HW2: Shabani – QM1 Due Sept 30th

1. Show that A is Hermitian: A = $aI + b\sigma_x + c\sigma_y + d\sigma_z$ where σs are Pauli matrices and I is unity matrix.
2. Problem 1.3 of the book

Problem 1.3 Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

(c) Sketch the graph of $\rho(x)$.

NORMALIZATION

We return now to the statistical interpretation of the wave function (Equation 1.3), which says that $|\Psi(x,t)|^2$ is the probability density for finding the particle at point x, at time t. It follows (Equation 1.16) that the integral of $|\Psi|^2$ over *all* x must be 1 (the particle's got to be *some*where):

$$\boxed{\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = 1.} \qquad (1.20)$$

3. Problem 1.4 of the book

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi, \tag{1.23}$$

and hence also (taking the complex conjugate of Equation 1.23)

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*, \tag{1.24}$$

so

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right]. \tag{1.25}$$

The integral in Equation 1.21 can now be evaluated explicitly:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty}. \tag{1.26}$$

But $\Psi(x,t)$ must go to zero as x goes to ($\pm$) infinity—otherwise the wave function would not be normalizable.[15] It follows that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = 0, \tag{1.27}$$

and hence that the integral is *constant* (independent of time); if Ψ is normalized at $t = 0$, it *stays* normalized for all future time. QED

Problem 1.4 At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A(x/a), & 0 \le x \le a, \\ A(b-x)/(b-a), & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are (positive) constants.

(a) Normalize Ψ (that is, find A, in terms of a and b).

(b) Sketch $\Psi(x,0)$, as a function of x.

(c) Where is the particle most likely to be found, at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

[15] A competent mathematician can supply you with pathological counterexamples, but they do not arise in physics; for us the wave function and all its derivatives go to zero at infinity.

4. Problem 1.14 of the book

You might as well take $\omega = 1$ (that sets the scale for time) and $A = 1$ (that sets the scale for length). Make a plot of x at 10,000 random times, and compare it with $\rho(x)$.
Hint: In Mathematica, first define

x[t_] := Cos[t]

then construct a table of positions:

snapshots = Table[x[π RandomReal[j]], {j, 10000}]

and finally, make a histogram of the data:

Histogram[snapshots, 100, "PDF", PlotRange → {0,2}]

Meanwhile, make a plot of the density function, $\rho(x)$, and, using **Show**, superimpose the two.

Problem 1.14 Let $P_{ab}(t)$ be the probability of finding the particle in the range $(a < x < b)$, at time t.

(a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t),$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x}\right).$$

What are the units of $J(x,t)$? *Comment:* J is called the **probability current**, because it tells you the rate at which probability is "flowing" past the point x. If $P_{ab}(t)$ is increasing, then more probability is flowing into the region at one end than flows out at the other.

(b) Find the probability current for the wave function in Problem 1.9. (This is not a very pithy example, I'm afraid; we'll encounter more substantial ones in due course.)

Problem 1.15 Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 \, dx = 0$$

for any two (normalizable) solutions to the Schrödinger equation (with the same $V(x)$), Ψ_1 and Ψ_2.

Problem 1.16 A particle is represented (at time $t = 0$) by the wave function

$$\Psi(x,0) = \begin{cases} A\left(a^2 - x^2\right), & -a \le x \le +a, \end{cases}$$