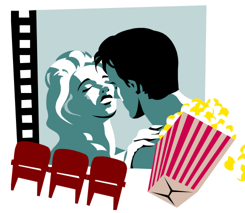

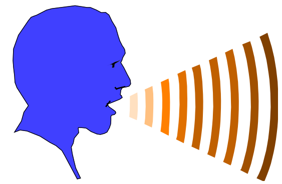

Image and Video Processing

Fourier Transform and Linear Filtering
Part 1: 2D Fourier Transform

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Outline of this lecture

- Part 1: 2D Fourier Transforms
- Part 2: 2D Convolution
- Part 3: Basic image processing operations:
  - Noise removal, image sharpening, and edge detection using linear filtering

2D Fourier Transform

- General concept of signals and transforms
  - Representation using basis functions
- Continuous Space Fourier Transform (CSFT)
  - 1D CTFT -> 2D CSFT
  - Concept of spatial frequency
  - Separable transform
  - Transform of separable signals
- Discrete Space Fourier Transform (DSFT) and DFT
  - 1D -> 2D

Signal in 2D Space

- General 2D continuous space signal: f(x,y)
  - Can have infinite support: x,y= (-infty,…, infty)
  - f(x,y) can generally take on complex values
- General 2D discrete space signal: f(m,n)
  - Can have infinite support: m,n= -infty,…, 0, 1,.., infty
  - f(m,n) can generally take on complex values
- Each color component of a digital image is a 2D real signal with finite support
  - MxN image: m=0,1,…,M-1, n=0,1,…,N-1
  - We will use first index for row, second index for column
  - We will consider a gray scale image or a single color component only
  - Same operations can be applied to each component

Separable Signals

- Separability
  - f(m,n) is separable if $f(m,n)= f_v(m)\, f_h(n)$
  - $f_v(m)$: changes vertically
  - $f_h(n)$: changes horizontally

- Separable image = Rank 1 matrix
  - Rank 1 matrix = product of 1D column vector and 1D row vector

$$H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T$$

Signal over 1 Dimensional (1D) Continuous Space

- $f(x),\ x \in (-\infty, \infty) = \mathrm{R}$ (real line)
  - $\mathrm{e.}g.x$ indicate time
- Real signal: $f(x)$ is real
- Complex signal: $f(x) = a(x) + j\,b(x)$

Inner Product and Norm

- Inner product of two 1D complex functions $f(x), g(x)$

$$< f(x), g(x) > = \int_{-\infty}^{\infty} f(x) g^*(x) dx$$

  – Projection of f(x) onto g(x) in the vector space

- L2 norm

$$|f(x)|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2} = (< f(x), f(x) >)^{1/2}$$

- If $< f(x), g(x) >=0$, we say $f(x)$ is orthogonal to $g(x)$

Transform Representation of 1D Signals

- Transforms are decompositions of a function f(x) into some basis functions Ø(x, u). u indicates which basis function. In this example, the basis fucntions are sinusoidal signals with different frequency.

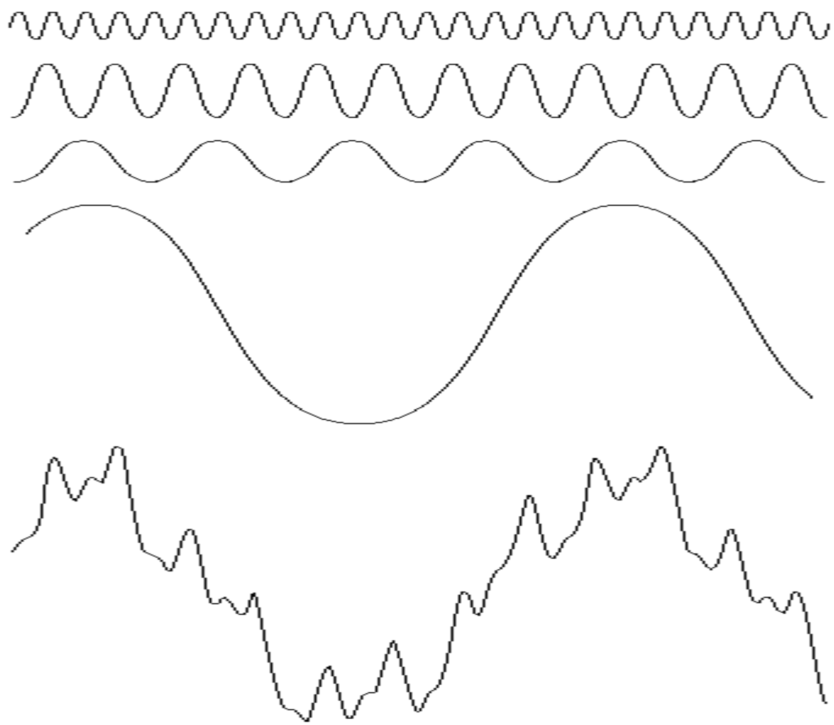
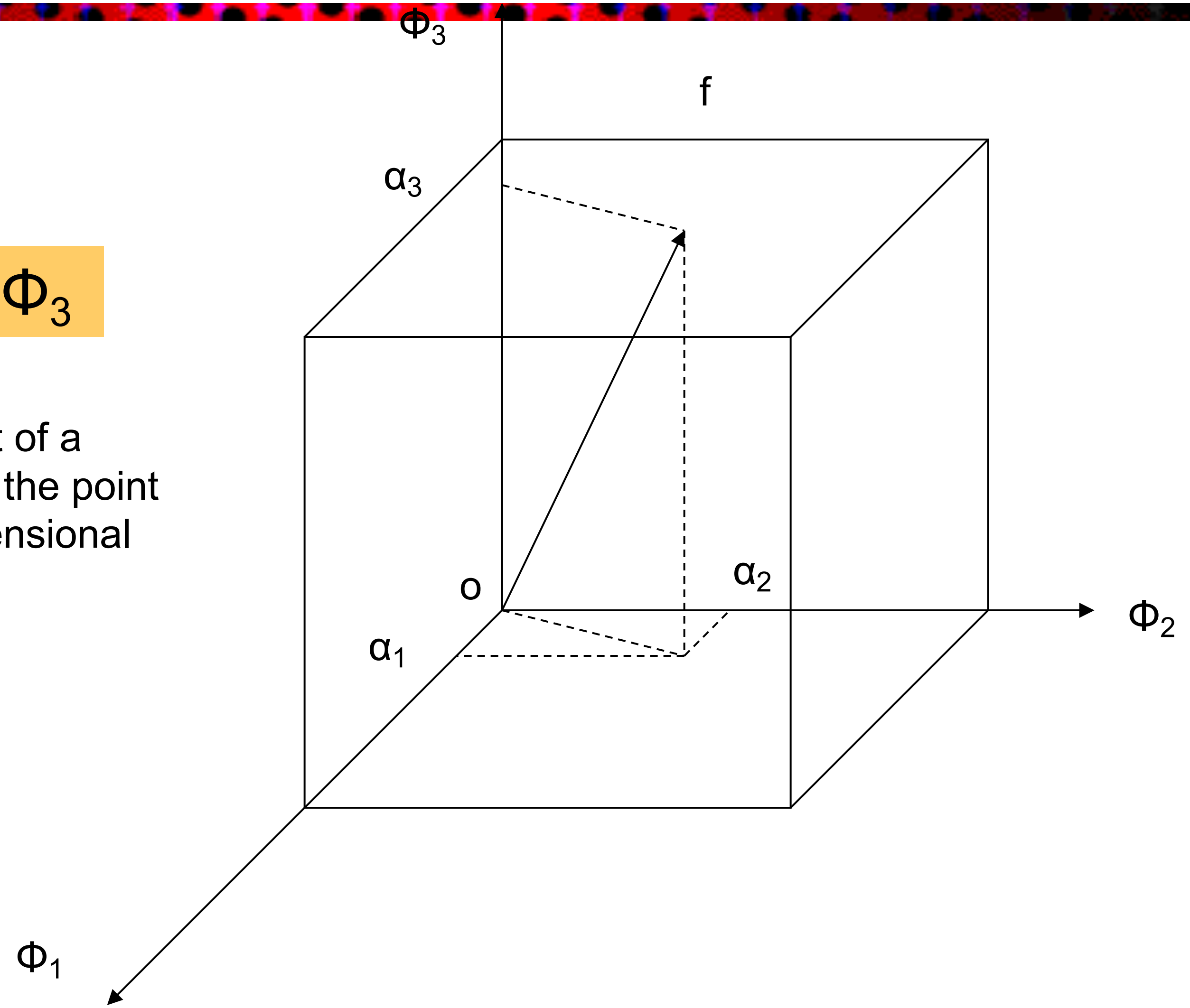


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Illustration of Decomposition in Vector Space

$f = \alpha_1\Phi_1+\alpha_2\Phi_2+\alpha_3\Phi_3$

Each signal can be thought of a point (or vector connecting the point to the origin) in a high dimensional vector space



Representation of 1D Signal Using An Orthonormal Basis

- Orthonormal basis function

$$\int_{-\infty}^{\infty} \phi(x,u_1)\phi^*(x,u_2)dx = \begin{cases} 1, & u_1 = u_2 \\ 0, & u_1 \neq u_2 \end{cases}$$

- Each basis has norm 1
- Different bases are orthogonal to each other

- Inverse transform: Representing *f(x)* as integral (limit of sum) of *φ(x,u)* for all *u*, with weight *F(u)*

$$f(x) = \int_{-\infty}^{\infty} F(u)\phi(x,u)du$$

- Forward transform: determining the weight through inner product

$$F(u) = \langle f(x), \phi(x,u) \rangle = \int_{-\infty}^{\infty} f(x)\phi^*(x,u)dx$$

1D Continuous Time Fourier Transform

- Basis functions (complex sinusoidal or exponential)

$$\phi(x,u) = e^{j2\pi ux}, \quad u \in (-\infty, +\infty).$$

u=frequency=
\# cycles per unit of x
$\omega = 2\pi$ u (radian freq.)

