HW 5 Solutions

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Quantum Mechanics I
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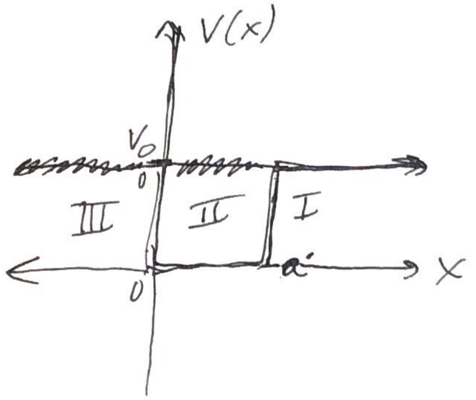
1. Calculate energy & wavefunction of:



$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x < a \\ V_0 & x \geq a \end{cases}$$

Start in section II, apply schrödinger eqn:

$\boxed{\text{II}}$ $$\frac{\partial^2 \psi}{\partial x^2} = -\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi \quad (V=0)$$

General solution $\psi_{II} = A\sin(kx) + B\cos(kx)$

Apply BC $\psi_{II}(0) = \psi_{III}(0) = 0$ bc infinite wall

$\Rightarrow \psi_{II}(0) = A\sin(0) + B\cos(0) = B$, so $B = 0$

$\Rightarrow \boxed{\psi_{II}(x) = A\sin(kx)}$

Now Section I:

$\boxed{\text{I}}$: $$\frac{\partial^2\psi}{\partial x^2} = \underbrace{-\frac{2m(E-V_0)}{\hbar^2}}_{\ell^2}\psi \quad (V=V_0)$$

$\ell^2 \rightarrow$ Note that ℓ^2 includes the minus sign, because $E < V_0$ and we need $\ell^2 > 0$

General solution: $\psi_I = Ce^{\ell x} + De^{-\ell x}$

As $x \to \infty$, first term would blow up, so must have $C=0$; then $\boxed{\psi_I(x) = De^{-\ell x}}$

And of course:

$\boxed{\text{III}}$ $\boxed{\psi_{III}(x) = 0}$

Now we can apply continuity @ $x=a$:

$\psi_{II}(a) = \psi_I(a) \Rightarrow A\sin(ka) = De^{-\ell a}$ ①

And continuity of derivative @ $x=a$:

$\psi_{II}'(a) = \psi_I'(a) \Rightarrow Ak\cos(ka) = -D\ell e^{-\ell a}$ ②

Let's divide $\frac{②}{①}$: $k\cot(ka) = -\ell \Rightarrow \boxed{\cot(ka) = -\frac{\ell}{k}}$

Now, ℓ & k are both only dependent on E + constants, so this is a transcendental equation that we could solve graphically. We can rewrite it more nicely with the following change of variables:

see that $k^2 + \ell^2 = \frac{2mE}{\hbar^2} + \left(-\frac{2m(E-V_0)}{\hbar^2}\right) = \frac{2mV_0}{\hbar^2}$, so call $z_0 = \frac{a}{\hbar}\sqrt{2mV_0} = a\sqrt{k^2+\ell^2}$

and $z = ka = \frac{a}{\hbar}\sqrt{2mE}$

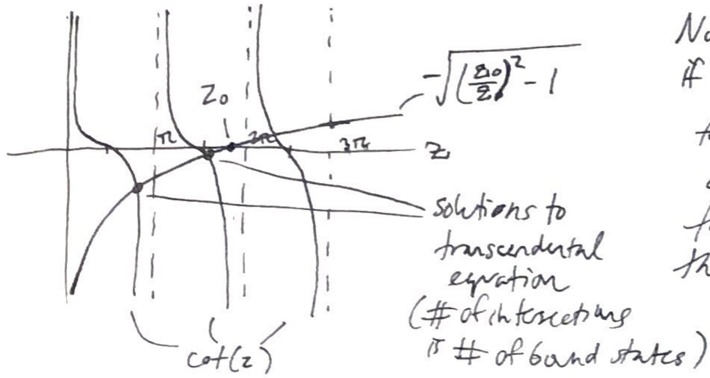
then we have $z_0^2 - z^2 = \frac{a^2}{\hbar^2} 2mV_0 - \frac{a^2}{\hbar^2} 2mE = \frac{a^2}{\hbar^2} 2m(V_0 - E) = a^2 l^2$

$$\Rightarrow \sqrt{z_0^2 - z^2} = la$$

Now rewrite $\cot(ka) = -\frac{l}{k}$ as:

$$\boxed{\cot(z) = -\frac{l}{k}\frac{a}{a} = -\frac{\sqrt{z_0^2 - z^2}}{z} = -\sqrt{\left(\frac{z_0}{z}\right)^2 - 1}}$$

This is now clearly just a function of z. We can sketch the two sides:



Notice that there are only real solutions if $\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} > 0$, so $z_0 > z$. Going back to the definitions, this is $\frac{a}{\hbar}\sqrt{2mV_0} > \frac{a}{\hbar}\sqrt{2mE}$, or $V_0 > E \Rightarrow E < V_0$; because of course the energies have to be within the dip of the well for bound states!

So our final solution is:

$$\psi(x) = \begin{cases} 0 & x < 0 \\ A\sin(kx) & 0 \le x < a \\ De^{-lx} & x > a \end{cases}$$

where k & l are found by solving $\cot(ka) = -\frac{l}{k}$

$$E = \frac{k^2\hbar^2}{2m}$$ (by rearranging k^2 definition)

We could relate A & D using the continuity BC at $x=a$, and then find the remaining constant via normalization, but the problem said we didn't have to worry about normalizing ☺