QM I Midterm I Solutions

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1. 3-level system: $\hat{H} = \epsilon \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}$

a) Yes, $\hat{H}$ is Hermitian. $\hat{H}^* = \epsilon \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix}$, $(\hat{H}^*)^T = \epsilon \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}$

we can see that $(\hat{H}^*)^T = \hat{H}$, Hermitian definition ✓

b) In Dirac notation, we can translate: $H_{ij} = \langle e_i | \hat{H} | e_j \rangle$ where e_i are the 3 states, $|1\rangle, |2\rangle, |3\rangle$. Then we can read off the matrix elements, seeing that all the 0 elements will disappear:

$$\hat{H} = \epsilon \left[-i|1\rangle\langle 2| + i|2\rangle\langle 1| + 2i|2\rangle\langle 3| - 2i|3\rangle\langle 2| \right]$$

To check this, you could go the other direction and reconstruct $\hat{H}$ by sandwiching this $\hat{H}$ in Dirac form between all possible pairs of states.

c) Eigenvalues:

$$\det \begin{pmatrix} -\lambda & -i\epsilon & 0 \\ i\epsilon & -\lambda & 2i\epsilon \\ 0 & -2i\epsilon & -\lambda \end{pmatrix} = -\lambda(\lambda^2 + \underbrace{4i^2\epsilon^2}_{-4\epsilon^2}) + \underbrace{i\epsilon(-\lambda i\epsilon)}_{\lambda\epsilon^2} = -\lambda^3 + 5\lambda\epsilon^2$$
$$= \lambda(5\epsilon^2 - \lambda^2) = 0$$
$$\Rightarrow \boxed{\lambda = 0, \sqrt{5}\epsilon, -\sqrt{5}\epsilon}$$

d) Eigenvector for largest eigenvalue, $\lambda = \sqrt{5}\epsilon$:

$H\psi = \lambda\psi$

$$\epsilon \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} = \sqrt{5}\epsilon \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

System of eqns (cancel ϵ):

$-ib = \sqrt{5}a \longrightarrow b = i\sqrt{5}a$

$ca + 2ic = \sqrt{5}b$

$-2ib = \sqrt{5}c \longrightarrow c = -\frac{2ib}{\sqrt{5}} = -\frac{2i(i\sqrt{5}a)}{\sqrt{5}} = 2a$

Choose $a = 1$, then: $v = \begin{pmatrix} 1 \\ i\sqrt{5} \\ 2 \end{pmatrix} A$ (A is normalization constant)

Normalize: $1 = (v^*)^T v = A^2 (1 \; -i\sqrt{5} \; 2) \begin{pmatrix} 1 \\ i\sqrt{5} \\ 2 \end{pmatrix} = A^2[1 + 5 + 4] = 10A^2 = 1 \Rightarrow A = \frac{1}{\sqrt{10}}$, $\boxed{v = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ i\sqrt{5} \\ 2 \end{pmatrix}}$

e) Normalize $\psi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$: $1 = |\psi|^2 = A^2 (1 \; 1 \; 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = A^2(1^2 + 1^2 + 1^2) = 3A^2 = 1 \Rightarrow A = \frac{1}{\sqrt{3}}$

$\psi = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$ in dirac notation

f) $\langle H \rangle = \langle \psi | H | \psi \rangle = \frac{\epsilon^2}{3} (1 \; 1 \; 1) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{\epsilon^2}{3}(1 \; 1 \; 1) \begin{pmatrix} -i \\ 3i \\ -2i \end{pmatrix} = \frac{\epsilon^2}{3}(-i + 3i - 2i) = \boxed{0 = \langle H \rangle}$

g) $\sigma_H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$; $\langle H^2 \rangle = \langle \psi | HH | \psi \rangle = \frac{\epsilon^2}{3}(1 \; 1 \; 1) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} \begin{pmatrix} -i \\ 3i \\ -2i \end{pmatrix} = \frac{\epsilon^2}{3}(1 \; 1 \; 1) \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \frac{14\epsilon^2}{3}$

$$\Rightarrow \sigma_H = \sqrt{\frac{14\epsilon^2}{3} - 0} = \boxed{\sqrt{\frac{14}{3}}\epsilon = \sigma_H}$$

2) Particle in infinite quantum well b/w $0 \leq x \leq a$:

$$\psi(x, t=0) = \sqrt{\frac{8}{5a}}\left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \sin\left(\frac{\pi x}{a}\right)$$

a) $\psi(x, t=t_0) = ?$

First notice that we can rewrite ψ in terms of just sins, using $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\psi(x,t=0) = \sqrt{\frac{8}{5a}}\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\right] = \sqrt{\frac{8}{5a}}\left[\sin\left(\frac{\pi x}{a}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{a}\right)\right]$$

these two terms are just the $n=1$ & $n=2$ states of the infinite well:

inf well: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$, and $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

So $\psi(x,t=0) = \sqrt{\frac{8}{5a}}\left[\sqrt{\frac{a}{2}}\psi_1 + \sqrt{\frac{a}{2}}\frac{1}{2}\psi_2\right] = \frac{2}{\sqrt{5}}\left[\psi_1 + \frac{1}{2}\psi_2\right]$