- Inverse transform

$$f(x) = F^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

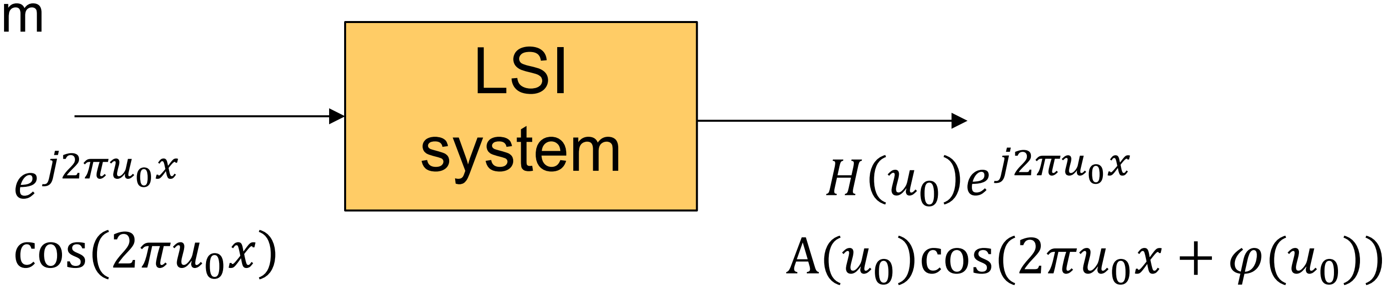
- Forward Transform: $F(u) =< f(x), \varphi(x,u) >$

$$F(u) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Here we use frequency (rather then radian frequency) to define FT. Inverse transform does not need the factor of $1/2\pi$

Why using complex exponential basis? (optional)

- Linear shift invariant (LSI) system: f(x) -> T(f(x))= g(x)
  - a f(x) -> a g(x)
  - $f_1(x)+f_2(x)$ -> $g_1(x)+g_2(x)$
- A LSI system is completely characterized by its impulse response
  - $\delta(x)$ -> $h(x)$
- $e^{j2\pi ux} = \cos(2\pi ux) + \mathrm{j}\sin(2\pi ux)$ are eigen functions of any LTI system



  - $H(u)$ is the frequency response of the system:
    - $H(u) = A(u)e^{j\varphi(u)}$ describes how the magnitude and phase of a sinusoid input with frequency u are changed!
  - $H(u)$ is the CTFT of $h(x)$

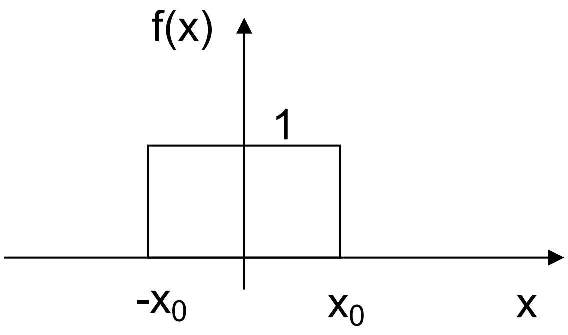
Complex number and function (review)

- Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$
$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

- A good review of basic math for signals and systems by Henry D. Pfister
  - http://pfister.ee.duke.edu/courses/ece485/math_review.pdf

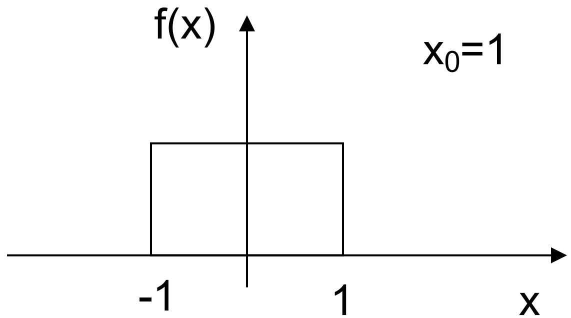
Example: CTFT of the Rectangle Function

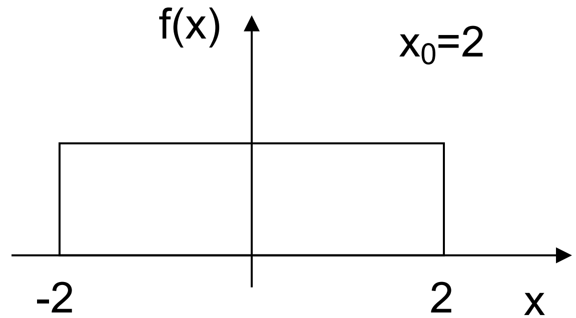


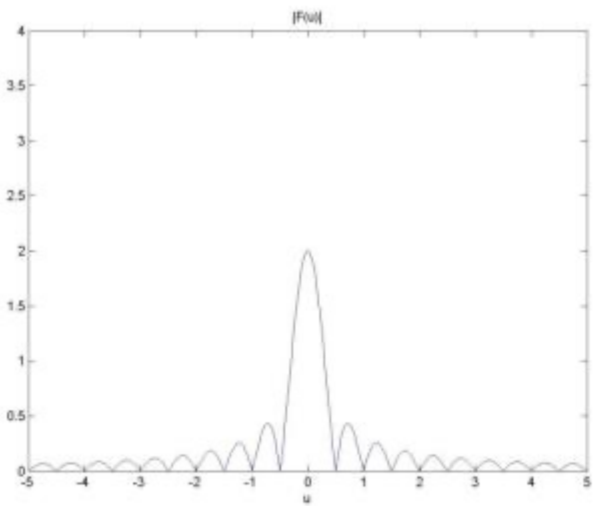
- Half complete on the board. To complete as a homework problem

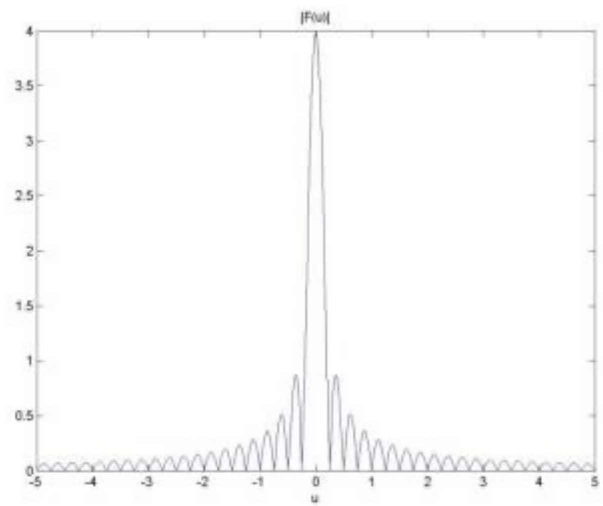
FT of the Rectangle Function

$$F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \operatorname{sinc}(2x_0 u) \quad where, \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$





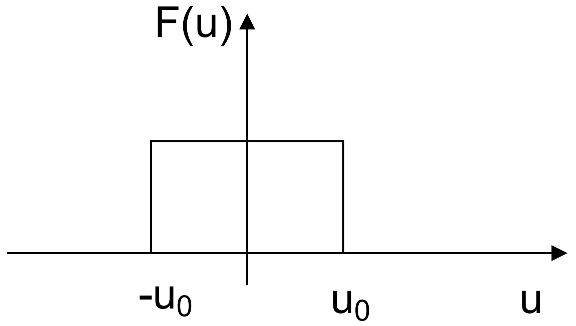




Note first zero occurs at u_0=1/(2 x_0)=1/pulse-width, other zeros are multiples of this.

IFT of Ideal Low Pass Signal

- What is f(x)?



- Duality of CTFT
  - If f(x) -> F(u), then F(x)-> f(-u)

Important Transform Pairs

$$f(x) = 1 \quad \Leftrightarrow \quad F(u) = \delta(u)$$

$$f(x) = e^{j2\pi f_0 x} \quad \Leftrightarrow \quad F(u) = \delta(u - f_0)$$

$$f(x) = \cos(2\pi f_0 x) \quad \Leftrightarrow \quad F(u) = \frac{1}{2}\left(\delta(u - f_0) + \delta(u + f_0)\right)$$

Type equation here.

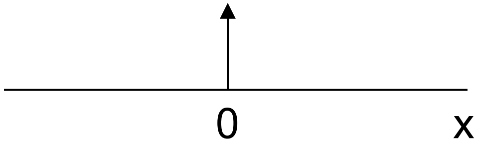
$$f(x) = \sin(2\pi f_0 x) \quad \Leftrightarrow \quad F(u) = \frac{1}{2j}\left(\delta(u - f_0) - \delta(u + f_0)\right)$$

$$f(x) = \begin{cases} 1, & |x| < x_0 \\ 0, & otherwise \end{cases} \quad \Leftrightarrow \quad F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \operatorname{sinc}(2x_0 u)$$

$$where, \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Delta function:

$$\delta(x) = \infty, if\ x = 0; = 0, if\ x \neq 0; \int_{-\infty}^{\infty} \delta(x) dx = 1$$

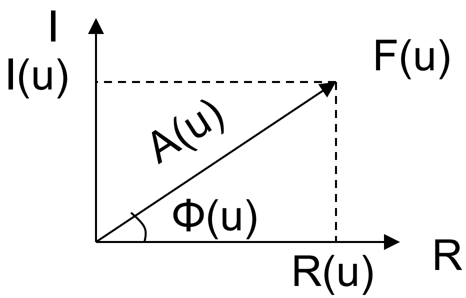


Representation of FT

- Generally, both f(x) and F(u) are complex
- Two representations
  - Real and Imaginary $F(u) = R(u) + jI(u)$
  - Magnitude and Phase

$$F(u) = A(u)e^{j\phi(u)}, \quad where$$

$$A(u) = \sqrt{R(u)^2 + I(u)^2}, \phi(u) = \tan^{-1}\frac{I(u)}{R(u)}$$



- Relationship

$$R(u) = A(u)\cos\phi(u), \quad I(u) = A(u)\sin\phi(u)$$

- Power spectrum

$$P(u) = A(u)^2 = F(u) \times F(u)^* = |F(u)|^2$$

What if f(x) is real?

