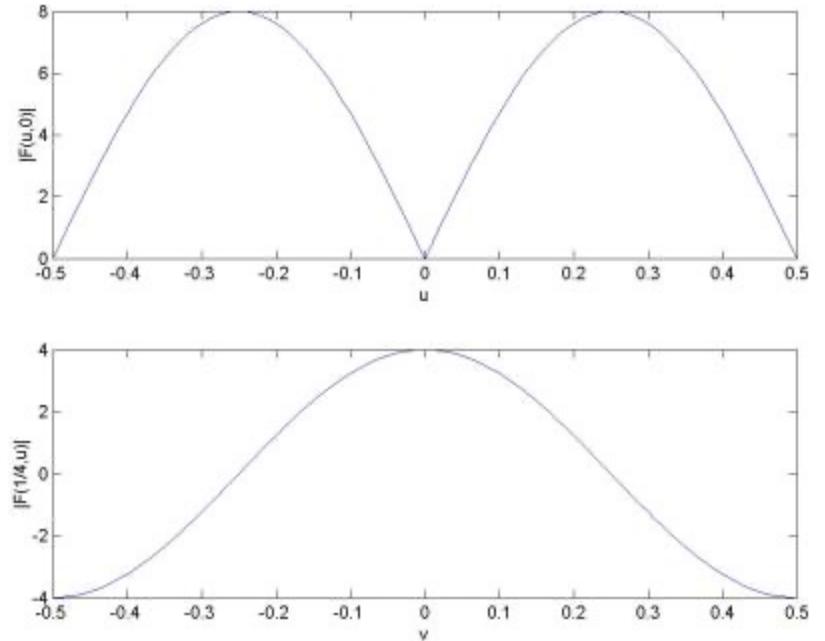
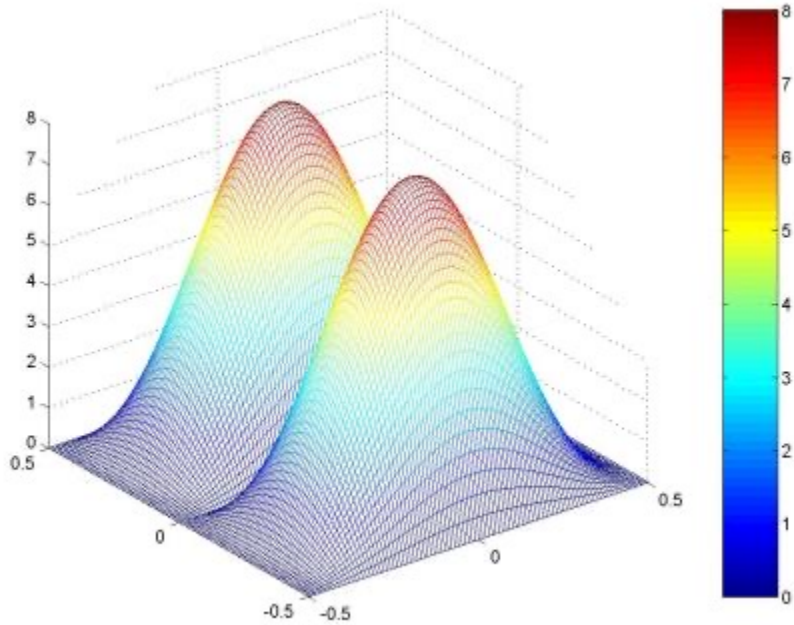
$$f(m,n) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$



$$\begin{aligned} F(u,v) &= 1e^{-j2\pi(-1*u-1*v)} + 2e^{-j2\pi(-1*u+0*v)} + 1e^{-j2\pi(-1*u+1*v)} \\ &\quad - 1e^{-j2\pi(1*u-1*v)} - 2e^{-j2\pi(1*u+0*v)} - 1e^{-j2\pi(1*u+1*v)} \\ &= j2 \sin 2\pi u e^{j2\pi v} + j4 \sin 2\pi u + j2 \sin 2\pi u e^{-j2\pi v} \\ &= j4 \sin 2\pi u (\cos 2\pi v + 1) \end{aligned}$$

Note: This signal is low pass in the horizontal direction v , and **band pass** in the vertical direction u .

