Homework 2 Solutions

TA: Kate Storey-Fisher
Prof: Javad Shabani
Quantum Mechanics I
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1. $A = aI + b\sigma_x + c\sigma_y + d\sigma_z$

$$= a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + c\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + d\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}a+d & b-ci\\ b+ci & a-d\end{pmatrix}$$

$$A^* = \begin{pmatrix}a+d & b+ci\\ b-ci & a-d\end{pmatrix} \qquad (A^*)^T = \begin{pmatrix}a+d & b-ci\\ b+ci & a-d\end{pmatrix}$$

$A = (A^*)^T$, so A is Hermitian!

2. (1.3 of Griffiths 2nd ed.) [Use integrals 1 & 2 from HW 3 Prob 2]

Gaussian: $\rho(x) = Ae^{-\lambda(x-a)^2}$

a) $$1 = \int_{-\infty}^{\infty}\rho(x)\,dx = A\int_{-\infty}^{\infty}e^{-\lambda(x-a)^2}dx \underset{\text{int 1}}{=} A\sqrt{\frac{\pi}{\lambda}} \Rightarrow \boxed{A = \sqrt{\frac{\lambda}{\pi}}}$$

b) $$\boxed{\langle x\rangle} = A\int_{-\infty}^{\infty}xe^{-\lambda(x-a)^2}dx \underset{\text{u. substitution, } u=x-a}{=} A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du \underset{\text{int 1}}{=} a\sqrt{\frac{\pi}{\lambda}}A = \boxed{a}$$

this term goes to zero bc odd.

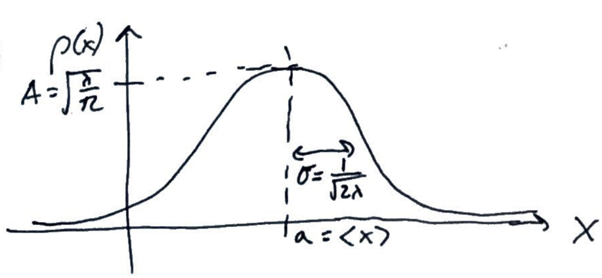
$$\boxed{\langle x^2\rangle} = A\int_{-\infty}^{\infty}x^2e^{-\lambda(x-a)^2}dx \underset{\text{u.sub}}{=} A\int_{-\infty}^{\infty}\Big[\underbrace{u^2e^{-\lambda u^2}}_{\text{int 2}} + \underbrace{2aue^{-\lambda u^2}}_{\text{odd, 0}} + \underbrace{a^2e^{-\lambda u^2}}_{a^2\sqrt{\frac{\pi}{\lambda}}}\Big]du$$

int 2: $\frac{1}{2\lambda}\int_{-\infty}^{\infty}du\,e^{-\lambda u^2} = \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}$

$$= \sqrt{\frac{\lambda}{\pi}}\left(\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + a^2\sqrt{\frac{\pi}{\lambda}}\right) = \boxed{a^2 + \frac{1}{2\lambda}}$$

$$\boxed{\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left(a^2 + \frac{1}{2\lambda}\right) - a^2} = \frac{1}{\sqrt{2\lambda}}}$$

c)

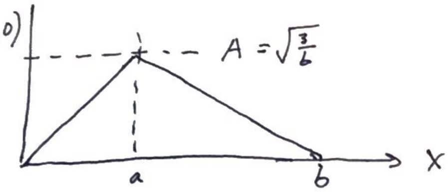


3. (1.4 from Griffith's 2^{nd} ed.)

a) $1 = \int |\Psi|^2 dx = \int_0^a A^2 \frac{x^2}{a^2} dx + \int_a^b A^2 \frac{(b-x)^2}{(b-a)^2} = \frac{A^2}{a^2}\left(\frac{1}{3}x^3\right)\Big]_0^a + \frac{A^2}{(b-a)^2}\left[-\frac{1}{3}(b-x)^3\right]_a^b$

$= \frac{A^2}{3}a + \frac{A^2}{3(b-a)^2}(b-a)^3 = \frac{A^2}{3}\big(a + (b-a)\big) = \frac{A^2}{3}b \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}}$

b) $\Psi(x,0)$



c) the particle is most likely to be found at $\boxed{a}$ the maximum of $\Psi(x,0)$.