- Real world signals f(x) are usually real
- F(u) is still complex, but has special properties

$$F^*(u) = F(-u)$$

$$R(u) = R(-u), A(u) = A(-u), P(u) = P(-u) : \text{even function}$$

$$I(u) = -I(-u), \phi(u) = -\phi(-u) : \text{odd function}$$

Properties of Fourier Transform

- Duality

$$f(t) \Leftrightarrow F(u)$$
$$F(t) \Leftrightarrow f(-u)$$

- Linearity

$$F\{a_1 f_1(x) + a_2 f_2(x)\} = a_1 F\{f_1(x)\} + a_2 F\{f_2(x)\}$$

- Scaling

$$F\{af(x)\} = aF\{f(x)\}$$

- Translation

$$f(x - x_0) \Leftrightarrow F(u)e^{-j2\pi x_0 u}, \quad f(x)e^{j2\pi u_0 x} \Leftrightarrow F(u - u_0)$$

- Convolution

$$f(x) \otimes g(x) = \int f(x - \alpha) g(\alpha) d\alpha$$

$$f(x) \otimes g(x) \Leftrightarrow F(u)G(u)$$

We will review convolution later!

Signal over 2D Continuous Space

- $f(x,y),\ x \in (-\infty,\infty),\ y \in (-\infty,\infty)$

- Inner product of two signals

$$< f(x,y), g(x,y) > = \iint_{-\infty}^{\infty} f(x,y) g^*(x,y) dx dy$$

- L2 norm

$$|f(x,y)|_2 = \left(\int_{-\infty}^{\infty} |f(x,y)|^2 dx \right)^{1/2} = (< f(x,y), f(x,y) >)^{1/2}$$

- A 2D signal can be decomposed into a set of 2D orthonormal basis functions $\varphi(x,y;u,v)$
  - (u,v): indices of the 2D basis functions

Two Dimension Continuous Space Fourier Transform (CSFT)

- Basis functions (separable functions)

$$\phi(x,y;u,v) = e^{j(2\pi ux + 2\pi vy)} = e^{j2\pi ux} e^{j2\pi vy}, \quad u,v \in (-\infty, +\infty).$$

  - $\varphi_2(x,y;u,v) = \varphi_1(x;u)\ \varphi_1(y;v)$
  - u : freq. along x, v : freq. along y

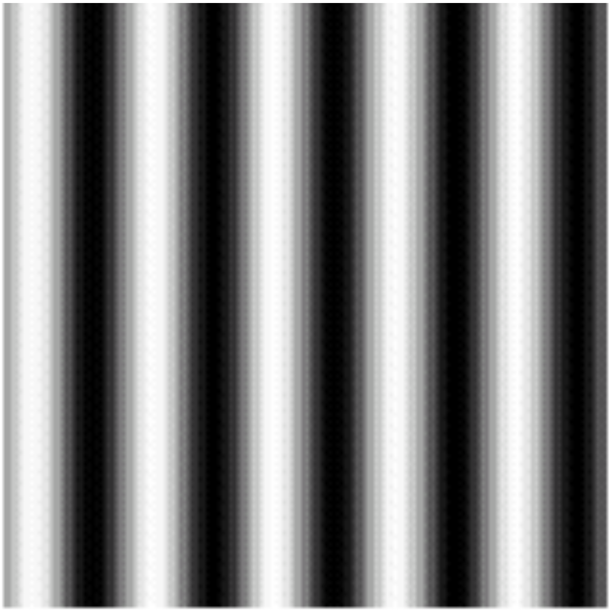
- Inverse Transform

$$f(x,y) = F^{-1}\{F(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)}\, dudv$$

- Forward Transform (inner product)

$$F(u,v) = F\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)}\, dxdy$$

Illustration of Spatial Frequency



$$f(x,y)=\sin(10\pi x)$$

$$f_x=5, f_y=0, f_m=5, \phi=0$$

f_x : unit= 1 cycle/image-width

f_y : unit=1 cycle/image-height

$$f(x,y)=\sin(10\pi x-20\pi y)$$

$$f_x=5, f_y=-10, f_m=\sqrt{125}, \phi=\text{atan}(-2)$$

fx and fy are used to represent the spatial freq. instead of u,v.

Need two components to describe spatial frequency: either frequency in two orthogonal directions (does not have to be horizontal and vertical) or amplitude and direction

$f_m = \left(f_x{}^2 + f_y{}^2\right)^{1/2}$; $\theta = atan\left(f_y/f_x\right)$ (direction orthogonal to the edge direction!)

Example 1

$$f(x,y) = \sin 4\pi x + \cos 6\pi y$$

$$F\{\sin 4\pi x\} = \iint \sin 4\pi x e^{-j2\pi(ux+vy)} dxdy$$

$$= \int \sin 4\pi x e^{-j2\pi ux} dx \int e^{-j2\pi vy} dy$$

$$= \int \sin 4\pi x e^{-j2\pi ux} dx \, \delta(v)$$

$$= \frac{1}{2j}(\delta(u-2) - \delta(u+2))\delta(v)$$

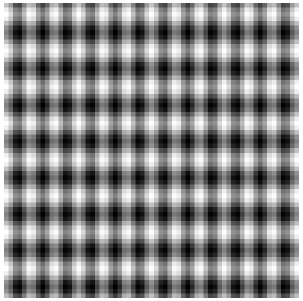
$$= \frac{1}{2j}(\delta(u-2,v) - \delta(u+2,v))$$

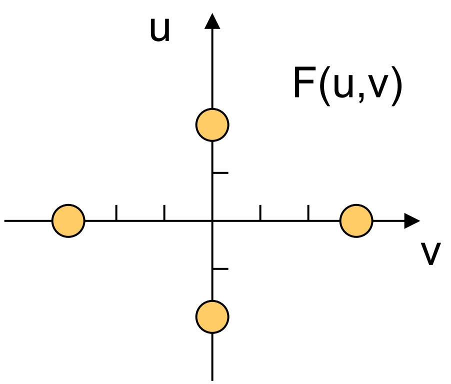
$$\text{where} \quad \delta(x,y) = \delta(x)\delta(y) = \begin{cases} \infty, & x = y = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Likewise,} \quad F\{\cos 6\pi y\} = \frac{1}{2}(\delta(u,v-3) + \delta(u,v+3))$$

Recall $F\{e^{j2\pi u_0 t}\} = \delta(u - u_0)$

f(x,y)





In this and following example, x / u is vertical, y /v is horizontal

Example 2

$$f(x,y)=\sin(2\pi x+3\pi y)=\frac{1}{2j}\left(e^{j(2\pi x+3\pi y)}-e^{-j(2\pi x+3\pi y)}\right)$$

$$F\left\{e^{j(2\pi x+3\pi y)}\right\}=\iint e^{j(2\pi x+3\pi y)}e^{-j2\pi(xu+yv)}dxdy$$

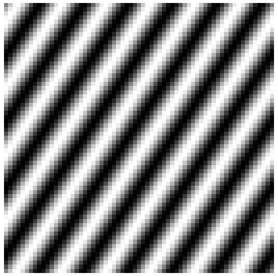
$$=\int e^{j2\pi x}e^{-j2\pi ux}dx\int e^{j3\pi y}e^{-j2\pi yv}dy$$

$$=\delta(u-1)\delta(v-\frac{3}{2})=\delta(u-1,v-\frac{3}{2})$$

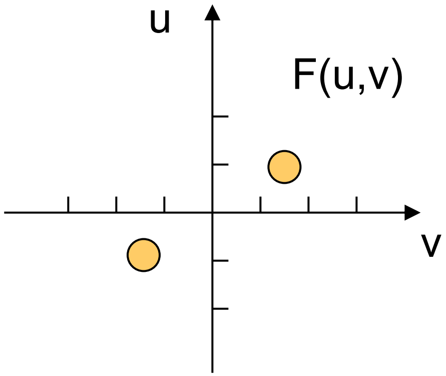
$$Likewise,\quad F\left\{e^{-j(2\pi x+3\pi y)}\right\}=\delta(u+1,v+\frac{3}{2})$$

$$Therefore,$$

$$F\{\sin(2\pi x+3\pi y)\}=\frac{1}{2j}\left(\delta(u-1,v-\frac{3}{2})-\delta(u+1,v+\frac{3}{2})\right)$$



```
[X,Y]=meshgrid(-2:1/16:2,-2:1/16:2);
f=sin(2*pi*X+3*pi*Y);
imagesc(f); colormap(gray)
Truesize, axis off;
```



A Little Teaser ☺

$$f(x,y)=\sin(10\pi x)$$

$$f_x=5, f_y=0, f_m=5, \phi=0$$

f_x : unit= 1 cycle/image-width

f_y : unit=1 cycle/image-height

What is its 2D FT?