Graph of $F(u,v)$



```

du = [-0.5:0.01:0.5];
dv = [-0.5:0.01:0.5];
Fu = abs(sin(2 * pi * du));
Fv = cos(2 * pi * dv);
F = 4 * Fu' * (Fv + 1);
mesh(du, dv, F);
colorbar;
Imagesc(F);
colormap(gray); truesize;
    
```

```

Using MATLAB freqz2:
f=[1,2,1;0,0,0;-1,-2,-1];
freqz2(f)
    
```

DSFT vs. 2D DFT

- 2D DSFT

$$F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m, n) e^{-j2\pi(mu+nv)}$$

- 2D DFT

$$F(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi(mk/M+nl/N)}$$

- (MxN) point 2D DFT are samples of DSFT for images at $u=k/M$, $v=l/N$, as long as M and N are equal to or greater than the height and width of the image, respectively. Can be computed using FFT algorithms. 1D FFT along rows, then 1D FFT along columns
 - Total complexity:
 - $M*N \log(N) + N*M \log(M)$ or $2M^2 \log(M)$ for $M=N$

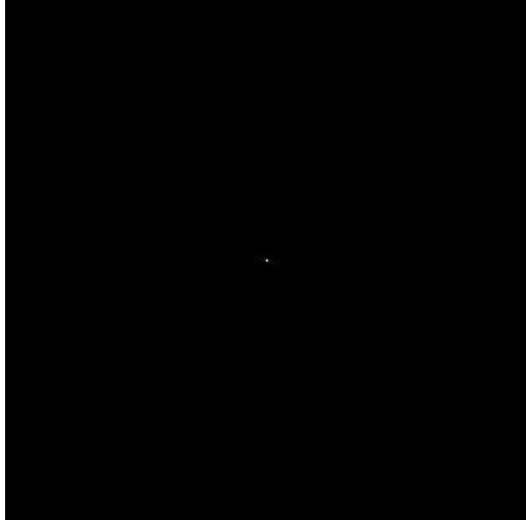
Display of DFT of Images



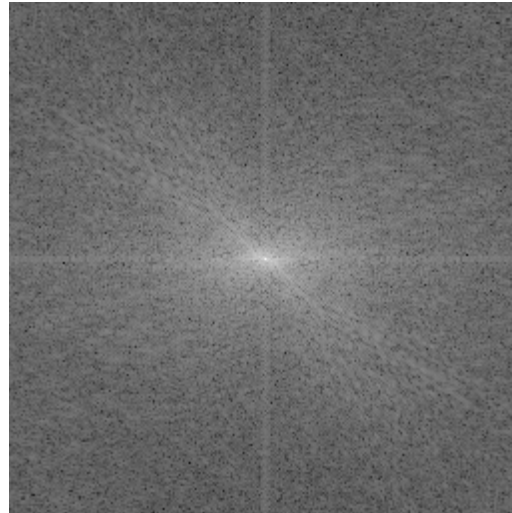
`imshow(img)`



`MFimg=abs(fft2(img)), imshow(MFimg,[])`



`SMFimg=fftshift(MFimg)`
`imshow(SMFimg,[])`

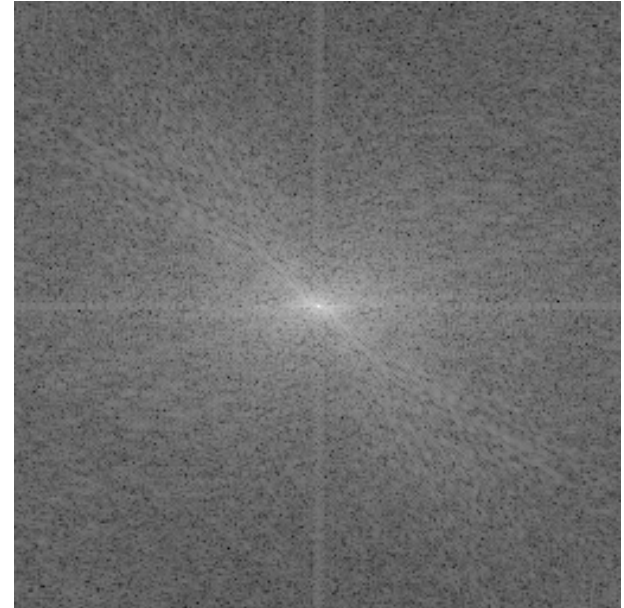


`LSMFimg=log(SMFimg+1);`
`imshow(LSMFimg,[])`

`fftshift` to shift the (0,0) freq.
to center

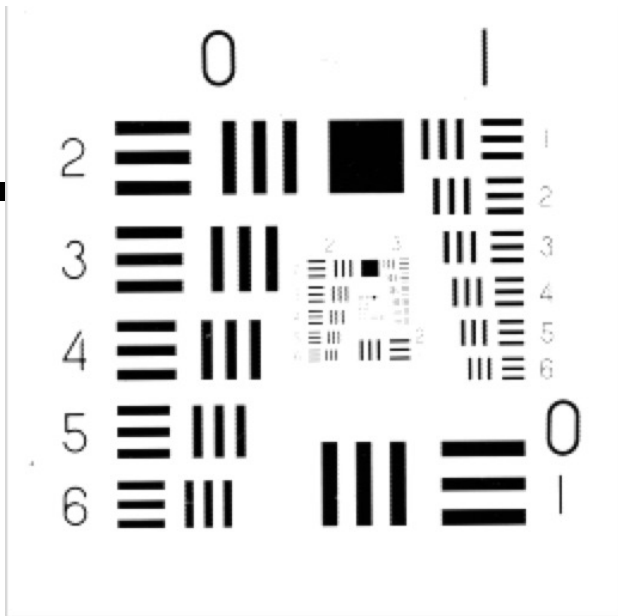
Log mapping to enhance
contrast (boost low values!)

DFT of Typical Images

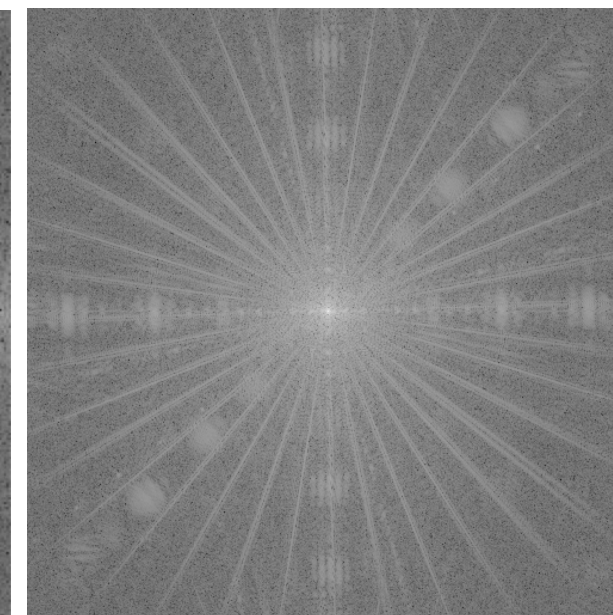
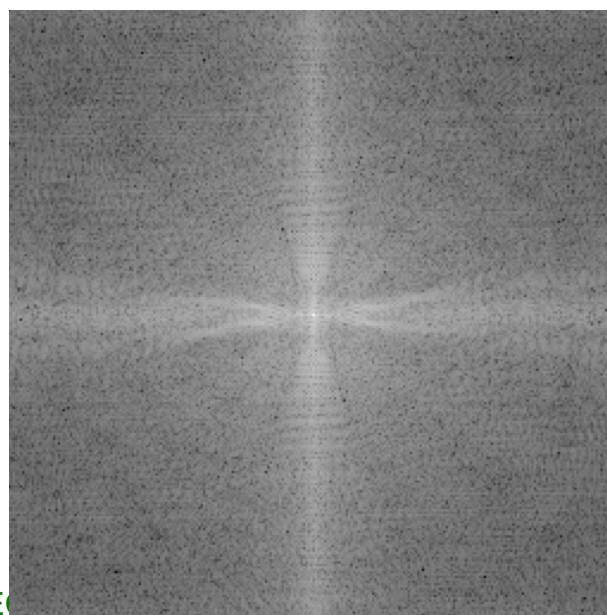
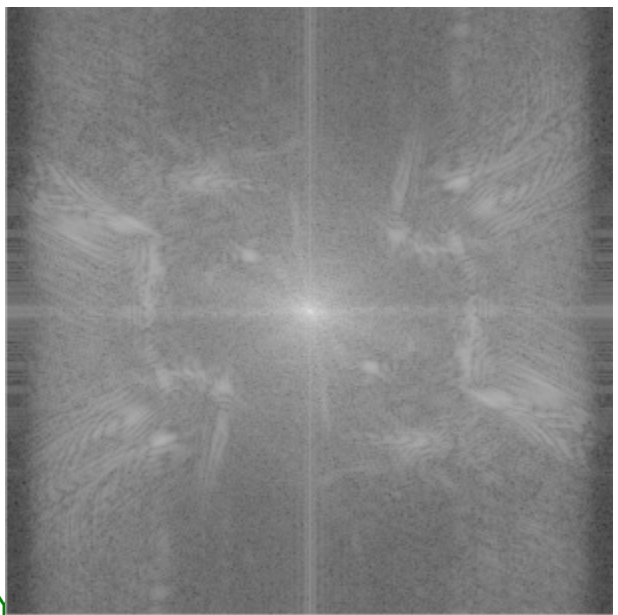


