Homework 1 Solutions

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1. Find eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_y \vec{a} = \lambda \vec{a} \Rightarrow (\sigma_y - \lambda I)a = 0 \Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - (-i^2) = \lambda^2 - 1 = 0, \boxed{\lambda = \pm 1}$

Eigenvector for $\lambda = 1$: $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \Rightarrow$ $-ia_2 = a_1$ → choose $a = 1$, so $a_2 = i$ → $\begin{pmatrix} 1 \\ i \end{pmatrix}$
$ia_1 = a_2$ (redundant)

Eigenvector for $\lambda = -1$: $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \Rightarrow$ $-ia_2 = -a_1$ → choose $a_1 = 1$, so $a_2 = -i$ → $\begin{pmatrix} 1 \\ i \end{pmatrix}$
$ia_1 = -a_2$ (redundant)

Normalize: $1 = A^2 (1 \mp i)\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = A^2(1 - i^2) = A^2 2 \Rightarrow A = \frac{1}{\sqrt{2}}$, $\boxed{\text{eigenvectors} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}}$

2. a) H_0 is a matrix with a global energy shift, so it takes the form $H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$ some const.

So: $\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \hat{H}_0 + \hat{H}_1 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} + \begin{pmatrix} \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon \end{pmatrix}$ Δ & $\tilde{\Delta}$ are just constants defining the real & imaginary components of H_{12}, H_{21}

$= \begin{pmatrix} E_0 + \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & E_0 - \epsilon \end{pmatrix}.$

We can then define things in terms of $\hat{H}$:

$H_{11} = E_0 + \epsilon \rightarrow H_{11} = 2E_0 - H_{22} \rightarrow E_0 = \frac{H_{11} + H_{22}}{2}$
$H_{22} = E_0 - \epsilon \rightarrow \epsilon = E_0 - H_{22}$

Plug this E_0 back in to find ϵ: $H_{11} = \frac{H_{11} + H_{22}}{2} + \epsilon \Rightarrow \epsilon = \frac{H_{11} - H_{22}}{2}$

We see that we can check this with $E_0 + \epsilon = H_{11}$ & $E_0 - \epsilon = H_{22}$, our original entries.

b) $\hat{H}_0$: can trivially see that the eigenvalues are E_0, but show:
$\hat{H}_0 a = \lambda a \Rightarrow \begin{vmatrix} E_0 - \lambda & 0 \\ 0 & E_0 - \lambda \end{vmatrix} = (E_0 - \lambda)(E_0 - \lambda) = 0, \boxed{\lambda = E_0}$ (doubly degenerate)

$\hat{H}_1$: $\hat{H}_1 a = \lambda a \Rightarrow \begin{vmatrix} \epsilon - \lambda & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon - \lambda \end{vmatrix} = (\epsilon - \lambda)(-\epsilon - \lambda) - (\Delta - i\tilde{\Delta})(\Delta + i\tilde{\Delta}) = 0$

$= \lambda^2 - \epsilon^2 - \Delta^2 - \cancel{i\tilde{\Delta}\Delta} + \cancel{i\Delta\tilde{\Delta}} + i^2\tilde{\Delta}^2 = \lambda^2 - \epsilon^2 - \Delta^2 - \tilde{\Delta}^2 = 0$

$\Rightarrow \boxed{\lambda = \pm\sqrt{\epsilon^2 + \Delta^2 + \tilde{\Delta}^2}}$

c) $\boxed{\hat{H}_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} + \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\tilde{\Delta} \\ i\tilde{\Delta} & 0 \end{pmatrix}} = \boxed{\epsilon\sigma_z + \Delta\sigma_x + \tilde{\Delta}\sigma_y}$

This problem is showing that we can write any 2-level $\hat{H}$ in terms of Pauli matrices! (If $\hat{H}$ is Hermitian)

3. If the system is invariant under a symmetry, this leads to
$H_B = U^\dagger H U$, and $H = H_B$ (they are the same operator, just acting on transformed coordinates.)

Let's compute the commutator assuming invariance:

$$[H, U] = HU - UH.$$

$$= (U H_B \underbrace{U^\dagger) U}_{1} - UH$$

Here we used: $H_B = U^\dagger H U$ because $UU^\dagger = 1$

$U H_B = U(U^\dagger H U) = HU$

$U H_B U^\dagger = H \underbrace{U U^\dagger}_{1} = H$

$$= UH_B - UH$$

using $H = H_B$

$$= 0$$

So if H_B & H are invariant, $[H, U] = 0$.