Properties of 2D FT (1)

- Linearity

$$F\{a_1 f_1(x,y) + a_2 f_2(x,y)\} = a_1 F\{f_1(x,y)\} + a_2 F\{f_2(x,y)\}$$

- Translation / modulation

$$f(x-x_0, y-y_0) \quad \Leftrightarrow \quad F(u,v)e^{-j2\pi(x_0 u + y_0 v)},$$

$$f(x,y)e^{j2\pi(u_0 x + v_0 y)} \quad \Leftrightarrow \quad F(u-u_0, v-v_0)$$

- Conjugation

$$f^*(x,y) \quad \Leftrightarrow \quad F^*(-u,-v)$$

Properties of 2D FT (2)

- Symmetry

$$f(x,y) \text{ is real} \quad \Leftrightarrow \quad |F(u,v)| = |F(-u,-v)|$$

- Convolution
  - Definition of convolution

$$f(x,y) \otimes g(x,y) = \iint f(x-\alpha, y-\beta) g(\alpha,\beta) d\alpha d\beta$$

  - Convolution theory

$$f(x,y) \otimes g(x,y) \quad \Leftrightarrow \quad F(u,v)G(u,v)$$

We will describe 2D convolution later!

Separability of 2D FT and Separable Signal

- Separability of 2D FT

$$F_2\{f(x,y)\} = F_y\{F_x\{f(x,y)\}\} = F_x\{F_y\{f(x,y)\}\}$$

  - where F_x, F_y are 1D FT along x and y.
  - one can do 1DFT for each row of original image, then 1D FT along each column of resulting image
- Separable Signal
  - $f(x,y) = f_x(x)f_y(y)$
  - $F(u,v) = F_x(u)F_y(v)$,
    - where $F_x(u) = F_x\{f_x(x)\}$, $F_y(u) = F_y\{f_y(y)\}$
  - For separable signals, one can simply compute two 1D transforms and take their product!

Example 1

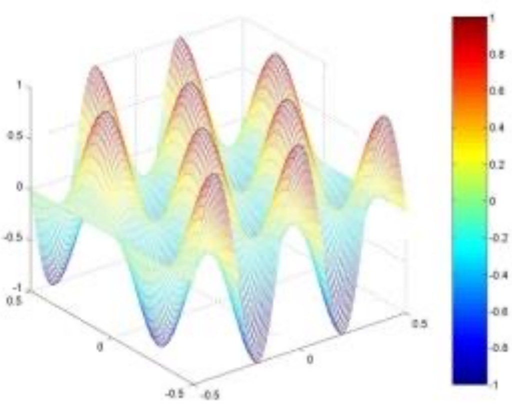
$$f(x,y)=\sin(3\pi x)\cos(5\pi y)$$

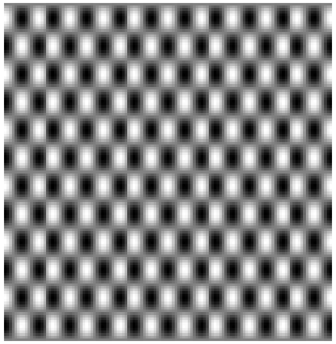
$$f_x(x)=\sin(3\pi x) \quad \Leftrightarrow \quad F_x(u)=\frac{1}{2j}\left(\delta(u-3/2)-\delta(u+3/2)\right)$$

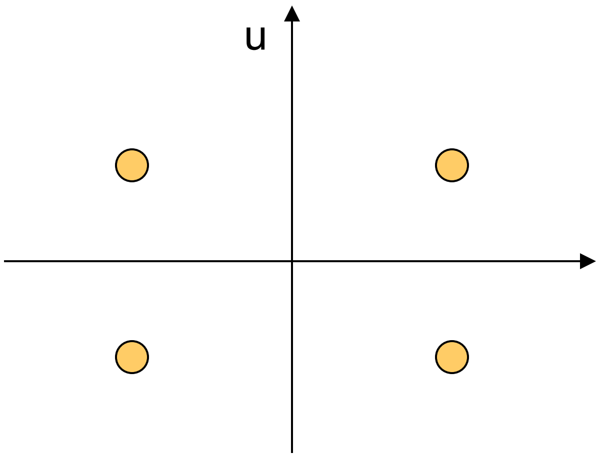
$$f_y(y)=\cos(5\pi y) \quad \Leftrightarrow \quad F_y(v)=\frac{1}{2}\left(\delta(v-5/2)+\delta(v+5/2)\right)$$

$$F(u,v)=Fx(u)Fy(v)$$

$$=\frac{1}{4j}\left(\delta(u-\frac{3}{2},v-\frac{5}{2})-\delta(u+\frac{3}{2},v-\frac{5}{2})+\delta(u-\frac{3}{2},v+\frac{5}{2})-\delta(u+\frac{3}{2},v+\frac{5}{2})\right)$$



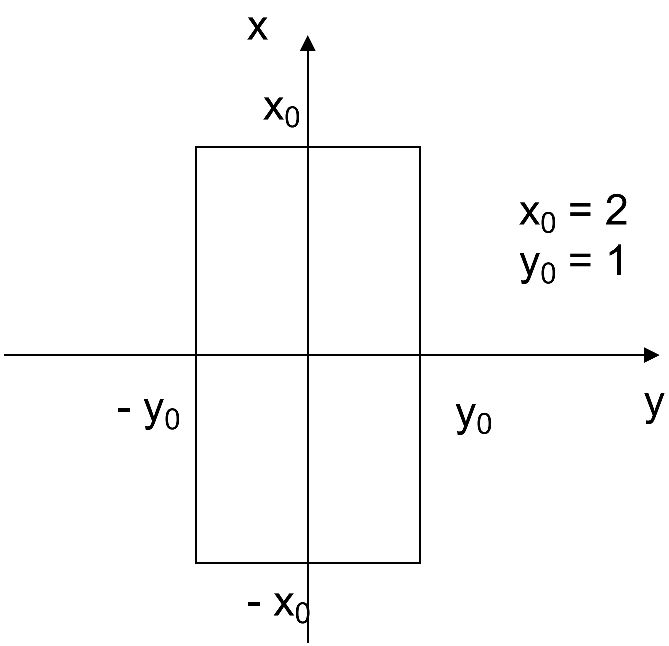




Example 2

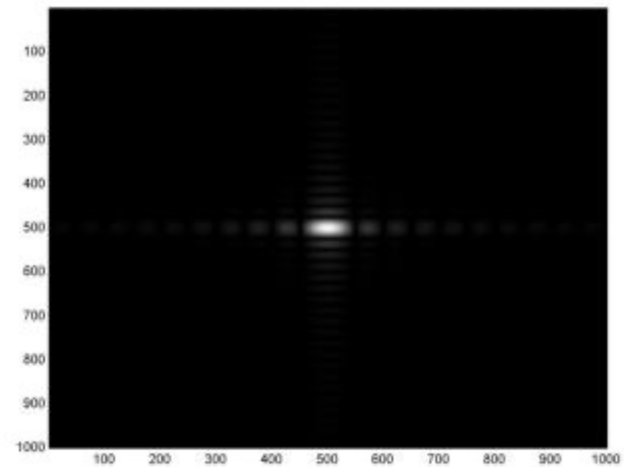
$$f(x,y) = \begin{cases} 1, & |x| \le x_0, |y| \le y_0 \\ 0, & otherwise \end{cases} \Rightarrow$$

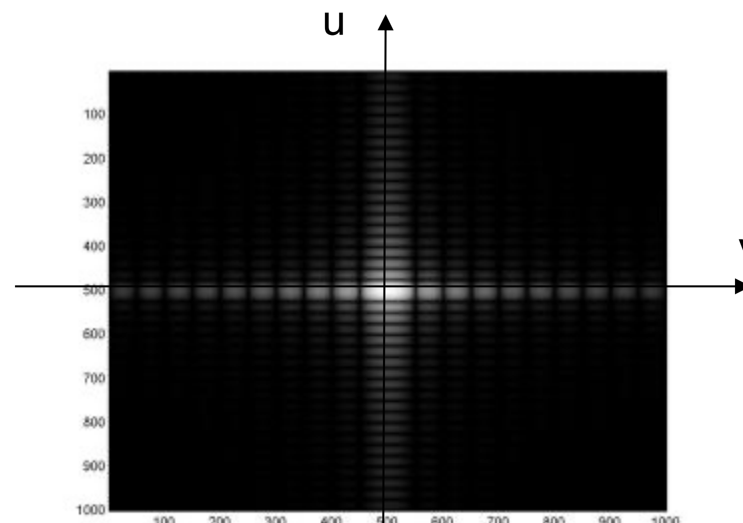
$$F(u,v) = 4x_0 y_0 \operatorname{sinc}(2x_0 u)\operatorname{sinc}(2y_0 v)$$



$x_0 = 2$
$y_0 = 1$

Is this signal separable?