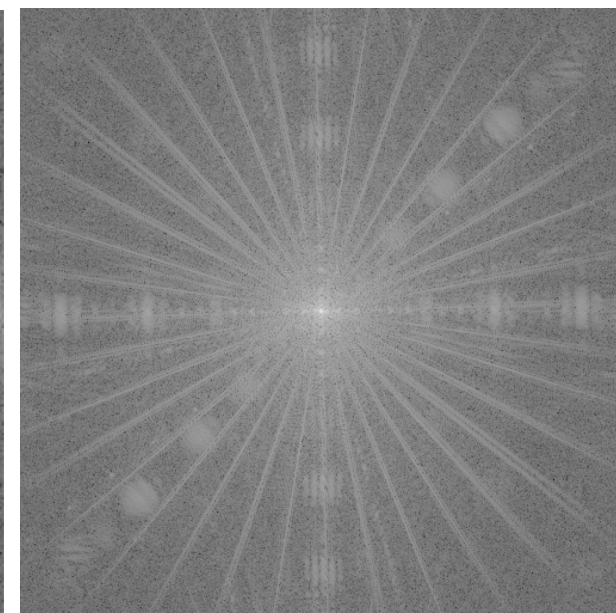
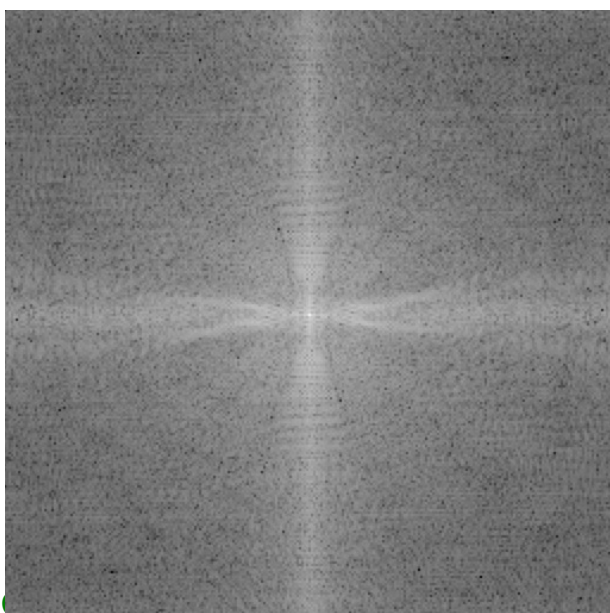
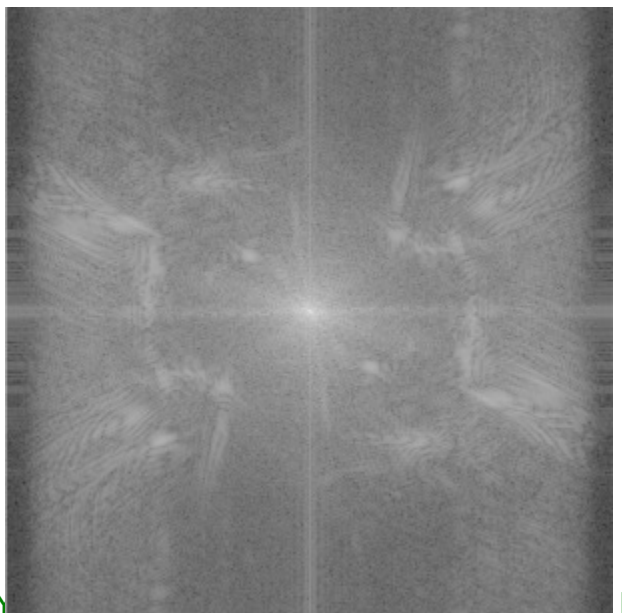
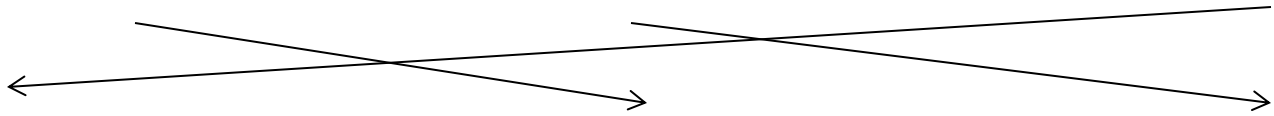
