# Homework 3 Solutions

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Quantum Mechanics I
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1. $$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\int \psi^* x \psi \, dx = \int\left[\frac{\partial \psi^*}{\partial t} x \psi + \psi^* x \frac{\partial \psi}{\partial t}\right] dx$$

Using Schröd. eqn:
$$= \int \frac{i\hbar}{2m} x \left[-\psi \frac{\partial^2 \psi^*}{\partial x^2} + \psi^* \frac{\partial^2 \psi}{\partial x^2}\right] dx$$

Recall $J = \frac{i\hbar}{2m}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right)$, probability density, so

$$\frac{dJ}{dx} = \frac{i\hbar}{2m}\left(\cancel{\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x}} + \psi \frac{\partial^2 \psi^*}{\partial x^2} - \cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}} - \psi^* \frac{\partial^2 \psi}{\partial x^2}\right) = \frac{i\hbar}{2m}\left(\psi \frac{\partial^2 \psi^*}{\partial x^2} - \psi^* \frac{\partial^2 \psi}{\partial x^2}\right)$$

So we have:
$$\frac{d\langle x\rangle}{dt} = \int \frac{i\hbar}{2m} x \frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) dx \underset{\text{integrate by parts, kill boundary term}}{=} -\int \frac{i\hbar}{2m} \underbrace{\frac{\partial}{\partial x}(x)}_{=1} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) dx$$

$$= -\int \frac{i\hbar}{2m}\left(\psi^* \frac{\partial \psi}{\partial x} + \underbrace{\frac{\partial \psi}{\partial x}\psi^*}_{\text{we integrated by parts again here}}\right) dx = -\frac{i\hbar}{m}\int \psi^* \frac{\partial}{\partial x}\psi \, dx$$

This is just the momentum operator, over m! So we have:
$$\boxed{\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}}$$

$$\langle p\rangle = \int \psi^*\left(-i\hbar \frac{\partial}{\partial x}\right)\psi \, dx = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x} dx$$

$$\frac{d\langle p\rangle}{dt} = -i\hbar \int\left[\frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial x} + \underbrace{\psi^* \frac{\partial^2 \psi}{\partial t \partial x}}_{\text{int. by parts, kill boundary}}\right] dx = -i\hbar \int\left[\frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial t}\frac{\partial \psi^*}{\partial x}\right] dx$$

Bring in $-i\hbar$:
$$= \int\left[\underbrace{\left(-i\hbar \frac{\partial \psi^*}{\partial t}\right)}_{=\left(i\hbar \frac{\partial \psi}{\partial t}\right)^*}\frac{\partial \psi}{\partial x} + \left(i\hbar \frac{\partial \psi}{\partial t}\right)\frac{\partial \psi^*}{\partial x}\right] dx \underset{\text{Schröd. eqn}}{=} \int\left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \underbrace{V\psi^*\right)\frac{\partial \psi}{\partial x}}_{\text{int by parts: } = -\psi \frac{\partial}{\partial x}(V\psi^*) = -\psi\left(\frac{\partial V}{\partial x}\psi^* + V\frac{\partial \psi^*}{\partial x}\right)} + \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right)\frac{\partial \psi^*}{\partial x}\right] dx$$

$$= \int\left[\cancel{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x}} \underbrace{- \psi \frac{\partial V}{\partial x}\psi}_{-\left\langle \frac{\partial V}{\partial x}\right\rangle^{*}} \cancel{- \psi V \frac{\partial \psi^*}{\partial x}} \underbrace{- \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial \psi^*}{\partial x}}_{\text{int by parts: } + \frac{\hbar^2}{2m}\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi^*}{\partial x^2} \text{, now cancels w/ first term}} + \cancel{V\psi \frac{\partial \psi^*}{\partial x}}\right] dx$$

So:
$$\boxed{\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle}$$

These equations are known as <u>Ehrenfest's Theorem</u>.