w/ contrast enhancement through log mapping

Important Transform Pairs

- All following signals are separable and the relations can be proved by applying the separability of the CSFT

$$f(x,y)=1 \quad \Leftrightarrow \quad F(u,v)=\delta(u,v), \quad where\, \delta(u,v)=\delta(u)\delta(v)$$

$$f(x)=e^{j2\pi(f_1x+f_2y)} \quad \Leftrightarrow \quad F(u)=\delta(u-f_1,v-f_2)$$

$$f(x,y)=\cos(2\pi(f_1x+f_2y)) \quad \Leftrightarrow \quad F(u)=\frac{1}{2}\left(\delta(u-f_1,v-f_2)+\delta(u+f_1,v+f_2)\right)$$

$$f(x,y)=\sin(2\pi(f_1x+f_2y)) \quad \Leftrightarrow \quad F(u)=\frac{1}{2j}\left(\delta(u-f_1,v-f_2)-\delta(u+f_1,v+f_2)\right)$$

$$f(x,y)=\begin{cases}1, & |x|<x_0,|y|<y_0\\ 0, & otherwise\end{cases} \quad \Leftrightarrow$$

$$F(u,v)=\frac{\sin(2\pi x_0u)}{\pi u}\frac{\sin(2\pi y_0v)}{\pi v}=4x_0y_0\,\mathrm{sinc}(2x_0u)\mathrm{sinc}(2y_0v)$$

$$where\ \mathrm{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$$

$$\frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\} \Leftrightarrow \exp\left\{-\frac{u^2+v^2}{2\beta^2}\right\}, \beta=\frac{1}{2\pi\sigma}$$

- Constant ⇔ impulse at (0,0) freq.
- Complex exponential ⇔ impulse at a particular 2D freq.
- 2D box ⇔ 2D sinc function
- Gaussian ⇔ Gaussian

Rotation

- Let $x = r\cos\theta, \quad y = r\sin\theta, \quad u = \rho\cos\omega, \quad v = \rho\sin\omega.$
- 2D FT in polar coordinate (r, θ) and (ρ, ϕ)

$$F(\rho,\phi) = \int_0^\infty \int_0^{2\pi} f(r,\theta) e^{-j2\pi(r\cos\theta\,\rho\cos\phi + r\sin\theta\,\rho\sin\phi)} r dr d\theta$$
$$= \iint f(r,\theta) e^{-j2\pi r\rho\cos(\theta-\phi)} r dr d\theta$$

- Property

$$f(r,\theta+\theta_0) \quad \Leftrightarrow \quad F(\rho,\phi+\theta_0)$$

- Proof: Homework!
- Note: In this notation: x is horizontal index, y is vertical

Example of Rotation

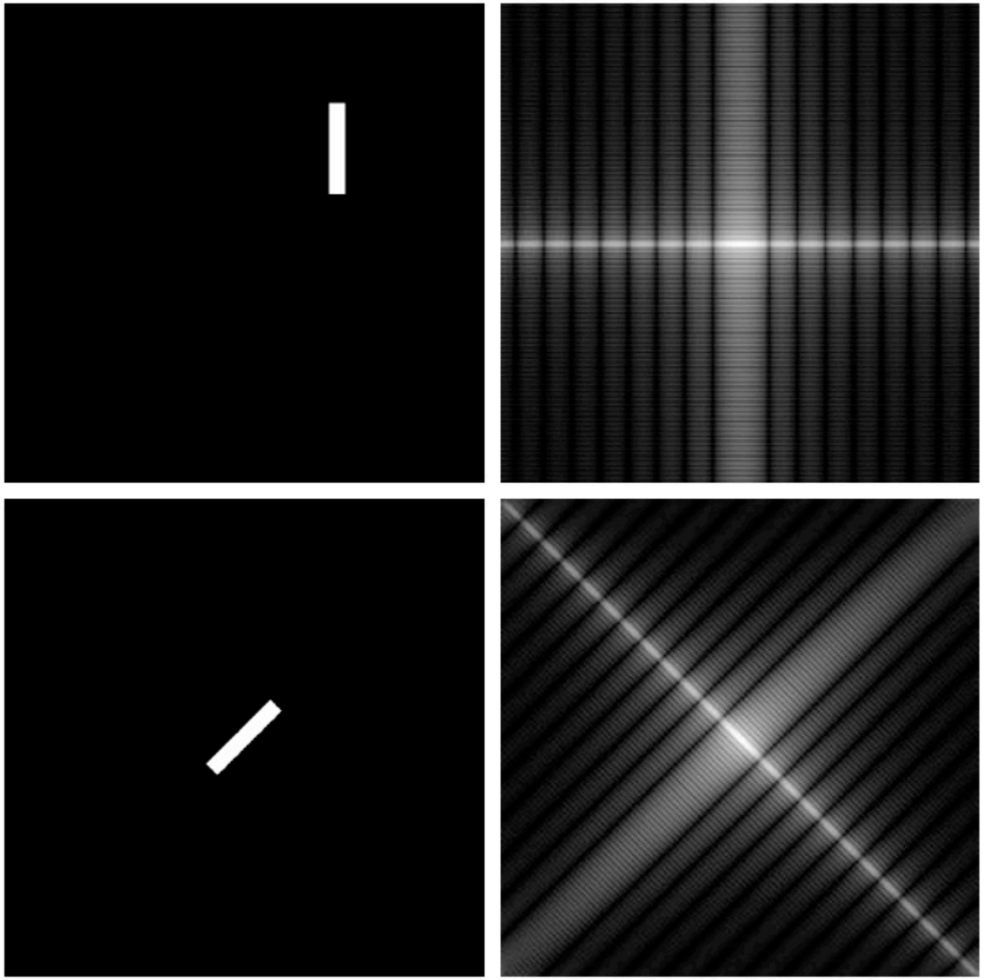


FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

From Gonzalez

Signal over 1D Discrete Time

- General: $f(n),\ n \in (-\infty, .., -1,0,1, \dots, \infty)$
- Finite length sequence:
  - only defined for $n \in (0,1, \dots, N-1)$
- Inner product of two signals

$$< f(n), g(n) > = \sum_{-\infty}^{\infty} f(n) g^*(n)$$

- L2 norm

$$|f(n)|_2 = \left(\sum_{-\infty}^{\infty} |f(n)|^2\right)^{1/2} = (< f(n), f(n) >)^{1/2}$$

- A signal can be decomposed into a set of 1D orthonormal discrete time basis functions $\varphi(n; u)$

1D Fourier Transform For Discrete Time Sequence (DTFT) (Review)

- Basis function $\varphi(n;u) = e^{j2\pi un}$, periodic with period 1
- Inverse Transform
$$f(n) = \int_{-1/2}^{1/2} F(u) e^{j2\pi un} du$$
  - Representing f(n) as a weighted sum of many complex sinusoidal signals with frequency u, F(u) is the weight
  - F(u) indicate the "amount" of sinusoidal component with freq. u, $e^{j2\pi un}$, in signal f(n)
- Forward Transform = inner product $< f(n), \varphi(n;u) >$
$$F(u) = \sum_{n=-\infty}^{\infty} f(n) e^{-j2\pi un}$$
- |u| = digital frequency = the number of cycles per integer sample. Period = 1/|u| (must be equal or greater than 2 samples-> |u|<=1/2)

Properties unique for DTFT

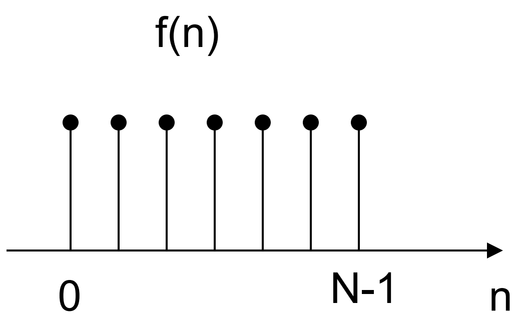
- Periodicity
  - F(u) = F(u+1) = f(u+k), k= integers
  - The FT of a discrete time sequence is only considered for u є (-½ , ½), and u = $\pm$½ is the highest discrete frequency
- Symmetry for real sequences

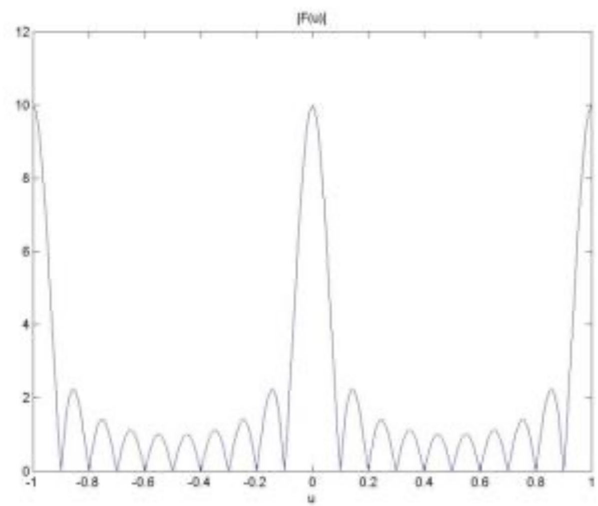
$$f(n) = f^*(n) \quad \Leftrightarrow \quad F(u) = F^*(-u)$$
$$\Rightarrow \quad |F(u)| = |F(-u)|$$
$$\Rightarrow \quad |F(u)| \; is\ symmetric$$

Example

$$f(n)=\begin{cases}1, & n=0,1,\ldots,N-1;\\ 0, & others\end{cases}$$

$$F(u)=\sum_{n=0}^{N-1} e^{-j2\pi nu}=\frac{1-e^{-j2\pi uN}}{1-e^{-j2\pi u}}=e^{-j\pi(N-1)u}\,\frac{\sin 2\pi u(N/2)}{\sin 2\pi u(1/2)}$$





N=10

Digital sinc: There are N/2-1 zeros in (0, ½], 1/N apart

1D Discrete Fourier Transform (DFT)

- DTFT of N-pt sequence

$$F(u)=\sum_{n=0}^{N-1} f(n)e^{-j2\pi un}$$

- N-pt DFT of N-pt sequence

$$F_N(k)=\sum_{n=0}^{N-1} f(n)e^{-j2\pi\frac{kn}{N}}=F\left(u=\frac{k}{N}\right)$$

- DFT is the sampled version of DTFT with samples at 0, 1/N, …, (N-1)/N, if f(n) has length <=N
- FFT: Fast algorithm for computing DFT
  - Direct computation takes N^2 operations
  - FFT takes ~N log(N) operations!