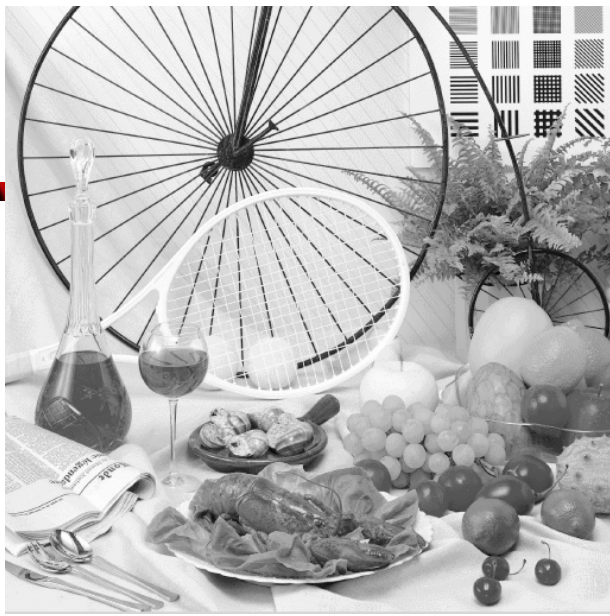
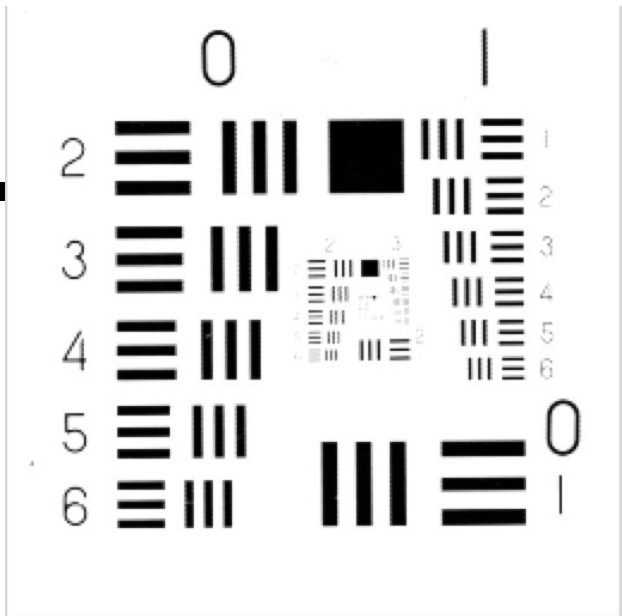
How to interpret the DFT image?
Why is it bright at center and has
some line structures?