To include time dependence, use: $\psi(x,t) = \sum_n c_n \psi_n e^{-itE_n/\hbar}$ where E_n's are these

and we can read off that $c_1 = \frac{2}{\sqrt{5}}$ & $c_2 = \frac{1}{\sqrt{5}}$:

$$\boxed{\psi(x,t) = \frac{2}{\sqrt{5}}\left(\psi_1 e^{-\frac{it}{\hbar}\underbrace{\left(\frac{\pi^2\hbar^2}{2ma^2}\right)}_{E_1}} + \frac{1}{2}\psi_2 e^{-\frac{it}{\hbar}\underbrace{\left(\frac{4\pi^2\hbar^2}{2ma^2}\right)}_{E_2}}\right)}$$

b) average energy $\langle H\rangle = \langle\psi|H|\psi\rangle = \sum|c_n|^2 E_n$ (very useful eqn, see Griffiths!)

$$\Rightarrow \boxed{\langle H\rangle = \frac{4}{5}\left[\frac{\pi^2\hbar^2}{2ma^2} + \left(\frac{1}{2}\right)^2\frac{4\pi^2\hbar^2}{2ma^2}\right] = \frac{4\pi^2\hbar^2}{5ma^2}}$$

The energy is independent of time so $\langle H\rangle$ at $t=0$ is the same as $\langle H\rangle$ at $t=t_0$.

c) $P(0 \leq x \leq \frac{a}{2}, t=t_0) = \int_0^{a/2}|\psi(x,t=t_0)|^2 dx$

$$= \frac{4}{5}\int_0^{a/2}\left(\psi_1^* e^{itE_1/\hbar} + \frac{1}{2}\psi_2^* e^{itE_2/\hbar}\right)\left(\psi e^{-itE_1/\hbar} + \frac{1}{2}\psi_2 e^{-itE_2/\hbar}\right)dx$$

$$= \frac{4}{5}\int_0^{a/2}\Big(\underbrace{\psi_1^*\psi_1}_{I_1} + \underbrace{\frac{1}{4}\psi_2^*\psi_2}_{I_2} + \underbrace{\frac{1}{2}\psi_1^*\psi_2 e^{-it(E_2-E_1)/\hbar} + \frac{1}{2}\psi_2^*\psi_1 e^{it(E_2-E_1)/\hbar}}_{I_3}\Big)dx$$

$$I_1 = \left(\frac{2}{a}\right)\int_0^{a/2}\sin^2\left(\frac{\pi x}{a}\right)dx = \left(\frac{2}{a}\right)\int_0^{a/2}\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right]dx = \left(\frac{2}{a}\right)\left[\frac{1}{2}x - \frac{1}{2}\left(\frac{a}{2\pi}\right)\sin\left(\frac{2\pi x}{a}\right)\right]_0^{a/2} = \frac{1}{2}$$

$$I_2 = \left(\frac{2}{a}\right)\frac{1}{4}\int_0^{a/2}\sin^2\left(\frac{2\pi x}{a}\right)dx = \frac{1}{2a}\int_0^{a/2}\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi x}{a}\right)\right]dx = \frac{1}{2a}\left[\frac{1}{2}x - \frac{1}{2}\left(\frac{a}{4\pi}\right)\sin\left(\frac{4\pi x}{a}\right)\right]_0^{a/2} = \frac{1}{8}$$

$$I_3 = \underbrace{\frac{1}{2}\left(e^{it(E_2-E_1)/\hbar} + e^{-it(E_2-E_1)/\hbar}\right)}_{\cos\left(\frac{t(E_2-E_1)}{\hbar}\right),\text{ by Euler's identity}}\left(\frac{2}{a}\right)\underbrace{\int_0^{a/2}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx}_{\text{using } \sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]}:$$

$$= \left(\frac{2}{a}\right)\cos\left(\frac{t(E_2-E_1)}{\hbar}\right)\int_0^{a/2}\frac{1}{2}\left[\cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right)\right]dx = \frac{1}{a}\cos\left(\frac{t(E_2-E_1)}{\hbar}\right)\underbrace{\left[\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right) - \left(\frac{a}{3\pi}\right)\sin\left(\frac{3\pi x}{a}\right)\right]_0^{\frac{a}{2}}}_{\left(\frac{a}{\pi}\right) + \left(\frac{a}{3\pi}\right) = \frac{4a}{3\pi}}$$

$$= \frac{2a}{3\pi}\cos\left(\frac{t\overbrace{(E_2-E_1)}^{\frac{3\pi^2\hbar^2}{2ma^2}}}{\hbar}\right)$$

$$\boxed{P = \frac{4}{5}\left(\frac{1}{2} + \frac{1}{8} + \frac{4a}{a\,3\pi}\cos\left(\frac{t(E_2-E_1)}{\hbar}\right)\right) = \frac{1}{2} + \frac{16}{15\pi}\cos\left(\frac{t}{\hbar}\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)\right)}$$