Signal over 2D Discrete Space

- General: $f(m,n), m,n \in (-\infty, .., -1,0,1, \dots, \infty)$
- Finite length sequence:
  - only defined for $m \in (0,1,\dots,M-1), n \in (0,1,\dots,N-1)$
- Inner product of two signals

$$< f(m,n), g(m,n) > = \sum_m \sum_n f(m,n) g^*(m,n)$$

- L2 norm

$$|f(m,n)|_2 = \left(\sum \sum |f(m,n)|^2 \right)^{1/2} = (< f(m,n), f(m,n) >)^{1/2}$$

- A signal can be decomposed into a set of 2D orthonormal discrete space basis functions $\varphi(m,n;u,v)$

Discrete Space Fourier Transform (DSFT) for Two Dimensional Signals

- Basis function $\varphi(m,n;u,v) = e^{j2\pi(um+vn)} = e^{j2\pi um}\, e^{j2\pi vn}$
- Inverse Transform

$$f(m,n) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} F(u,v) e^{j2\pi(mu+nv)} du dv$$

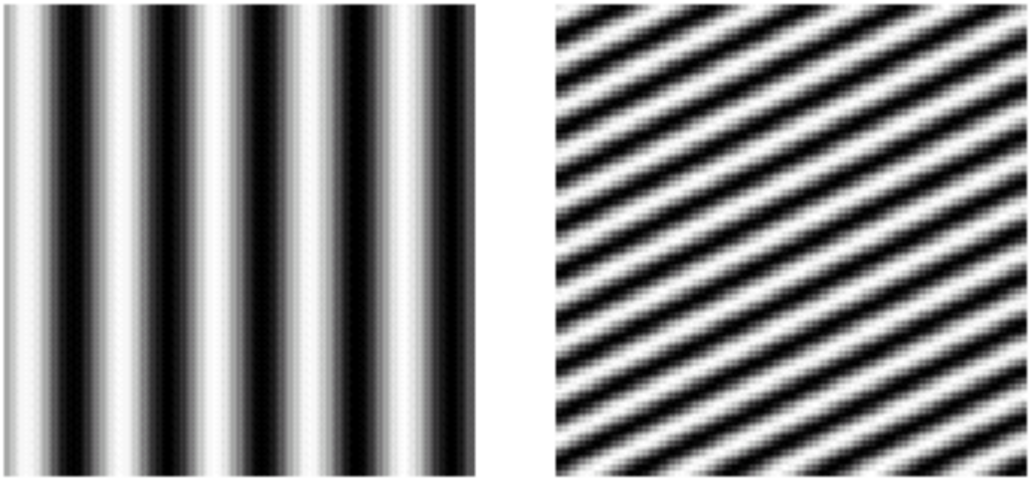
  - Representing an image as weighted sum of many 2D complex sinusoidal images $e^{j2\pi(mu+nv)}$ for different *u*,*v*

- Forward Transform = inner product

$$F(u,v) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n) e^{-j2\pi(mu+nv)}$$

- *u*: number of cycles per sample along m direction
- *v*: number of cycles per sample along n direction

Spatial Frequency for Digital Images



(a) (b)

Figure 2.1 Two-dimensional sinusoidal signals: (a) $(f_x, f_y) = (5, 0)$; (b) $(f_x, f_y) = (5, 10)$. The horizontal and vertical units are the width and height of the image, respectively. Therefore, $f_x = 5$ means that there are five cycles along each row.

$$f_s = \sqrt{125}, \quad \varphi = \arctan(2)$$

Analog vs. digital frequency:

If the image has 256x256 pixels, fx=5 cycles per width (analog frequency) -> u= 5 cycles/256 pixels =5/256 cycles/sample (digital frequency)

When both horizontal and vertical frequency are non-zero, we see directional patterns. fs is the frequency along the direction with the maximum change (orthogonal to the lines)

Note that 1/u may not correspond to integer.

If u=a/b, digital period= b. Ex u=3/8, Digital Period = 8

Periodicity

$$F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)e^{-j2\pi(mu+nv)}$$

- *F(u,v)* is periodic in both u, v with period 1, i.e., for all integers k, l:
  - F(u+k, v+l) = F(u, v)
  - Only need to consider the rectangle area defined by (-1/2,1/2)
- To see this consider

$$\begin{aligned} &e^{-j2\pi(m(u+k)+n(v+l))} \\ &= e^{-j2\pi(mu+nv)}e^{-j2\pi(mk+nl)} \\ &= e^{-j2\pi(mu+nv)} \end{aligned}$$

Example: Delta Function

- Fourier transform of a delta function

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(m,n) e^{-j2\pi(mu+nv)} = 1$$

$$\delta(m,n) \Leftrightarrow 1$$

  - Delta function contains all frequency components with equal weights!

1D delta signal

$$\delta_1(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$$

2D delta signal

$$\delta(m,n) = \delta_1(m)\delta_1(n)$$

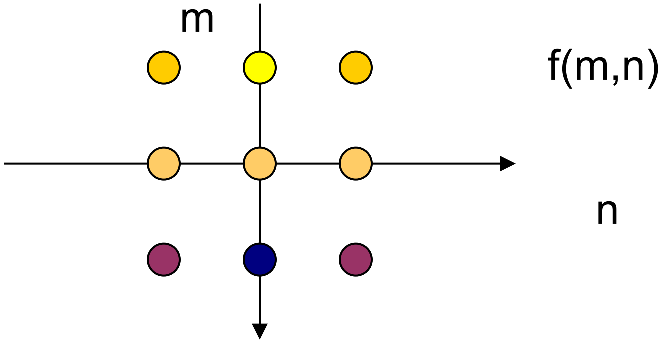
- Inverse Fourier transform of a delta function

$$f(m,n) = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} \delta(u,v) e^{j2\pi(mu+nv)} du dv = 1$$

$$1 \Leftrightarrow \delta(u,v)$$

Example

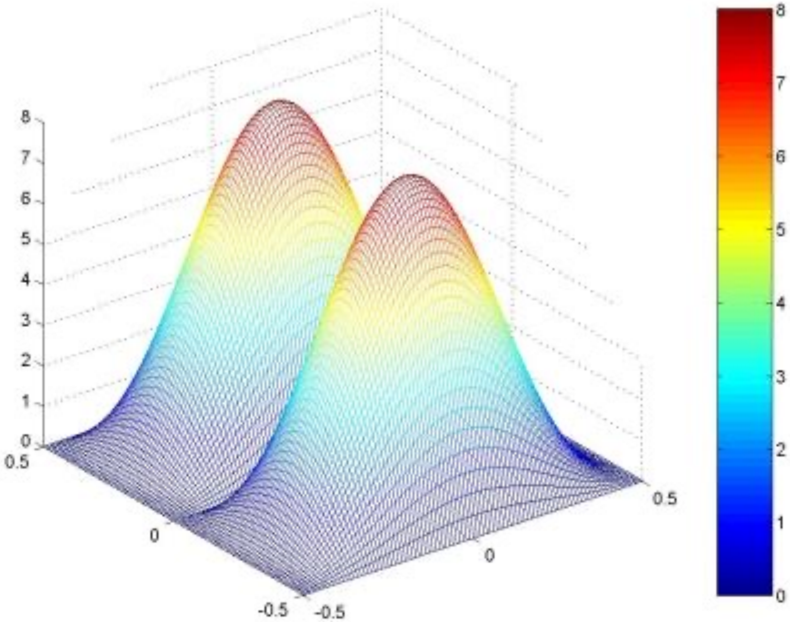
$$f(m,n)=\begin{bmatrix}1 & 2 & 1\\ 0 & 0 & 0\\ -1 & -2 & -1\end{bmatrix}$$

