ℓ_1-regularized least squares

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Outline

Soft-thresholding

Alternating direction method of multipliers

Notations

- $x = [x_1, \cdots, x_n]^\top$
- $\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$
- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- $f(x) = \frac{1}{2}\|y - x\|_2^2 + \lambda\|x\|_1 = \sum_{i=1}^{n} \frac{1}{2}(y_i - x_i)^2 + \lambda\,|x_i|$

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- $|x|$ is differentiable except at $x = 0$ i.e. $\partial|x| = \operatorname{sgn}(x)$ for $x \neq 0$.

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- $x^* - y + \lambda \operatorname{sgn}(x^*) = 0 \Rightarrow y = x^* + \lambda \operatorname{sgn}(x^*)$

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let's plot $y = x^* + \lambda \operatorname{sgn}(x^*)$

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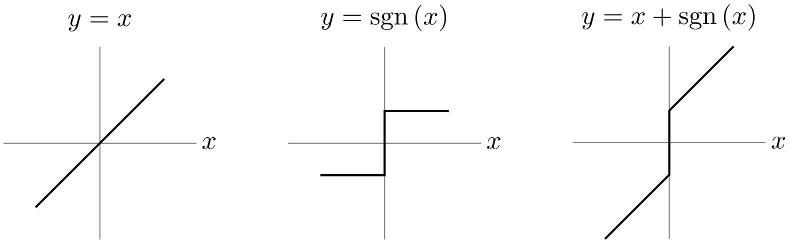

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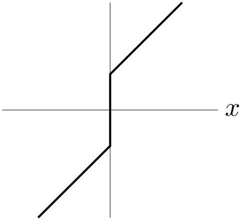

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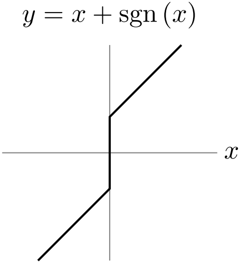

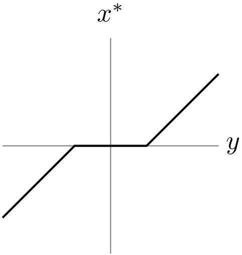

- $x^* = \operatorname{sgn}(y) \times \max\{|y| - \lambda, 0\}$

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- convex equality constrained optimization problem

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- recover $x^* = \arg\min_x L(x, u^*)$.

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- barely works in practice! needs lots of strong assumptions!

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- solves large problems! often slow, needs lots of strong assumptions!

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$$L(x, z, u) = f(x) + g(z) + u^\top (Ax + Bz - c) + \frac{\rho}{2} \|Ax + Bz - c\|_2^2$$

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