d) $\boxed{P(\text{left of } a)} = \int_0^a A^2 \frac{x^2}{a^2} dx \underset{\text{plug in } A}{=} \frac{3}{ba^2}\int_0^a x^2 dx = \frac{3}{ba^2}\left(\frac{x^3}{3}\right)\Big]_0^a = \boxed{\frac{a}{b}}$

case $b = a$: $\boxed{P = 1}$ ✓, case $b = 2a$: $\boxed{P = \frac{1}{2}}$ ✓ ← Check by looking at graph

e) $\boxed{\langle x \rangle} = \int_0^a x\left(\frac{A^2x^2}{a^2}\right)dx + \int_a^b x \frac{A^2(b-x)^2}{(b-a)^2} dx$

$= \frac{A^2}{a^2}\underbrace{\int_0^a x^3 dx}_{\frac{x^4}{4}\big]_0^a = \frac{a^4}{4}} + \frac{A^2}{(b-a)^2}\underbrace{\int x(b-x)^2 dx}$

u substitution: $u = x - b$, $du = dx$, $x = u + b$; $b \to 0$, $a \to a - b$

$\int (u+b)u^2 du = \int u^3 du + \int b u^2 du$

$= \frac{u^4}{4}\Big]_{a-b}^0 + b\left(\frac{u^3}{3}\right)\Big]_{a-b}^0 = -\frac{(a-b)^4}{4} - \frac{b(a-b)^3}{3}$

plugged in A^2

$= \frac{3}{b}\left[\frac{a^4}{4} - \frac{(b-a)^4}{4(b-a)^2} + \frac{b(b-a)^3}{3(b-a)^2}\right] = \frac{3}{b}\left[\frac{a^4}{4} - \frac{(b-a)^2}{4} + \frac{b(b-a)}{3}\right]$

$= \frac{3}{b}\left[\cancel{\frac{a^4}{4}} - \frac{b^2 - 2ab + \cancel{a^2}}{4} + \frac{b^2 - \cancel{ab}}{3}\right] = \frac{3}{b}\left(\frac{b^2}{12} + \frac{ab}{6}\right) = \frac{b}{4} + \frac{a}{2} = \boxed{\frac{2a+b}{4}}$

4. a) $\left(P_{ab}(t) = \int_a^b \Psi^* \Psi dx\right)$ $\frac{dP_{ab}}{dt} = \frac{d}{dt}\int_a^b \Psi^*\Psi dx = \int_a^b \left(\Psi \frac{d\Psi^*}{dt} + \Psi^* \frac{d\Psi}{dt}\right) dx$

$\frac{dP_{ab}}{dt} =$ use Schröd eqn: $= \int_a^b \left(-\frac{i\hbar}{2m}\Psi\frac{\partial^2\Psi^*}{\partial x^2} + \cancel{\frac{i}{\hbar}V\Psi\Psi^*} + \frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}V\Psi^*\Psi}\right) dx$

$= \int_a^b \frac{i\hbar}{2m}\left(-\Psi\frac{\partial^2\Psi^*}{\partial x^2} + \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right) dx$

Integrate by parts — keeping boundary term! $= \frac{i\hbar}{2m}\left[-\Psi\frac{\partial\Psi^*}{\partial x}\Big]_a^b + \cancel{\int_a^b \frac{\partial\Psi}{\partial x}\frac{\partial\Psi^*}{\partial x}dx} + \Psi^*\frac{\partial\Psi}{\partial x}\Big]_a^b - \cancel{\int_a^b \frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}dx}\right]$

$= \frac{i\hbar}{2m}\left[-\Psi\frac{\partial\Psi^*}{\partial x}\Big|_a^b + \Psi^*\frac{\partial\Psi}{\partial x}\Big]_a^b\right] = J(a,t) - J(b,t) \Rightarrow \boxed{\frac{dP_{ab}(t)}{dt} = J(a,t) - J(b,t)}$

The units of J are s^{-1} inverse time, as P is unitless and $\frac{d}{dt}$ has units s^{-1}!

b) $\Psi(x,t) = Ae^{-a\left[\frac{mx^2}{\hbar} + it\right]} \rightarrow \frac{\partial\Psi}{\partial x} = -\frac{2Amx}{\hbar}e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$, $\frac{\partial\Psi^*}{\partial x} = -\frac{2Amx}{\hbar}e^{-a\left[\frac{mx^2}{\hbar} - it\right]}$

$\boxed{J(x,t) = \frac{i\hbar}{2m}\left(\frac{2A^2mx}{\hbar^2}\right)\left[-e^{-\frac{2amx^2}{\hbar}} + e^{-\frac{2amx^2}{\hbar}}\right]} = \boxed{0}$