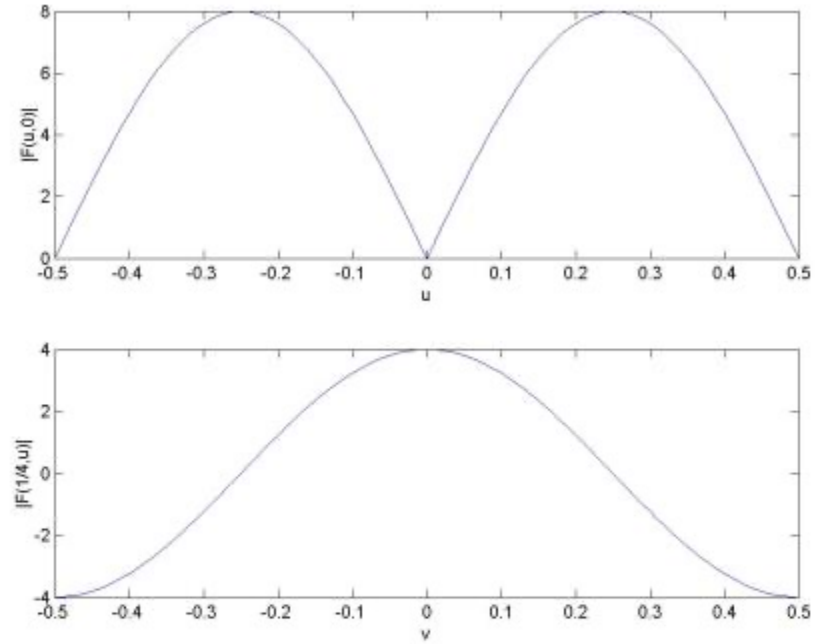


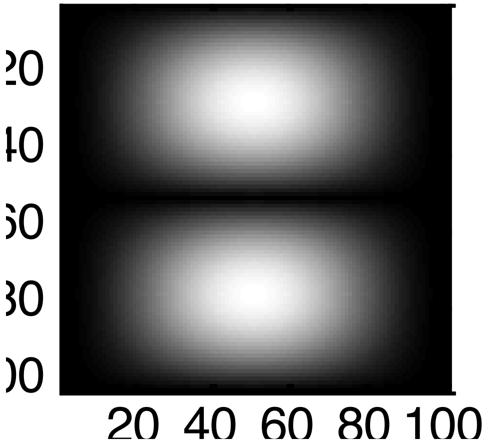
$$\begin{aligned}F(u,v) &= 1e^{-j2\pi(-1*u-1*v)}+2e^{-j2\pi(-1*u+0*v)}+1e^{-j2\pi(-1*u+1*v)}\\ &\quad -1e^{-j2\pi(1*u-1*v)}-2e^{-j2\pi(1*u+0*v)}-1e^{-j2\pi(1*u+1*v)}\\ &= j2\sin 2\pi u e^{j2\pi v}+j4\sin 2\pi u+j2\sin 2\pi u e^{-j2\pi v}\\ &= j4\sin 2\pi u(\cos 2\pi v+1)\end{aligned}$$

Note: This signal is low pass in the horizontal direction v, and band pass in the vertical direction u.

Graph of F(u,v)







```
du = [-0.5:0.01:0.5];
dv = [-0.5:0.01:0.5];
Fu = abs(sin(2 * pi * du));
Fv = cos(2 * pi * dv);
F = 4 * Fu' * (Fv + 1);
mesh(du, dv, F);
colorbar;
Imagesc(F);
colormap(gray); truesize;
```

Using MATLAB freqz2:

f=[1,2,1;0,0,0;-1,-2,-1];

freqz2(f)

DSFT of Separable Signals

- Separable signal:
  - $h(m,n)=h_x(m)\ h_y(n)$

$$H=\begin{bmatrix}1 & 0 & -1\\ 2 & 0 & -2\\ 1 & 0 & -1\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}=h_x h_y^T$$

  - All the rows are scaled versions of each other, all the columns are scaled versions of each other
  - Image can be represented as the product of a 1D column vector and a 1D row vector
- 2D DSFT of separable signal = product of 1D DSFT of each 1D component
  - $H(u,v)= H_x(u)\ H_y(v)$
  - $H_x(u)$: 1D FT of h_x
  - $H_y(v)$: 1D FT of h_y

Using Separable Processing to Compute DSFT (Example 1)

- 3x3 averaging filter

$$H = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- How to compute its DSFT?
- Is this signal separable?

Solution

- 3x3 averaging filter

$$H = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Recognizing that the filter is separable

$$H = 1/9\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1/3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} 1/3 = h_1 h_1^T$$

$$1/3\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} <--> H_1(v) = (1e^{j2\pi v} + 1 + 1e^{-j2\pi v})/3 = (1 + 2\cos 2\pi v)/3$$

$$H(u,v) = H_1(u)H_1(v) = (1+\cos 2\pi u)(1+\cos 2\pi v)/9$$

Example 2

- A separable signal (filter)
- $H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
- Recognizing that the filter is separable

$$H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T$$

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} <-> Fy(v) = 1e^{j2\pi v} + 0 + (-1)e^{-j2\pi v} = 2j \sin 2\pi v$$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} <-> Fx(u) = 1e^{j2\pi u} + 2 + e^{-j2\pi u} = 2 + 2\cos 2\pi u$$

$$F(u,v) = Fx(u)Fy(v) = 4j(1 + \cos 2\pi u)\sin 2\pi v$$

DSFT of Special Signals

- Constant <-> pulse at (0,0)
- Rectangle <-> 2D digital sinc
- Sinc <-> Rectangle (ideal low pass)
- Complex sinusoid with freq (u_0,v_0) <-> pulse at (u_0,v_0)
- …
- Can be shown easily by making use of the fact that the signal is separable

What about rotation?

- No theoretical proof, however, roughly it is still true
- Rotation in space <-> Rotation in freq.
- Example:
- f(m,n)=δ(m) (horizontal line) <-> F(u,v) = δ(v) (vertical line)
- What about f(m,n)= δ(m-n) (diagonal line)?
  - <-> F(u,v)= δ(u+v) (antidiagonal line!)
  - Homework

2D Discrete Fourier Transform

- 2D DSFT

$$F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)e^{-j2\pi(mu+nv)}$$

- 2D DFT

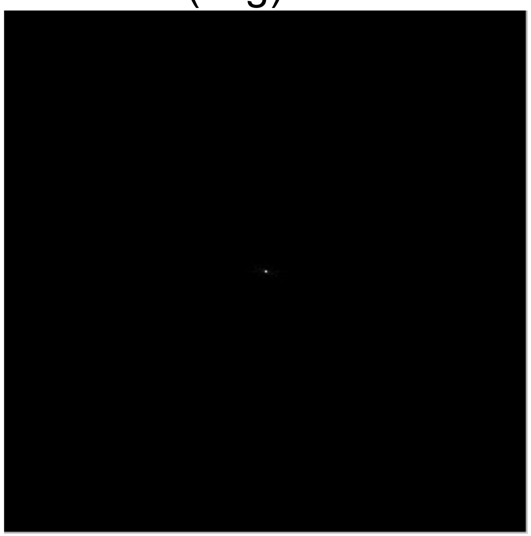
$$F(k,l)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(mk/M+nl/N)}$$

- (MxN) point 2D DFT are samples of DSFT for images at u=k/M, v=l/N, as longs M and N are equal to or greater than the height and width of the image, respectively. Can be computed using FFT algorithms. 1D FFT along rows, then 1D FFT along columns
  - Total complexity:
    - M*N log(N)+ N*M log(M) or $2M^2$log(M) for M=N
- DFT are usually used to compute DSFT!

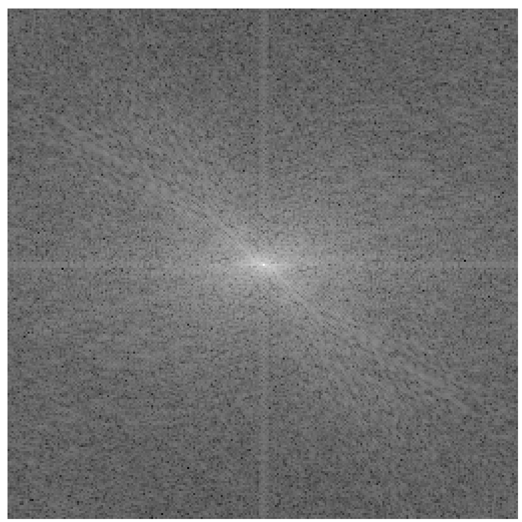
Display of DFT of Images

Imshow(img)

MFimg=abs(fft2(img)), imshow(MFimg,[ ])



SMFimg=fftshift(MFimg)
imshow(SMFimg,[ ])

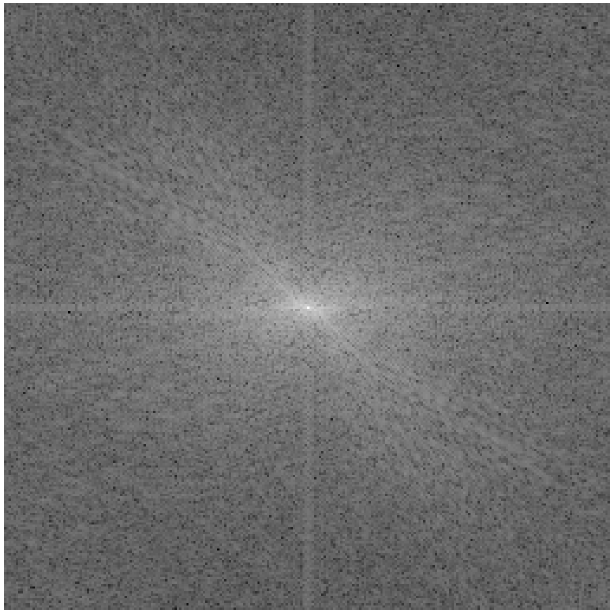


LSMFimg=log(SMFimg+1);
Imshow(LSMFimg,[ ])

fftshift to shift the (0,0) freq. to center

Log mapping to enhance contrast (boost low values!)