$|F(u,v)|$ (obtained using 2D FFT)
`ff=abs(fft2(f));`
`imagesc(fftshift(log(ff+1)));`
Log mapping to enhance contrast
Fftshift to shift the (0,0) freq. to center



Which one below is the DFT of which one above?





DSFT of Separable Signals

- Separable signal:
 - $h(m,n)=h_x(m) h_y(n)$

$$H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T$$

- All the rows are scaled versions of each other, all the columns are scaled versions of each other
 - Image can be represented as the product of a 1D column vector and a 1D row vector
- 2D DSFT of separable signal = product of 1D DSFT of each 1D component
 - $H(u,v)= H_x(u) H_y(v)$
 - $H_x(u)$: 1D FT of h_x
 - $H_y(v)$: 1D FT of h_y

Using Separable Processing to Compute DSFT (Example 1)

- 3x3 averaging filter

$$H = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- How to compute its DSFT?
- Is this signal separable?

Solution

- 3x3 averaging filter

$$H = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Recognizing that the filter is separable

$$H = 1/9 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} 1/3 = h_1 h_1^T$$

$$1/3 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \leftrightarrow H_1(v) = (1e^{j2\pi v} + 1 + 1e^{-j2\pi v}) / 3 = (1 + 2 \cos 2\pi v) / 3$$

$$H(u, v) = H_1(u)H_1(v) = (1 + \cos 2\pi u)(1 + \cos 2\pi v) / 9$$

Example 2

- A separable signal (filter)

- $H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

- Recognizing that the filter is separable

$$H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T$$

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \leftrightarrow Fy(v) = 1e^{j2\pi v} + 0 + (-1)e^{-j2\pi v} = 2j \sin 2\pi v$$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \leftrightarrow Fx(u) = 1e^{j2\pi u} + 2 + e^{-j2\pi u} = 2 + 2 \cos 2\pi u$$

$$F(u, v) = Fx(u)Fy(v) = 4j(1 + \cos 2\pi u) \sin 2\pi v$$

DSFT of Special Signals

- Constant \leftrightarrow pulse at $(0,0)$
- Rectangle \leftrightarrow 2D digital sinc
- Sinc \leftrightarrow Rectangle (ideal low pass)
- Complex sinusoid with freq (u_0, v_0) \leftrightarrow pulse at (u_0, v_0)
- ...
- Can be shown easily by making use of the fact that the signal is separable

What about rotation?

- No theoretical proof, however, roughly it is still true
- Rotation in space \leftrightarrow Rotation in freq.
- Example:
- $f(m,n)=\delta(m)$ (horizontal line) \leftrightarrow $F(u,v) = \delta(v)$ (vertical line)
- What about $f(m,n)=\delta(m-n)$ (diagonal line)?
 - \leftrightarrow $F(u,v)=\delta(u+v)$ (antidiagonal line!)
 - Homework

What you should know?

- General concept of signal decomposition / transform
 - Representing a signal as a linear combination of chosen basis signals
 - Orthonormal transform: the basis functions have unit norm and are orthogonal to each other
 - Forward transform: generating coefficients by inner product
 - Inverse transform: reconstructing signal by sum of weighted basis signals
- 2D signals
 - Meaning of spatial frequency: represented by two orthogonal frequencies
 - Horizontal and vertical, magnitude and direction
 - Separable signals
 - Product of a 1D column vector and a 1D row vector
 - All the rows are scaled versions of each other, all columns are scaled versions of each other
- 2D CSFT and DSFT and DFT
 - Difference between CSFT, DSFT and DFT
 - CSFT: continuous in space and frequency
 - DSFT: discrete in space, continuous in frequency in $(-1/2, 1/2)$
 - 2D DFT: discrete and finite length in space and frequency (equal to samples of DSFT for finite length image)
 - Many properties of 1D CTFT, DTFT, DFT carry over, but there are a few things unique to 2D
 - 2D FT of separable signal = product of 1D FT
 - Rotation in space \leftrightarrow rotation in frequency plane