HW7: Shabani – QM1 Due on Tue Dec 7th

Q1:

(a) Work out all of the **canonical commutation relations** for components of the operators **r** and **p**: $[x, y]$, $[x, p_y]$, $[x, p_x]$, $[p_y, p_z]$, and so on. *Answer*:

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [p_i, p_j] = 0. \qquad [4.10]$$

(b) Show that

$$\frac{d}{dt}\langle \mathbf{r} \rangle = \frac{1}{m}\langle \mathbf{p} \rangle, \quad \text{and} \quad \frac{d}{dt}\langle \mathbf{p} \rangle = \langle -\nabla V \rangle. \qquad [4.11]$$

(Each of these, of course, stands for *three* equations—one for each component.) *Hint*: Note that Equation 3.148 is valid in three dimensions.

(c) Formulate Heisenberg's uncertainty principle in three dimensions. *Answer*:

$$\sigma_x \sigma_{p_x} \geq \hbar/2, \quad \sigma_y \sigma_{p_y} \geq \hbar/2, \quad \sigma_z \sigma_{p_z} \geq \hbar/2, \qquad [4.12]$$

but there is no restriction on, say, $\sigma_x \sigma_{p_y}$.

Q2:

Problem 4.2 Use separation of variables in Cartesian coordinates to solve the infinite cubical well (or "particle in a box"):

$$V(x, y, z) = \begin{cases} 0, & \text{if } x, y, z \text{ are all between 0 and } a; \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Find the stationary state wave functions and the corresponding energies.

(b) Call the distinct energies $E_1, E_2, E_3, \ldots$, in order of increasing energy. Find E_1, E_2, E_3, E_4, E_5, and E_6. Determine the degeneracy of each of these energies (that is, the number of different states that share the same energy). Recall (Problem 2.42) that degenerate bound states do not occur in *one* dimension, but they are common in three dimensions.

(c) What is the degeneracy of E_{14}, and why is this case interesting?

Q3:

(a) Prove that if f is simultaneously an eigenfunction of L^2 and of L_z (Equation 4.104), the square of the eigenvalue of L_z cannot exceed the eigenvalue of L^2. *Hint*: Examine the expectation value of L^2.

(b) As it turns out (see Equations 4.118 and 4.119), the square of the eigenvalue of L_z never even *equals* the eigenvalue of L^2 (except in the special case $l = m = 0$). Comment on the implications of this result. Show that it is enforced by the uncertainty principle (Equation 4.100), and explain how the special case gets away with it.

Equations are listed here:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m, \tag{4.118}$$

where

$$l = 0, 1/2, 1, 3/2, \ldots; \quad m = -l, -l+1, \ldots, l-1, l. \tag{4.119}$$

$$\sigma_{L_x}\sigma_{L_y} \geq \frac{\hbar}{2}|\langle L_z\rangle|. \tag{4.100}$$

Q4:

(a) Prove that for a particle in a potential $V(\mathbf{r})$ the rate of change of the expectation value of the orbital angular momentum $\mathbf{L}$ is equal to the expectation value of the torque:

$$\frac{d}{dt}\langle \mathbf{L}\rangle = \langle \mathbf{N}\rangle,$$

where

$$\mathbf{N} = \mathbf{r} \times (-\nabla V).$$

(This is the rotational analog to Ehrenfest's theorem.)

(b) Show that $d\langle \mathbf{L}\rangle/dt = 0$ for any spherically symmetric potential. (This is one form of the quantum statement of **conservation of angular momentum**.)