[Int 1 & Int 2 given in problem]

2. Gaussian wavepacket: $\psi(x) = N\exp\left(-\frac{1}{2}\frac{x^2}{a^2}\right) = Ne^{-\alpha x^2}$ with $\alpha = \frac{1}{2a^2}$

Normalization: $1 = \int \psi^*(x)\psi(x) = N^2 \int e^{-2\alpha x^2} dx \underset{\uparrow \text{Int 1}}{=} N^2\sqrt{\frac{\pi}{2\alpha}} \Rightarrow N^2 = \sqrt{\frac{2\alpha}{\pi}} = \frac{1}{a\sqrt{\pi}}$

$$\langle x\rangle = N^2\int_{-\infty}^{\infty} x e^{-2\alpha x^2} dx = N^2\left[-\frac{1}{4\alpha}e^{-2\alpha x^2}\right]_{-\infty}^{\infty} = N^2[0-0] = 0$$

$$\langle x^2\rangle = N^2\int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} dx \underset{\uparrow \text{Int 2}}{=} N^2 \frac{1}{4\alpha}\int_{-\infty}^{\infty} e^{-2\alpha x^2} \underset{\uparrow \text{Int 1}}{=} \frac{N^2}{4\alpha}\sqrt{\frac{\pi}{2\alpha}} \underset{\uparrow \text{plug in } N \,\&\, \alpha}{=} \frac{a^2}{2}$$

$$\boxed{\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{2} - 0^2} = \frac{a}{\sqrt{2}}}$$

$$\langle p\rangle = -N^2 i\hbar \int e^{-\alpha x^2}\underbrace{\frac{d}{dx}\left(e^{-\alpha x^2}\right)}_{-2\alpha x e^{-\alpha x^2}} dx = 2N\alpha i\hbar \int x e^{-2\alpha x^2} dx$$

$$= 2N\alpha i\hbar \left[-\frac{1}{4\alpha}e^{-2\alpha x^2}\right]_{-\infty}^{\infty} = 2N\alpha i\hbar(0-0) = 0$$

$$\langle p^2\rangle = -N^2\hbar^2 \int e^{-\alpha x^2}\underbrace{\frac{d^2}{dx^2}\left(e^{-\alpha x^2}\right)}_{=\frac{d}{dx}\left(-2\alpha x e^{-\alpha x^2}\right) = -2\alpha\left(e^{-\alpha x^2} - 2\alpha x^2 e^{-\alpha x^2}\right)} dx = 2N^2\hbar^2\alpha\int\left[e^{-2\alpha x^2} - 2\alpha x^2 e^{-2\alpha x^2}\right]dx$$

$$\underset{\text{Int 1\&2}}{=} 2N^2\hbar^2\alpha\left[\sqrt{\frac{\pi}{2\alpha}} - \underbrace{2\alpha\left(\frac{1}{4\alpha}\right.}_{\frac{1}{2}}\underbrace{\left.\int_{-\infty}^{\infty} e^{-2\alpha x^2}dx\right)}_{\text{Int 1: } \sqrt{\frac{\pi}{2\alpha}}}\right] = N^2\hbar^2\alpha\sqrt{\frac{\pi}{2\alpha}} \underset{\uparrow \text{plug in } N \,\&\, \alpha}{=} \frac{\hbar^2}{2a^2}$$

$$\boxed{\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar^2}{2a^2} - 0^2} = \frac{1}{a\sqrt{2}}\hbar}$$

Heisenberg uncertainty product:

$$\boxed{\sigma_x\sigma_p = \left(\frac{a}{\sqrt{2}}\right)\left(\frac{1}{a\sqrt{2}}\hbar\right) = \frac{\hbar}{2}}$$

So the Gaussian is a <u>minimum-uncertainty</u> wavepacket!

3. Take an operator $\hat{A}$ with eigenfunctions $f$ & eigenvalues $\lambda_A$: $\hat{A}f = \lambda_A f$.

Now apply operator $\hat{B}$: $\hat{B}(\hat{A}f) = \hat{B}(\lambda_A f) = \lambda_A(\hat{B}f)$.

Assume $\hat{A}$ & $\hat{B}$ commute, $[\hat{A},\hat{B}]=0$, so $\hat{B}(\hat{A}f) = \hat{A}(\hat{B}f)$. We just found that $\hat{B}(\hat{A}f) = \lambda_A(\hat{B}f)$, so we require also $\hat{A}(\hat{B}f) = \lambda_A(Bf)$. This is just the eigenvalue eqn., so $\hat{B}f$ must be an eigenfunction of $\hat{A}$. Only constant multiples of $f$ can be eigenfunctions of $\hat{A}$, so we must have $\hat{B}f = bf$ ($b$ is a const.) This is just the eigenvalue eqn. for $\hat{B}$! So $f$ are also eigenfncs. of $\hat{B}$, and $\hat{A}$ & $\hat{B}$ have the same set of eigenfunctions.

From uncertainty principle $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$, if $[A,B]=0$, $\sigma_A^2\sigma_B^2 \geq 0$ & there's no limit on the uncertainty