DFT of Typical Images

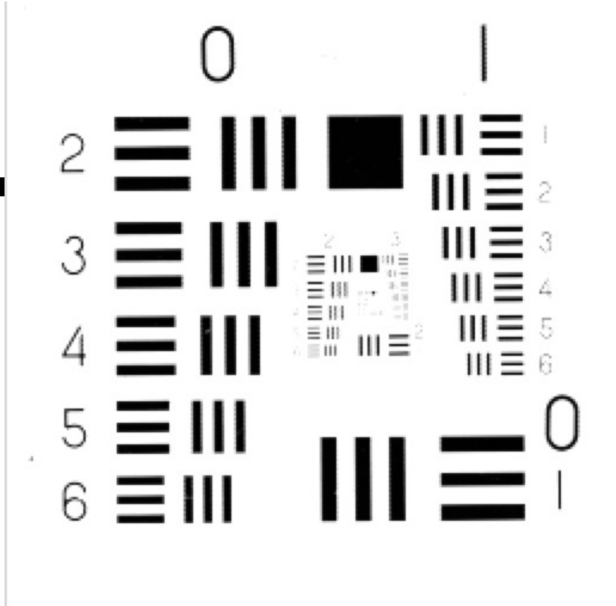


How to interpret the DFT image?
Why is it bright at center and has some line structures?
Horizontal edges: changes vertically in all frequency
Vertical edges: changes horizontally

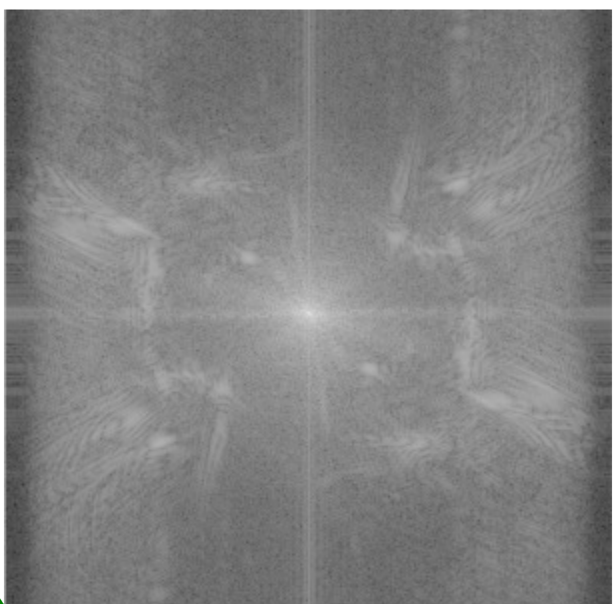
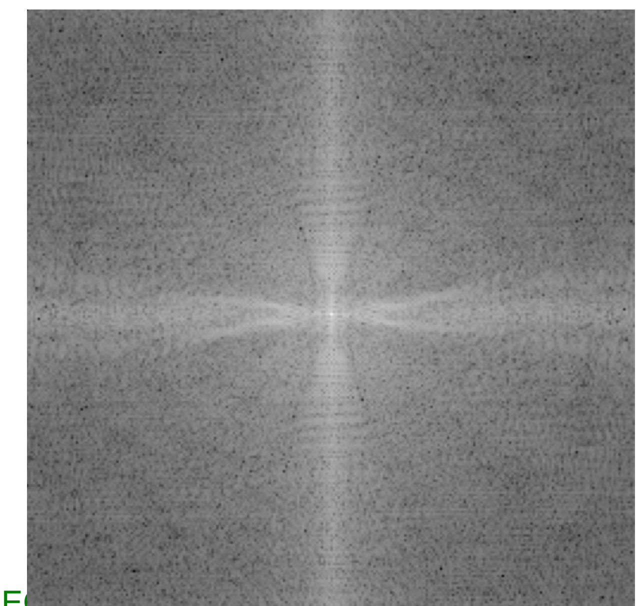
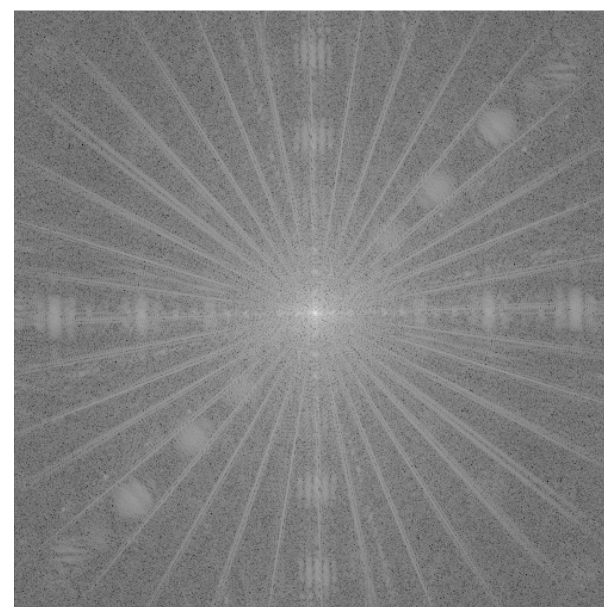
|F(u,v)| (obtained using 2D FFT)

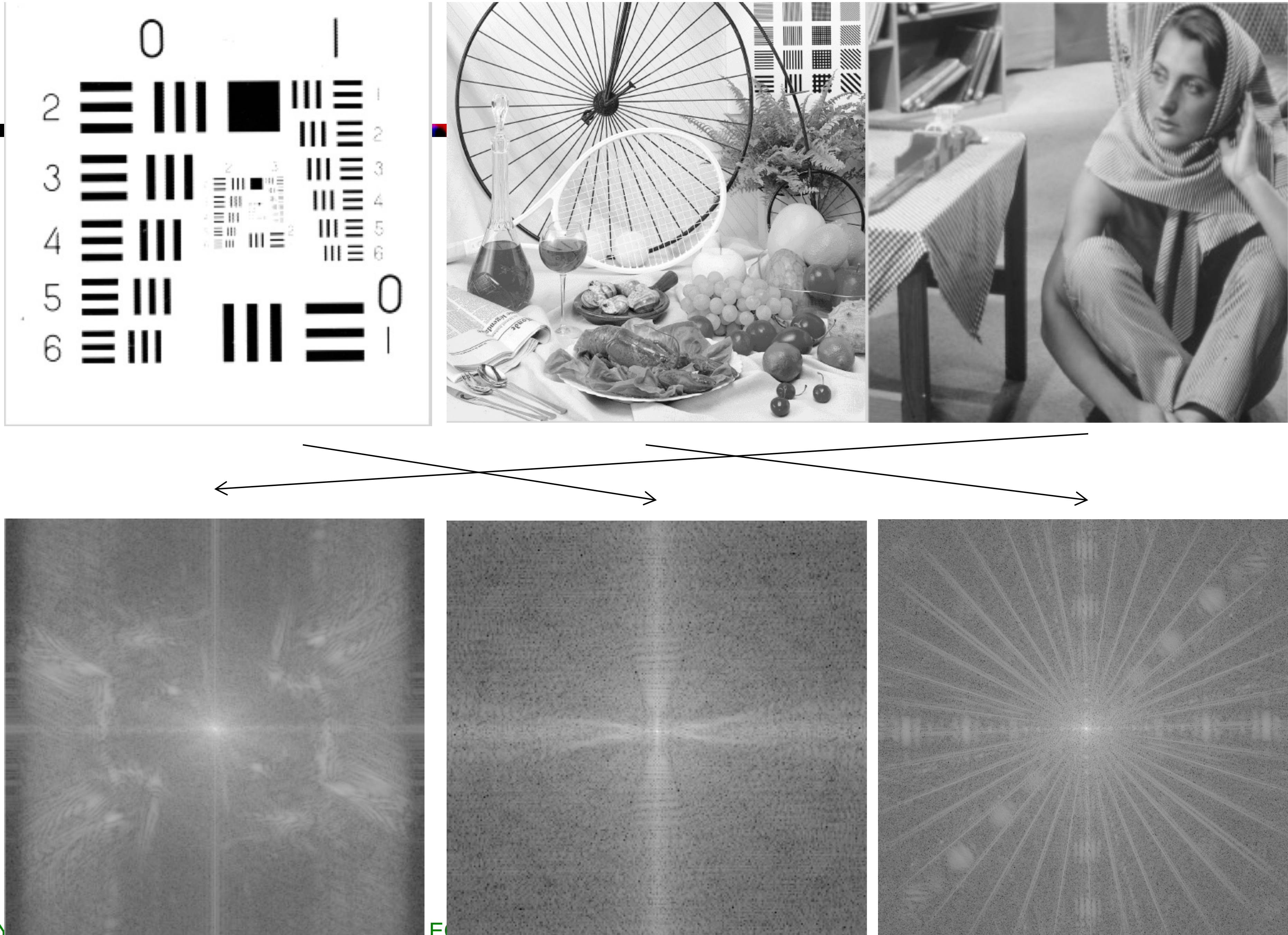
```
ff=abs(fft2(f));
imagesc(fftshift(log(ff+1)));
```

Log mapping to enhance contrast
Fftshift to shift the (0,0) freq. to center




Which one below is the DFT of which one above?



What you should know?

- General concept of signal decomposition / transform
  - Representing a signal as a linear combination of chosen basis signals
  - Orthonormal transform: the basis functions have unit norm and are orthogonal to each other
  - Forward transform: generating coefficients by inner product
  - Inverse transform: reconstructing signal by sum of weighted basis signals
- 2D signals
  - Meaning of spatial frequency: represented by two orthogonal frequencies
    - Horizontal and vertical, magnitude and direction
  - Separable signals
    - Product of a 1D column vector and a 1D row vector
    - All the rows are scaled versions of each other, all columns are scaled versions of each other
- 2D CSFT and DSFT and DFT
  - Difference between CSFT, DSFT and DFT
    - CSFT: continuous in space and frequency
    - DSFT: discreate in space, continuous in frequency in (-1/2, ½)
    - 2D DFT: discrete and finite length in space and frequency (equal to samples of DSFT for finite length image)
  - Many properties of 1D CTFT, DTFT, DFT carry over, but there are a few things unique to 2D
    - 2D FT of separable signal = product of 1D FT
    - Rotation in space <-> rotation in frequency plane